CHAPTER 4 ITERATIVE ALGORITHMS FOR SOLVING THE SYSTEM OF MIXED EQUILIBRIUM, FIXED POINT AND VARIATIONAL INCLUSIONS PROBLEMS

4.1 Strong Convergence Theorem for Inverse-Strongly Monotone operators

In this section, we prove a strong convergence theorem for solving a common solution of the set of solutions of fixed point for an infinite family of nonexpansive mappings, the set of solution of a system of mixed equilibrium problems and the set of solutions of the variational inclusion for an β-inverse-strongly monotone mapping in a real Hilbert space. we show for some application and numerical example.

Theorem 4.1.1. *Let H be a real Hilbert space, C a close convex subset of H and B be an β-inverse-strongly monotone operator. Let $\varphi : C \longrightarrow \mathbb{R}$ be a convex and lower semicontinuous function, $f : C \longrightarrow C$ be a contraction mapping with coefficient α $(0 < \alpha < 1)$, $M : H \longrightarrow 2^H$ be a maximal monotone operator. Let A be a strongly positive linear bounded operator of H into itself with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$ and $\lambda \in (0, 2\beta)$. Let $\{T_n\}$ be a family of nonexpansive mappings of H into itself such that*

$$\theta := \cap_{n=1}^{\infty} F(T_n) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap I(B, M) \neq \emptyset.$$

Suppose that $\{x_n\}$ is a sequence generated by the following algorithm for $x_0 \in C$ arbitrarily and

$$\begin{cases} u_n = K^{F_N}_{r_n,n} \cdot K^{F_{N-1}}_{r_{n-1},n} \cdot K^{F_{N-2}}_{r_{n-2},n} \cdot \ldots \cdot K^{F_2}_{r_2,n} \cdot K^{F_1}_{r_1,n} \cdot x_n, \ \forall n \in N \\ x_{n+1} = P_C[\epsilon_n \gamma f(x_n) + (I - \epsilon_n A) W_n J_{M,\lambda}(u_n - \lambda B u_n)] \end{cases} \quad (4.1.1)$$

for all n = 1,2,3,..., where

$$
\begin{aligned}
K^{F_i}_{r_{i,n}}(x) &= \{u_n \in C := F_i(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_{i,n}}\langle y - u_n, u_n - x_n\rangle \geq 0, \\
&\quad \forall y \in C\},\ i = 1, 2, 3, \ldots, N,
\end{aligned}
$$

and the following conditions are satisfied

(C1): $\{\epsilon_n\} \subset (0,1),\ \lim_{n\to 0}\epsilon_n = 0,\ \sum_{n=1}^{\infty}\epsilon_n = \infty,\ \sum_{n=1}^{\infty}|\epsilon_{n+1} - \epsilon_n| < \infty;$

(C2): $\{r_{k,n}\} \subset [c, d]$ *with* $c, d \in (0, 2\sigma)$ *and* $\sum_{n=1}^{\infty}|r_{k,n+1} - r_{k,n}| < \infty.$

Then, the sequence $\{x_n\}$ *converges strongly to* $q \in \theta$*, where* $q = P_\theta(\gamma f + I - A)(q)$ *which solves the following variational inequality:*

$$\langle(\gamma f - A)q, p - q\rangle \leq 0, \quad \forall p \in \theta, \tag{4.1.2}$$

which is the optimality condition for the minimization problem

$$\min_{q\in\theta} \frac{1}{2}\langle Aq, q\rangle - h(q), \tag{4.1.3}$$

where h is a potential function for γf *(i.e.,* $h'(q) = \gamma f(q)$ *for* $q \in H$*).*

Proof. Since condition (C1), we may assume without loss of generality, then $\epsilon_n \in (0, \| A \|^{-1})$ for all n. By Lemma 2.6.9 we have $\|I - \epsilon_n A\| \leq 1 - \epsilon_n\bar{\gamma}$. Next, we will assume that $\|I - A\| \leq \|1 - \bar{\gamma}\|$.

Next, we will divide the proof into six steps.

Step 1. First, will show that $\{x_n\}$ and $\{u_n\}$ are bounded. Since B is β - inverse strongly monotone mappings, we have

$$
\begin{aligned}
\|(I - \lambda B)x - (I - \lambda B)y\|^2 &= \|Ix - \lambda Bx - Iy + \lambda By\|^2 \\
&= \|x - y - \lambda Bx + \lambda By\|^2 \\
&= \|(x - y) - \lambda(Bx + By)\|^2 \\
&\leq \|x - y\|^2 - 2\lambda\langle x - y\rangle\langle Bx + By\rangle \\
&\quad +\lambda^2\|Bx - By\|^2 \\
&\leq \|x - y\|^2 - 2\lambda\beta\|Bx + By\|^2 \\
&\quad +\lambda^2\|Bx - By\|^2 \\
&\leq \|x - y\|^2 + \lambda(\lambda - 2\beta\|Bx + By\|^2 \\
&\leq \|x - y\|^2 \qquad (4.1.4)
\end{aligned}
$$

if $0 < \lambda < 2\beta$ *and* $0 < r_n < 2\sigma$, *then* $I - \lambda B$ is all nonexpansive.

Put $y_n = J_{M,\lambda}(u_n - \lambda B u_n),\ n \geq 0$. It follows that

$$\begin{aligned} \|y_n - q\| &= \|J_{M,\lambda}(u_n - \lambda B u_n) - J_{M,\lambda}(q - \lambda B q)\| \\ &\leq \|(u_n - \lambda B u_n) - (q - \lambda B q)\| \\ &\leq \|u_n - q\|. \end{aligned} \tag{4.1.5}$$

By Lemma 2.6.13, we have

$$\begin{aligned} u_n &= K^{F_N}_{r_n,n} \cdot K^{F_{N-1}}_{r_{n-1},n} \cdot K^{F_{N-2}}_{r_{n-2},n} \cdot \ldots \cdot K^{F_2}_{r_2,n} \cdot K^{F_1}_{r_1,n} \cdot x_n,\ for\ n \geq 0 \\ \tau^k_n &= K^{F_k}_{r_k,n} \cdot K^{F_{k-1}}_{r_{k-1},n} \cdot \ldots \cdot K^{F_2}_{r_2,n} \cdot K^{F_1}_{r_1,n},\quad for\ k \in \{0,1,2,\ldots,N\} \end{aligned}$$

and $\tau^0_n = I$ for all $n \in N$, $\quad q = \tau^{F_k}_{r_k,n} q, \quad u_n = \tau^N_{r_k,N} x_n$ Then, we have

$$\begin{aligned} \|u_n - q\|^2 &= \|\tau^N_{r_k,n} x_n - \tau^{F_k}_{r_k,n} q\|^2 \\ &= \|x_n - q\|^2. \end{aligned} \tag{4.1.6}$$

