CHAPTER 3 ITERATIVE ALGORITHMS FOR FIXED POINT PROBLEMS AND VARIATIONAL INEQUALITY PROBLEMS

In this section, we show a strong convergence theorem for finding the least norm of fixed points for strict pseudo mappings, a common element of the set of fixed points and the solution set of variational inequality in Hilbert spaces.

3.1 [An Algorithm for Minimum-Norm of Fixed Point for Nonexpansive Mappings]

Theorem 3.1.1. *Let C be a closed convex of a real Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping. Let A be an α-inverse strongly monotone and $\Omega := F(S) \cap VI(C, A) \neq \emptyset$. Assume that a sequence $\{\alpha_n\} \subset (0, 1)$ satisfies the conditions:*

- *(i)* $\lim_{n\to\infty} \alpha_n = 0$;
- *(ii)* $\Sigma_{n=0}^{\infty}\alpha_n = +\infty$.

Then the sequence $\{x_n\}$ generated by the following algorithm

$$x_{n+1} = (1 - \alpha_n)[\lambda SP_C(I - \lambda A)x_n + (1 - \lambda)x_n] \tag{3.1.1}$$

converges strongly to a fixed point of S which is a minimal norm and the unique solution of the variational inequality:

$$x^* \in \Omega, \langle x^*, x - x^*\rangle \geq 0, \forall x \in \Omega.$$

Proof First, we prove that the sequence $\{x_n\}$ is bounded. Let $q \in \Omega$. By (3.1.1), we have

$$\begin{aligned}
\|x_{n+1} - q\| &= \|(1 - \alpha_n)[\lambda SP_C(I - \lambda A)x_n + (1 - \lambda)x_n] - q\| \\
&\leq \|(1 - \alpha_n)[(1 - \lambda)(x_n - q) + \lambda(SP_C(I - \lambda A)x_n - q)] - \alpha_n q\| \\
&\leq \|(1 - \alpha_n)[(1 - \lambda)\|(x_n - q)\| + \lambda\|(x_n - q)\|] - \|\alpha_n q\| \\
&\leq (1 - \alpha_n)\|x_n - q\| + \alpha_n\|q\| \\
&\leq \max\{\|x_n - q\|, \|q\|\}.
\end{aligned}$$

By induction, it follows that

$$\|x_n - q\| \leq \max\{\|x_0 - q\|, \|q\|\},$$

for all $n \geq 0$. Then $\{x_n\}$ is bounded. Therefore, $\{SP_C(I - \lambda A)x_n\}$ is also bounded.

Let $y_n = \frac{(1-\alpha_n)\lambda SP_C(I-\lambda A)x_n}{\alpha_n+(1-\alpha_n)\lambda}$, then the iterative sequence (3.1.1) is equivalent to

$$x_{n+1} = (\alpha_n + (1 - \alpha_n)\lambda)y_n + (1 - \alpha_n - (1 - \alpha_n)\lambda)x_n. \tag{3.1.2}$$

Since $\lim_{n\to\infty}(\alpha_n + (1 - \alpha_n)\lambda) = \lambda$, then

$$\begin{aligned}
\|y_n - q\| &= \Big\|\frac{(1 - \alpha_n)\lambda SP_C(I - \lambda A)x_n}{\alpha_n + (1 - \alpha_n)\lambda} - q\Big\| \\
&= \Big\|\frac{(1 - \alpha_n)\lambda SP_C(I - \lambda A)x_n - (\alpha_n + (1 - \alpha_n)\lambda)q}{\alpha_n + (1 - \alpha_n)\lambda}\Big\| \\
&= \Big\|\frac{(1 - \alpha_n)\lambda SP_C(I - \lambda A)x_n - \alpha_n q - (1 - \alpha_n)\lambda q}{\alpha_n + (1 - \alpha_n)\lambda}\Big\| \\
&\leq \frac{(1 - \alpha_n)\lambda\|x_n - q\| - \alpha_n\|q\|}{\alpha_n + (1 - \alpha_n)\lambda} \\
&= \frac{\alpha_n}{\alpha_n + (1 - \alpha_n)\lambda}\|q\| + (1 - \frac{\alpha_n}{\alpha_n + (1 - \alpha_n)\lambda})\|x_n - q\| \\
&\leq \max\{\|x_n - q\|, \|q\|\}.
\end{aligned}$$

Thus, $\{y_n\}$ is bounded. Hence by nonexpansiveness of S and P_C, we have

$$\begin{aligned}
&\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \\
= &\Big\|\frac{(1 - \alpha_{n+1})\lambda SP_C(I - \lambda A)x_{n+1}}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda} - \frac{(1 - \alpha_n)\lambda SP_C(I - \lambda A)x_n}{\alpha_n + (1 - \alpha_n)\lambda}\Big\| - \|x_{n+1} - x_n\| \\
\leq &\frac{(1 - \alpha_{n+1})\lambda}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda}\|SP_C(I - \lambda A)x_{n+1} - SP_C(I - \lambda A)x_n\| \\
&+ \Big|\frac{(1 - \alpha_{n+1})\lambda}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda} - \frac{(1 - \alpha_n)\lambda}{\alpha_n + (1 - \alpha_n)\lambda}\Big|\|SP_C(I - \lambda A)x_n\| \\
\leq &\frac{(1 - \alpha_{n+1})\lambda}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda}\|Sx_{n+1} - Sx_n\| \\
&+ \Big|\frac{(1 - \alpha_{n+1})\lambda}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda} - \frac{(1 - \alpha_n)\lambda}{\alpha_n + (1 - \alpha_n)\lambda}\Big|\|SP_C(I - \lambda A)x_n\| \\
&-\|x_{n+1} - x_n\| \\
\leq &\Big(\frac{(1 - \alpha_{n+1})\lambda}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda} - 1\Big)\|x_{n+1} - x_n\| \\
&+\Big|\frac{(1 - \alpha_{n+1})\lambda}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda} - \frac{(1 - \alpha_n)\lambda}{\alpha_n + (1 - \alpha_n)\lambda}\Big|\|SP_C(I - \lambda A)x_n\|.
\end{aligned}$$

From $\{x_n\}$ and $\{SP_C(I-\lambda A)x_n\}$ are bounded sequences and $\lim_{n\to\infty}\alpha_n = 0$, then

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \leq 0.$$

By Lemma 2