Hence, we get

$$\|y_n - q\| \leq \|x_n - q\|. \tag{4.1.7}$$

From (4.1.1), we deduce that

$$\begin{aligned} \|x_{n+1} - q\| &= \|P_C(\epsilon_n \gamma f(x_n) + (I - \epsilon_n A) W_n y_n) - P_C q\| \\ &\leq \|\epsilon_n(\gamma f(x_n) - Aq) + (I - \epsilon_n A)(W_n y_n - q)\| \\ &\leq \epsilon_n \|\gamma f(x_n) - Aq)\| + (1 - \epsilon_n \bar{\gamma})\|(y_n)) - q\| \\ &\leq \epsilon \gamma \epsilon_n \|x_n - q\| + \epsilon_n \|\gamma f(g) - Aq\| \\ &\quad + (1 - \epsilon_n \bar{\gamma})\|x_n - q\| \qquad (4.1.8) \\ &= (1 - (\bar{\gamma} - \gamma\epsilon)\epsilon_n)\|x_n - q\| - \epsilon_n \|\gamma f(q) - Aq\| \\ &= (1 - (\bar{\gamma} - \gamma\epsilon)\epsilon_n)\|x_n - q\| + (\bar{\gamma} - \gamma\epsilon)\epsilon_n \frac{\|\gamma f(q) - Aq\|}{\bar{\gamma} - \gamma\epsilon} \\ &\ \vdots \\ &\leq \max\Big\{\|x_n - q\|,\ \frac{\|\gamma f(q) - Aq\|}{\bar{\gamma} - \gamma\epsilon}\Big\}. \end{aligned}$$

It follows by induction that

$$\|x_n - q\| \leq \max\{\|x_0 - q\|,\ \frac{\|\gamma f(q) - Aq\|}{\bar{\gamma} - \gamma\epsilon}\}, n \geq 0. \tag{4.1.9}$$

Therefore $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{Bu_n\}$, $\{f(x_n)\}$ and $\{AW_ny_n\}$.

Step 2. We claim that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and $\lim_{n\to\infty}\|y_{n+1}-y_n\|=0$. From (4.1.1), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|P_C(\epsilon_n\gamma f(x_n)+(I-\epsilon_nA)W_ny_n)-P_C(\epsilon_{n-1}\gamma f(x_{n-1})\\ &\quad +(I-\epsilon_{n-1}A)W_ny_{n-1}\| \qquad (4.1.10)\\ &\leq \|(I-\epsilon_nA)(W_ny_n-W_ny_{n-1})-(\epsilon_n-\epsilon_{n-1})AW_ny_{n-1}+\\ &\quad \gamma\epsilon_n(f(x_n)-f(x_{n-1}))+\gamma(\epsilon_n-\epsilon_{n-1})f(x_{n-1})\|\\ &\leq (1-\epsilon_n\bar{\gamma})\|y_n-y_{n-1}\|+|\epsilon_n-\epsilon_{n-1}|\|AW_ny_n\|+\gamma\epsilon\epsilon_n\|x_n-x_{n-1}\|\\ &\quad +\gamma|\epsilon_n-\epsilon_{n-1}|\|f(x_{n-1})\|. \qquad (4.1.11)\end{aligned}$$

Since $I-\lambda B$ are nonexpansive, we also have

$$\begin{aligned}\|y_n-y_{n-1}\| &= \|J_{M,\lambda}(u_n-\lambda Bu_n)-J_{M,\lambda}(u_{n-1}-\lambda Bu_{n-1})\|\\ &\leq \|(u_n-\lambda Bu_n)-(u_{n-1}-\lambda Bu_{n-1})\| \qquad (4.1.12)\\ &\leq \|u_n-u_{n-1}\|.\end{aligned}$$

On the other hand, from $u_{n-1}=\tau^N_{r_k,n-1}x_{n-1}$ and $u_n=\tau^N_{r_k,n}x_n$, it follows that

$$F(u_{n-1},y)+\varphi(y)-\varphi(u_{n-1})+\frac{1}{r_{n-1}}\langle y-u_{n-1},u_{n-1}-x_{n-1}\rangle \geq 0, \quad \forall y\in C \qquad (4.1.13)$$

and

$$F(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0, \quad \forall y\in C. \qquad (4.1.14)$$

Substituting $y=u_n$ into (4.1.13) and $y=u_{n-1}$ into (4.1.14), we get

$$F(u_{n-1},u_n)+\varphi(u_n)-\varphi(u_{n-1})+\frac{1}{r_{n-1}}\langle u_n-u_{n-1},u_{n-1}-x_{n-1}\rangle \geq 0 \qquad (4.1.15)$$

and

$$F(u_n,u_{n-1})+\varphi(u_{n-1})-\varphi(u_n)+\frac{1}{r_n}\langle u_{n-1}-u_n,un-x_n\rangle \geq 0. \qquad (4.1.16)$$

From (A2), we obtain

$$\langle u_n-u_{n-1},\frac{u_{n-1}-x_{n-1}}{r_{n-1}}-\frac{u_n-x_n}{r_n}\rangle \geq 0, \qquad (4.1.17)$$

and then

$$\langle u_n - u_{n-1}, u_{n-1} - x_{n-1} - \frac{r_{n-1}}{r_n}(u_n - x_n)\rangle \geq 0, \tag{4.1.18}$$

so

$$\langle u_n - u_{n-1}, u_{n-1} - u_n + u_n - x_{n-1} - \frac{r_{n-1}}{r_n}(u_n - x_n)\rangle \geq 0. \tag{4.1.19}$$

It follows that

$$\langle u_n - u_{n-1}, u_{n-1} - u_n + u_n - x_n - \frac{r_{n-1}}{r_n}(u_n - x_n)\rangle \geq 0,$$

$$\langle u_n - u_{n-1}, u_{n-1} - u_n\rangle + \langle u_n - u_{n-1}, (1 - \frac{r_{n-1}}{r_n})(u_n - x_n)\rangle \geq 0. \tag{4.1.20}$$

Without loss of generality, let us assume that there exists a real number c such that $r_{n-1} > c > 0$, for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \|u_n - u_{n-1}\|^2 &\leq \Big\langle u_n - u_{n-1}, \Big(1 - \frac{r_{n-1}}{r_n}\Big)(u_n - x_n)\Big\rangle \\ &\leq \|u_n - u_{n-1}\|\Big\{|1 - \frac{r_{n-1}}{r_n}\Big|\|u_n - x_n\|\Big\} \end{aligned}$$

and hence

$$\begin{aligned} \|u_n - u_{n-1}\| &\leq \|x_n - x_{n-1}\| + \frac{1}{r_n}|r_n - r_{n-1}|\|u_n - x_n\| \\ &\leq \|x_n - x_{n-1}\| + \frac{M_1}{c}|r_n - r_{n-1}|, \end{aligned} \tag{4.1.21}$$

where $M_1 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. Substituting(4.1.21) into (4.1.12), we have

$$\|y_n - y_{n-1}\| \leq \|x_n - x_{n-1}\| + \frac{M_1}{c}|r_n - r_{n-1}|. \tag{4.1.22}$$

Substituting 4.1.22 into 4.1.10, we get

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1 - \epsilon_n\bar{\gamma})\Big(\|x_n - x_{n-1}\| + \frac{M_1}{c}|r_n - r_{n-1}|\Big) \\ &\quad + |\epsilon_n - \epsilon_{n-1}|\|AW_n y_{n-1}\| + \gamma\epsilon\epsilon_n\|x_n - x_{n-1}\| \\ &\quad + \gamma|\epsilon_n - \epsilon_{n-1}|\|f(x_{n-1})\| \\ &= (1 - \epsilon_n\bar{\gamma})\|x_n - x_{n-1}\| + (1 - \epsilon_n\bar{\gamma})\frac{M_1}{c}|r_n - r_{n-1}| \\ &\quad + |\epsilon_n - \epsilon_{n-1}|\|AW_n y_{n-1}\| \\ &\quad + \gamma\epsilon\epsilon_n\|x_n - x_{n-1}\| + \gamma|\epsilon_n - \epsilon_{n-1}|\|f(x_{n-1})\| \qquad (4.1.23) \\ &\leq (1 - (\bar{\gamma} - \gamma\epsilon)\epsilon_n)\|x_n - x_{n-1}\| + \frac{M_1}{c}|r_n - r_{n-1}| \\ &\quad + |\epsilon_n - \epsilon_{n-1}|\|AW_n y_{n-1}\| + \gamma|\epsilon_n - \epsilon_{n-1}|\|f(x_{n-1})\| \\ &\leq (1 - (\bar{\gamma} - \gamma\epsilon)\epsilon_n)\|x_n - x_{n-1}\| + \frac{M_1}{c}|r_n - r_{n-1}| + M_2|\epsilon_n - \epsilon_{n-1}|, \end{aligned}$$

where $M_2 = \sup\big\{\max\{\|AW_ny_{n-1}\|, \|f(x_{n-1})\| : n \in \mathbb{N}\}\big\}$. Since conditions (C1)-(C2) by Lemma 2.1.23, we have $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. From (4.1.22), we also have that $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$.

Step 3. Next, we show that $\lim_{n\to\infty} \|Bu_n - Bq\| = 0$.

For $q \in \theta$ and $q = J_{M,\lambda}(q - \lambda Bq)$. By (4.1.4) and (4.1.6), we get

$$\begin{aligned}
\|y_n - q\|^2 &= \|J_{M,\lambda}(u_n - \lambda Bu_n) - J_{M,\lambda}(q - \lambda Bq)\|^2 \\
&\leq \|(u_n - \lambda Bu_n) - (q - \lambda Bq)\|^2 \\
&\leq \|u_n - q\|^2 + \lambda(\lambda - 2\beta)\|Bu_n - Bq\|^2 \\
&\leq \|x_n - q\|^2 + \lambda(\lambda - 2\beta)\|Bu_n - Bq\|^2. \qquad (4.1.24)
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|P_C(\epsilon_n\gamma f(x_n) + (I - \epsilon_n A)W_ny_n) - P_C(q)\|^2 \\
&\leq \|\epsilon_n(\gamma f(x_n) - Aq) + (I - \epsilon_n A)(W_ny_n - q)\|^2 \\
&\leq \big(\epsilon_n\|\gamma f(x_n) - Aq\| + (1 - \epsilon_n\bar{\gamma})\|y_n - q\|\big)^2 \\
&\leq \epsilon_n\|\gamma f(x_n) - Aq\|^2 + (1 - \epsilon_n\bar{\gamma})\|y_n - q\|^2 \\
&\quad + 2\epsilon_n(1 - \epsilon_n\bar{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\| \qquad (4.1.25) \\
&\leq \epsilon_n\|\gamma f(x_n) - Aq\|^2 + 2\epsilon_n(1 - \epsilon_n\bar{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\| \\
&\quad + (1 - \epsilon_n\bar{\gamma})\Big(\|x_n - q\|^2 + \lambda(\lambda - 2\beta)\|Bu_n - Bq\|^2\Big) \\
&\leq \epsilon_n\|\gamma f(x_n) - Aq\|^2 + 2\epsilon_n(1 - \epsilon_n\bar{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\| \\
&\quad + \|x_n - q\|^2 + (1 - \epsilon_n\bar{\gamma})\lambda(\lambda - 2\beta)\|Bu_n - Bq\|^2.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&(1 - \epsilon_n\bar{\gamma})\lambda(2\beta - \lambda)\|Bu_n - Bq\|^2 \\
\leq\ & \epsilon_n\|\gamma f(x_n) - Aq\|^2 + \|x_n - x_{n+1}\|\big(\|x_n - q\| + \|x_{n+1} - q\|\big) + \xi_n, \qquad (4.1.26)
\end{aligned}$$

where $\xi_n = 2\epsilon_n(1 - \epsilon_n\bar{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\|$. By conditions (C1),(C3) and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, then we obtain that $\|Bu_n - Bq\| \to 0$ as $n \to \infty$.

Step 4. We show the followings:

(i) $\lim_{n\to\infty} \|x_n - u_n\| = 0$;

(ii) $\lim_{n\to\infty} \|u_n - y_n\| = 0$;

(iii) $\lim_{n\to\infty} \|y_n - W_n y_n\| = 0$.

Since $K_{r_n}(x)$ is firmly nonexpansive and Lemma 2.1.12(iii), we observe that

$$\begin{aligned} \|u_n - q\|^2 &= \|\tau^N_{r_n,n}\, x_n - \tau^N_{r_n,n}\, q\|^2 \\ &\leq \langle x_n - q, u_n - q\rangle \\ &= \frac{1}{2}\Big(\|x_n - q\|^2 + \|u_n - q\|^2 - \|x_n - q - u_n - q\|^2\Big) \qquad (4.1.27) \\ &\leq \frac{1}{2}\Big(\|x_n - q\|^2 + \|u_n - q\|^2 - \|x_n - u_n\|^2\Big) \end{aligned}$$

it follows that

$$\|u_n - q\|^2 \leq \|x_n - q\|^2 - \|x_n - u_n\|^2.$$

Since $J_{M,\lambda}$ is 1-inverse-strongly monotone and by Lemma 2.1.12(iii), we compute

$$\begin{aligned} \|y_n - q\|^2 &= \|J_{M,\lambda}(u_n - \lambda Bu_n) - J_{M,\lambda}(q - \lambda Bq)\|^2 \\ &\leq \langle (u_n - \lambda Bu_n) - (q - \lambda Bq), y_n - q\rangle \\ &= \frac{1}{2}\Big(\|(u_n - \lambda Bu_n) - (q - \lambda Bq)\|^2 + \|y_n - q\|^2 \\ &\quad -\|(u_n - \lambda Bu_n) - (q - \lambda Bq) - (y_n - q)\|^2\Big) \qquad (4.1.28) \\ &\leq \frac{1}{2}\Big(\|u_n - q\|^2 + \|y_n - q\|^2 - \|(u_n - y_n) - \lambda(Bu_n - Bq)\|^2\Big) \\ &= \frac{1}{2}\Big(\|u_n - q\|^2 + \|y_n - q\|^2 - \|u_n - y_n\|^2 \\ &\quad +2\lambda\langle u_n - y_n, Bu_n - Bq\rangle - \lambda^2\|Bu_n - Bq\|^2\Big), \end{aligned}$$

which implies that

$$\|y_n - q\|^2 \leq \|u_n - q\|^2 - \|u_n - y_n\|^2 + 2\lambda\|u_n - y_n\|\|Bu_n - Bq\|. \qquad (4.1.29)$$

Substitute (4.1.29) into (4.1.25), we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \epsilon_n\|\gamma f(x_n) - Aq\|^2 + \|y_n - q\|^2 + 2\epsilon_n(1 - \epsilon_n\bar{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\| \\ &\leq \epsilon_n\|\gamma f(x_n) - Aq\|^2 \\ &\quad +\Big(\|u_n - q\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\|u_n - y_n\|\|Bu_n - Bq\|\Big) \\ &\quad +2\epsilon_n(1 - \epsilon_n\bar{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\|. \qquad (4.1.30) \end{aligned}$$

Then, we derive

$$\begin{aligned}
\|x_n - u_n\|^2 + \|u_n - y_n\|^2 &\le \epsilon_n\|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
&\quad +2\lambda\|u_n - y_n\|\|Bu_n - Bq\| \\
&\quad +2\epsilon_n(1 - \epsilon_n\bar{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\| \\
&= \epsilon_n\|\gamma f(x_n) - Aq\|^2 \\
&\quad +\|x_n - x_{n+1}\|(\|x_n - q\| + \|x_{n+1} - q\|) \\
&\quad +2\lambda\|u_n - y_n\|\|Bu_n - Bq\| \\
&\quad +2\epsilon_n(1 - \epsilon_n\bar{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\|. \qquad (4.1.31)
\end{aligned}$$

By condition (C1), $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$ and $\lim_{n\to\infty}\|Bu_n - Bq\| = 0$. So, we have $\|x_n - u_n\| \to 0,\ \|u_n - y_n\| \to 0$ as $n \to \infty$. It follows that

$$\|x_n - y_n\| \le \|x_n - u_n\| + \|u_n - y_n\| \to 0, \text{ as } n \to \infty. \qquad (4.1.32)$$

From (4.1.1), we have

$$\begin{aligned}
\|x_n - W_ny_n\| &\le \|x_n - W_ny_{n-1}\| + \|W_ny_{n-1} - W_ny_n\| \\
&\le \big\|P_C\big(\epsilon_{n-1}\gamma f(x_{n-1}) + (I - \alpha_{n-1}A)W_ny_{n-1}\big) - P_C(W_ny_{n-1})\big\| \\
&\quad +\|y_{n-1} - y_n\| \qquad (4.1.33) \\
&\le \epsilon_{n-1}\|\gamma f x_{n-1} - AW_ny_{n-1}\| + \|y_{n-1} - y_n\|.
\end{aligned}$$

By condition (C1) and $\lim_{n\to\infty}\|y_{n-1}-y_n\| = 0$, then we obtain that $\|x_n-W_ny_n\| \to 0$ as $n \to \infty$.

Hence, we have

$$\begin{aligned}
\|x_n - W_nx_n\| &\le \|x_n - W_ny_n\| + \|W_ny_n - W_nx_n\| \\
&\le \|x_n - W_ny_n\| + \|y_n - x_n\|. \qquad (4.1.34)
\end{aligned}$$

By (4.1.32) and $\lim_{n\to\infty}\|x_n - W_ny_n\| = 0$ we obtain $\|x_n - W_nx_n\| \to 0$ as $n \to \infty$. Moreover, we also that

$$\|y_n - W_ny_n\| \le \|y_n - x_n\| + \|x_n - W_ny_n\|.$$

By (4.1.32) and $\lim_{n\to\infty}\|x_n-W_ny_n\| = 0$ then we obtain $\|y_n-W_ny_n\| \to 0$ as $n \to \infty$.

Step 5. We show that $q \in \theta := \bigcap_{n=1}^{\infty} F(T_n) \cap (\bigcap_{k=1}^{N} SMEP(F_k))$ and $\limsup_{n\to\infty} \langle (\gamma f - A)q, W_n y_n - q\rangle \leq 0$. It is easy to see that $P_\theta(\gamma f + (I-A))$ is a contraction of H into itself. Indeed, since $0 < \gamma < \frac{\bar{\gamma}}{\epsilon}$, we have that

$$\begin{aligned}
& \|P_\theta(\gamma f + (I-A))x - P_\theta(\gamma f + (I-A))y\| \\
\leq\ & \gamma\|f(x) - f(y)\| + \|I - A\|\|x - y\| \\
\leq\ & \gamma\epsilon\|x - y\| + (1 - \bar{\gamma})\|x - y\| \qquad (4.1.35) \\
\leq\ & (1 - \bar{\gamma} + \gamma\epsilon)\|x - y\|.
\end{aligned}$$

Hence H is complete, there exists a unique fixed point $q \in H$ such that $q = P_\theta(\gamma f + (I-A))(q)$. By Lemma 2.1.12 we obtain that $\langle (\gamma f - A)q, w - q\rangle \leq 0$ for all $w \in \theta$.

Next, we show that $\limsup_{n\to\infty} \langle (\gamma f - A)q, W_n y_n - q\rangle \leq 0$, where $q = P_\theta(\gamma f + I - A)(q)$ is the unique solution of the variational inequality $\big\langle (\gamma f - A)q, w - q\big\rangle \geq 0,\ \ \forall w \in \theta$. We can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty} \langle (\gamma f - A)q, W_n y_n - q\rangle = \lim_{i\to\infty} \langle (\gamma f - A)q, W_n y_{n_i} - q\rangle. \qquad (4.1.36)$$

As $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to w. We may assume without loss of generality that $y_{n_i} \rightharpoonup w$.

We claim that $w \in \theta$. Since $\|y_n - W_n y_n\| \to 0, \|x_n - W_n x_n\| \to 0$ and $\|x_n - y_n\| \to 0$ and by Lemma 2.6.24, we have $w \in \bigcap_{n=1}^{\infty} F(T_n)$.

Next, we show that $w \in \bigcap_{k=1}^{\infty} SMEP(F_k)$.

Since $u_n = \tau_{r_k,n}^{N}\, x_n$, for k = 1, 2, 3, …, N, we know that

$$F_k(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C. \qquad (4.1.37)$$

It follows by (A2) that

$$\varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F_k(y, u_n), \quad \forall y \in C. \qquad (4.1.38)$$

Hence, for k = 1, 2, 3, …, N, we get

$$\varphi(y) - \varphi(u_{n_i}) + \frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \geq F_k(y, u_{n_i}), \quad \forall y \in C. \qquad (4.1.39)$$

For $t \in (0, 1]$ and $y \in H$, let $y_t = ty + (1-t)w$. From (4.1.39), we have

$$0 \ \geq\ \varphi(y_t) + \varphi(u_{n_i}) - \frac{1}{r_{n_i}}\langle y_t - u_{n_i}, u_{n_i} - x_{n_i}\rangle + F_k(y_t, u_{n_i}) \qquad (4.1.40)$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, from (A4) and the weakly lower semicontinuity of φ, $\frac{(u_{n_i}-x_{n_i})}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$. From (A1), (A4) and we have

$$\begin{aligned} 0 &= F_k(y_t, y_t) - \varphi(y_t) + \varphi(y_t) \\ &\leq tF_k(y_t, y) + (1-t)F_k(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t) \\ &\leq t[F_k(y_t, y) + \varphi(y) - \varphi(y_t)]. \end{aligned}$$

Deviding by t, we get

$$F_k(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0.$$

The weakly lower semicontinuity of φ for k = 1, 2, 3, …, N,

$$F_k(w, y) + \varphi(y) \geq \varphi(w).$$

So, we have

$$F_k(w, y) + \varphi(y) - \varphi(w) \geq 0, \ \forall k = 1, 2, 3, \dots, N.$$

This implies that $w \in \bigcap_{k=1}^{N} SMEP(F_k)$.

Lastly, we show that $w \in I(B, M)$. In fact, since B is a β-inverse-strongly monotone, hence B is a monotone and Lipschitz continuous mapping. It follows from Lemma 2.6.22 that $M + B$ is a maximal monotone. Let $(v, g) \in G(M + B)$, since $g - Bv \in M(v)$. Again since $y_{n_i} = J_{M,\lambda}(u_{n_i} - \lambda B u_{n_i})$, we have $u_{n_i} - \lambda B u_{n_i} \in (I + \lambda M)(y_{n_i})$, that is, $\frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda B u_{n_i}) \in M(y_{n_i})$. By virtue of the maximal monotonicity of $M + B$, we have

$$\langle v - y_{n_i}, g - Bv - \frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda B u_{n_i})\rangle \geq 0,$$

and hence

$$\begin{aligned} \langle v - y_{n_i}, g\rangle &\geq \Big\langle v - y_{n_i}, Bv + \frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda B u_{n_i})\Big\rangle \\ &= \langle v - y_{n_i}, Bv - By_{n_i}\rangle + \langle v - y_{n_i}, By_{n_i} - Bu_{n_i}\rangle \qquad (4.1.41) \\ &\quad + \Big\langle v - y_{n_i}, \frac{1}{\lambda}(u_{n_i} - y_{n_i})\Big\rangle. \end{aligned}$$

It follows from $\lim_{n\to\infty} \|u_n - y_n\| = 0$, we have $\lim_{n\to\infty} \|Bu_n - By_n\| = 0$ and $y_{n_i} \rightharpoonup w$ that

$$\limsup_{n\to\infty} \langle v - y_n, g\rangle = \langle v - w, g\rangle \geq 0. \qquad (4.1.42)$$

It follows from the maximal monotonicity of $B + M$ that $\theta \in (M + B)(w)$, that is, $w \in I(B, M)$. Therefore, $w \in \theta$. It follows that

$$\begin{align}
\limsup_{n\to\infty}\langle(\gamma f - A)q, W_ny_n - q\rangle &= \lim_{i\to\infty}\langle(\gamma f - A)q, W_ny_{n_i} - q\rangle \tag{4.1.43}\\
&= \langle(\gamma f - A)q, w - q\rangle \le 0. \tag{4.1.44}
\end{align}$$

Step 6. Finally, we prove $x_n \to q$. By using (4.1.1) and together with Schwarz inequality, we have

$$\begin{align}
\|x_{n+1} - q\|^2 &= \|P_C\big(\epsilon_n\gamma f(x_n) + (I - \epsilon_nA)W_ny_n\big) - P_C(q)\|^2 \nonumber\\
&\le \|\epsilon_n(\gamma f(x_n) - Aq) + (I - \epsilon_nA)(W_ny_n - q)\|^2 \nonumber\\
&\le (I - \epsilon_nA)^2\|(W_ny_n - q)\|^2 + \epsilon_n^2\|\gamma f(x_n) - Aq\|^2 \nonumber\\
&\quad +2\epsilon_n\langle(I - \epsilon_nA)(W_ny_n - q), \gamma f(x_n) - Aq\rangle \nonumber\\
&\le (1 - \epsilon_n\bar{\gamma})^2\|y_n - q\|^2 + \epsilon_n^2\|\gamma f(x_n) - Aq\|^2 \nonumber\\
&\quad +2\epsilon_n\langle W_ny_n - q, \gamma f(x_n) - Aq\rangle - 2\epsilon_n^2\langle A(W_ny_n - q), \gamma f(x_n) - Aq\rangle \nonumber\\
&\le (1 - \epsilon_n\bar{\gamma})^2\|x_n - q\|^2 + \epsilon_n^2\|\gamma f(x_n) - Aq\|^2 \tag{4.1.45}\\
&\quad +2\epsilon_n\langle W_ny_n - q, \gamma f(x_n) - \gamma f(q)\rangle \nonumber\\
&\quad +2\epsilon_n\langle W_ny_n - q, \gamma f(q) - Aq\rangle - 2\epsilon_n^2\langle A(W_ny_n - q), \gamma f(x_n) - Aq\rangle \nonumber\\
&\le (1 - \epsilon_n\bar{\gamma})^2\|x_n - q\|^2 + \epsilon_n^2\|\gamma f(x_n) - Aq\|^2 \tag{4.1.46}\\
&\quad +2\epsilon_n\|W_ny_n - q\|\|\gamma f(x_n) - \gamma f(q)\| \nonumber\\
&\quad +2\epsilon_n\langle W_ny_n - q, \gamma f(q) - Aq\rangle - 2\epsilon_n^2\langle A(W_ny_n - q), \gamma f(x_n) - Aq\rangle \nonumber\\
&\le (1 - \epsilon_n\bar{\gamma})^2\|x_n - q\|^2 + \epsilon_n^2\|\gamma f(x_n) - Aq\|^2 + 2\gamma\epsilon\epsilon_n\|y_n - q\|\|x_n - q\| \nonumber\\
&\quad +2\epsilon_n\langle W_ny_n - q, \gamma f(q) - Aq\rangle - 2\epsilon_n^2\langle A(W_ny_n - q), \gamma f(x_n) - Aq\rangle \nonumber\\
&\le (1 - \epsilon_n\bar{\gamma})^2\|x_n - q\|^2 + \epsilon_n^2\|\gamma f(x_n) - Aq\|^2 + 2\gamma\epsilon\epsilon_n\|x_n - q\|^2 \nonumber\\
&\quad +2\epsilon_n\langle W_ny_n - q, \gamma f(q) - Aq\rangle - 2\epsilon_n^2\langle A(W_ny_n - q), \gamma f(x_n) - Aq\rangle \nonumber\\
&\le \big((1 - \epsilon_n\bar{\gamma})^2 + 2\gamma\epsilon\epsilon_n\big)\|x_n - q\|^2 + \epsilon_n\Big\{\epsilon_n\|\gamma f(x_n) - Aq\|^2 \nonumber\\
&\quad +2\langle W_ny_n - q, \gamma f(q) - Aq\rangle - 2\epsilon_n\|A(W_ny_n - q)\|\|\gamma f(x_n) - Aq\|\Big\} \nonumber\\
&= \big(1 - 2(\bar{\gamma} - \gamma\epsilon)\epsilon_n\big)\|x_n - q\|^2 + \epsilon_n\Big\{\epsilon_n\|\gamma f(x_n) - Aq\|^2 \nonumber\\
&\quad +2\langle W_ny_n - q, \gamma f(q) - Aq\rangle - 2\epsilon_n\|A(W_ny_n - q)\| \, |\gamma f(x_n) - Aq\| \nonumber\\
&\quad +\epsilon_n\bar{\gamma}^2\|x_n - q\|^2\Big\}. \tag{4.1.47}
\end{align}$$

Since $\{x_n\}$ is bounded, where $\eta \ge \|\gamma f(x_n) - Aq\|^2 - 2\|A(W_ny_n - q)\|\|\gamma f(x_n) -$

$Aq\| + \bar{\gamma}^2\|x_n - q\|^2$ for all $n \geq 0$. It follows that

$$\|x_{n+1} - q\|^2 \leq \big(1 - 2(\bar{\gamma} - \gamma\epsilon)\epsilon_n\big)\|x_n - q\|^2 + \epsilon_n\delta_n, \tag{4.1.48}$$

where $\delta_n = 2\langle W_n y_n - q, \gamma f(q) - Aq\rangle + \eta\alpha_n$. Since $\limsup_{n\to\infty}\langle(\gamma f - A)q, W_n y_n - q\rangle \leq 0$, we get $\limsup_{n\to\infty}\delta_n \leq 0$. Applying Lemma 2.1.23, we can conclude that $x_n \to q$. This completes the proof. □

4.2 Some Applications to Minimization Problems

In this section, we apply the iterative scheme (4.1.1) for finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping.

Using Theorem 4.1.1, we first prove a strongly convergence theorem for finding a common fixed point of a nonexpansive mapping and a strictly pseudo-contraction, which is the solution of minimization problem.

Theorem 4.2.1. *Let H be a real Hilbert space, C be a closed convex subset of H and B be an β-inverse-strongly monotone, $\varphi : C \to \mathbb{R}$ is convex and lower semicontinuous function, $f : C \to C$ be a contraction with coefficient α $(0 < \alpha < 1)$ and A be a strongly positive linear bounded operator of H into itself with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{T_n\}$ be a family of nonexpansive mappings of H into itself and let S be a κ-strictly pseudo-contraction of C into itself such that*

$$\theta := \cap_{n=1}^{\infty} F(T_n) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap F(S) \neq 0.$$

Suppose $\{x_n\}$ is a sequence generated by the following algorithm for $x_0, u_n \in C$ arbitrarily:

$$\begin{cases} u_n = K^{F_N}_{r_n,n} \cdot K^{F_{N-1}}_{r_{n-1},n} \cdot K^{F_{N-2}}_{r_{n-2},n} \cdot \ldots \cdot K^{F_2}_{r_2,n} \cdot K^{F_1}_{r_1,n} \cdot x_n, \ \forall n \in N \\ x_{n+1} = P_C[\epsilon_n \gamma f(x_n) + (I - \epsilon_n A)W_n(1-\lambda)x_n + \lambda S_{x_n}] \end{cases} \tag{4.2.1}$$

for all $n = 0, 1, 2, \ldots$, and the conditions (C1)-(C2) in Theorem 4.1.1 are satisfied.

Then, the sequence $\{x_n\}$ converges strongly to $q \in \theta$, where $q = P_\theta(\gamma f + I - A)(q)$ which solves the following variational inequality:

$$\langle(\gamma f - A)q, p - q\rangle \leq 0, \ \forall p \in \theta$$

which is the optimality condition for the minimization problem

$$\min_{q\in\theta} \frac{1}{2}\langle Aq, q\rangle - h(q), \tag{4.2.2}$$

where h is a potential function for γf (i.e., $h'(q) = \gamma f(q)$ for $q \in H$).

Proof. Put $B \equiv I - T$, then B is $\frac{1-\kappa}{2}$ inverse-strongly monotone and $F(S) = I(B, M)$ and $J_{M,\lambda}(x_n - \lambda B x_n) = (1-\lambda)x_n + \lambda T x_n$. So by Theorem 4.1.1, we obtain the desired result. □

4.3 Numerical Example

Now, we give a real numerical example in which the condition satisfy the ones of theorem 4.1.1 and some numerical experiment results to explain the main result theorem 4.1.1 as follows:

Example. Let $H = R, C = [-1, 1], T_n = I,\ \lambda_n = \beta \in (0, 1),\ n \in N,\ F_k(x, y) = 0, \forall x, y \in C, r_{n,n} = 1,\ k \in \{1, 2, 3, ..., N\},\ \varphi(x) = 0,\ \forall x \in C,\ B = A = I,\ f(x) = \frac{1}{5}x,\ \forall x \in H,\ \lambda = \frac{1}{2}$ with contraction coefficient $\alpha = \frac{1}{10},\ \epsilon_n = \frac{1}{n}$ for every n $\in N$ and $\gamma = 1$. Then $\{x_n\}$ is the sequence generated by

$$x_{n+1} = (\frac{1}{2} - \frac{3}{10n})x_n \tag{4.3.1}$$

and $x_n \longrightarrow 0$ as $n \longrightarrow \infty$, where 0 is the unique solution of the minimization problem

$$\min_{x\in C} = \frac{2}{5}x^2 + q.$$

Proof. We prove the Example 4.3.1 by step 1, step 2, step 3. By step 4, we give two numerical experiment results which can directly explain the sequence $\{x_n\}$ strongly converges to 0.

Step 1. We show

$$K^{F_N}_{r_n,n}x = P_C x,\ \forall x \in H, F_N \in \{1, 2, 3, ..., N\}, \tag{4.3.2}$$

where

$$P_C x = \begin{cases} \frac{x}{|x|}, & x \in H \backslash C \\ x, & x \in C. \end{cases} \tag{4.3.3}$$

Indeed, since $F_k(x, y) = 0,\ \forall x, y \in C,\ n \in \{1, 2, 3, ..., N\}$, due to the definition of K_r(x), $\forall x \in H$, as lemma 2.6.13, we have

$$K_r(x) = \Big\{u \in C : \langle y - u, u - x\rangle \geq 0, \forall y \in C\Big\}.$$

Also by the equivalent property (2.4.1) of the nearest projection P_C from $H \longrightarrow C$, we obtain this conclusion, when we take $x \in C$, $K^{F_N}_{r_n,n}x = P_C x = Ix$. By (iii) in lemma 2.6.13, we have

$$\bigcap_{k=1}^{N} SMEP(F_k) = C. \tag{4.3.4}$$

Step 2. We show

$$W_n = I. \tag{4.3.5}$$

Indeed. By (2.4.10), we have

$$W_1 = U_{11} = \lambda_1 T_1 U_{12} + (1 - \lambda_1)I = \lambda_1 T_1 + (1 - \lambda_1)I, \tag{4.3.6}$$

$$\begin{aligned} W_2 = U_{21} &= \lambda_1 T_1 U_{22} + (1 - \lambda_1)I = \lambda_1 T_1(\lambda_2 T_2 U_{23} + (1 - \lambda_2)I) + (1 - \lambda_1)I \\ &= \lambda_1\lambda_2 T_1 T_2 + \lambda_1(1 - \lambda_2)T_1 + (1 - \lambda_1)I, \\ W_3 = U_{31} &= \lambda_1 T_1 U_{32} + (1 - \lambda_1)I = \lambda_1 T_1(\lambda_2 T_2 U_{33} + (1 - \lambda_2)I) + (1 - \lambda_1)I \\ &= \lambda_1\lambda_2 T_1 T_2 U_{33} + \lambda_1(1 - \lambda_2)T_1 + (1 - \lambda_1)I, \\ &= \lambda_1\lambda_2 T_1 T_2(\lambda_3 T_3 U_{34} + (1 - \lambda_3 I)) + \lambda_1(1 - \lambda_2)T_1 + (1 - \lambda_1)I, \\ &= \lambda_1\lambda_2\lambda_3 T_1 T_2 T_3 + \lambda_1\lambda_2(1 - \lambda_3 T_1 T_2 + \lambda_1(1 - \lambda_2)T_1 + (1 - \lambda_1)I. \end{aligned}$$

Compute in this way by (2.4.10), we obtain

$$\begin{aligned} W_n = U_{n1} &= \lambda_1\lambda_2\cdots\lambda_n T_1 T_2 \cdots T_n + \lambda_1\lambda_2\cdots\lambda_{n-1}(1 - \lambda_n)T_1 T_2 \cdots T_{n-1} \\ &\quad + \lambda_1\lambda_2\cdots\lambda_{n-2}(1 - \lambda_{n-1})T_1 T_2 \cdots T_{n-2} + \cdots + \lambda_1(1 - \lambda_2)T_1 + (1 - \lambda_1)I. \end{aligned}$$

Since $T_n = I,\ \lambda_n = \beta,\ n \in N$, thus

$$W_n = [\beta^n + \beta^{n-1}(1-\beta) + \cdots + \beta(1-\beta) + (1-\beta)]I = I.$$

Step 3. We show

$$x_{n+1} = (\frac{1}{2} - \frac{3}{10n})x_n \ and \ x_{n+1} \longrightarrow 0,\ as\ n \longrightarrow \infty, \tag{4.3.7}$$

where 0 is the unique solution of the minimization problem

$$\min_{x \in C} = \frac{2}{5}x^2 + q.$$

Indeed, we can see A = I is a strongly position bounded linear operator with coefficient $\bar{\gamma} = \frac{1}{2}$, γ is a real number such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, so we can take $\gamma = 1$. Due to (4.3.1), (4.3.3) and (4.3.5), we can obtain an special sequence $\{x_n\}$ of (4.1.1) in theorem 4.1.1 as follows:

$$x_{n+1} = (\frac{1}{2} - \frac{3}{10n})x_n.$$

Since $T_n = I,\ n \in N$, so,

$$\cap_{n=1}^{\infty} F(T_n) = H,$$

combining with (4.3.4), we have

$$\theta := \cap_{n=1}^{\infty} F(T_n) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap I(B, M) = C = [-1, 1].$$

By Lemma 2.1.23, it is obviously that $z_n \longrightarrow 0$, 0 is the unique solution of the minimization problem

$$\min_{x \in C} = \frac{2}{5}x^2 + q,$$

where q is a constant number.

Step 4. We give the numerical experiment results using software Mathlab 7.0 and get the table 1 to table 2, which show that the iteration process of the sequence $\{x_n\}$ is a monotone decreasing sequence and converges to 0, but the more the iteration steps are, the more showily the sequence $\{x_n\}$ converges to 0.

Now we turn to realizing (4.1.1) for approximating a fixed point of T. We take the initial valued $x_1 = 1$ and $x_1 = 1/2$, respectively. All the numerical results are given in Tables 4.1 and 4.2. The corresponding graph appears in Figure 4.1 (i) and (ii).

Table 4.1: This table shows the value of sequence $\{x_n\}$ on each iteration steps (initial value $x_1 = 1$)

n	x_n	n	x_n
1	1.000000000000000	31	0.000000000054337
2	0.200000000000000	32	0.000000000026643
3	0.070000000000000	33	0.000000000013072
4	0.028000000000000	34	0.000000000006417
$\vdots$	$\vdots$	$\vdots$	$\vdots$
19	0.000000301580666	39	0.000000000000184
20	0.000000146028533	40	0.000000000000091
21	0.000000070823839	41	0.000000000000045
$\vdots$	$\vdots$	$\vdots$	$\vdots$
29	0.000000000226469	47	0.000000000000001
30	0.000000000110892	48	0.000000000000000

Table 4.2: This table shows the value of sequence $\{x_n\}$ on each iteration steps (initial value $x_1 = \frac{1}{2}$)

n	x_n	n	x_n
1	0.500000000000000	31	0.000000000027168
2	0.100000000000000	32	0.000000000013321
3	0.035000000000000	33	0.000000000006536
4	0.014000000000000	34	0.000000000003208
$\vdots$	$\vdots$	$\vdots$	$\vdots$
19	0.000000150790333	39	0.000000000000092
20	0.000000073014267	40	0.000000000000045
21	0.000000035411919	41	0.000000000000022
$\vdots$	$\vdots$	$\vdots$	$\vdots$
29	0.000000000113235	46	0.000000000000001
30	0.000000000055446	47	0.000000000000000

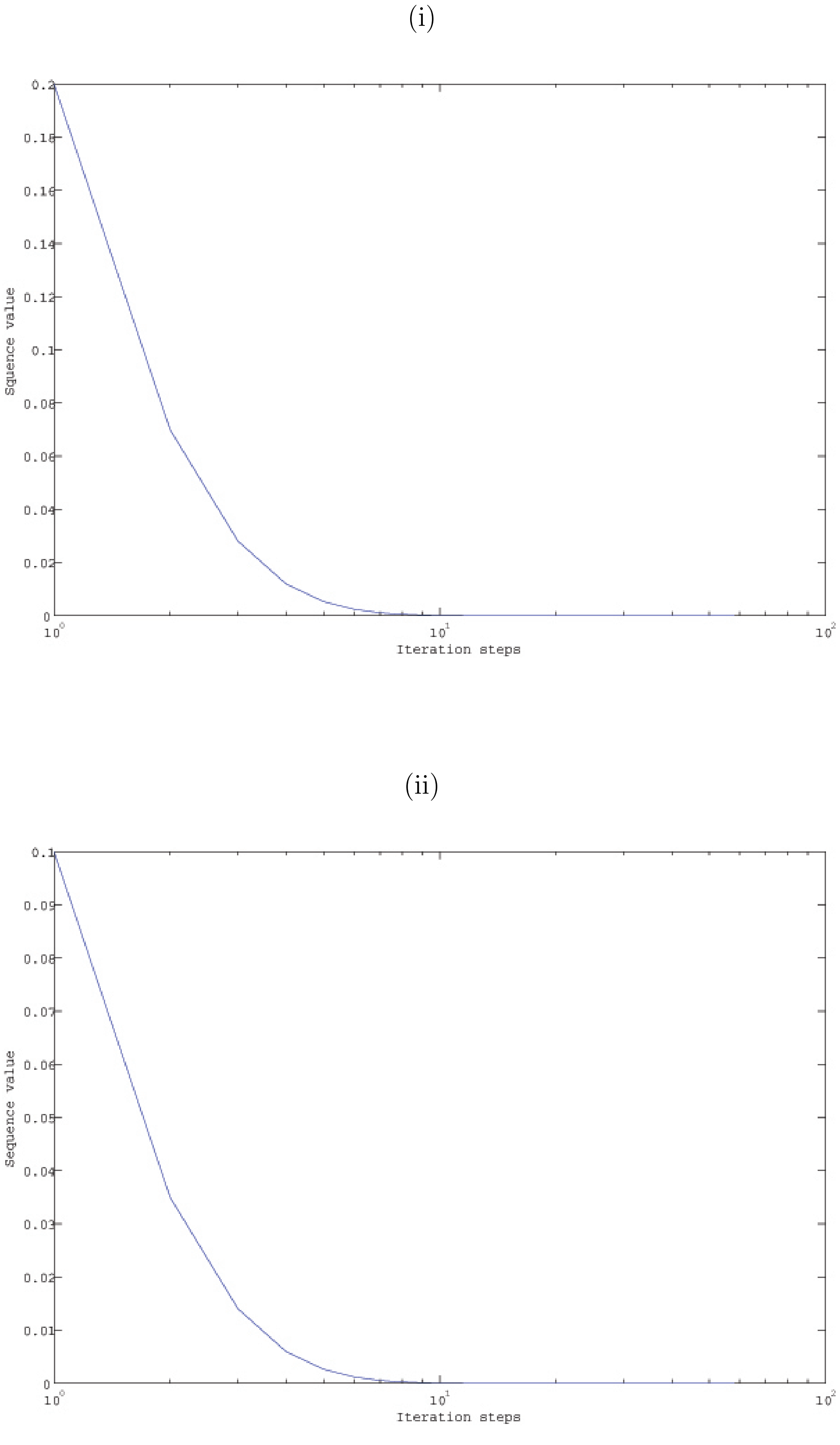

Figure 4.1 The iteration comparison chart of different initial values.
(i) $x_1 = 1$ and (ii) $x_1 = \frac{1}{2}$.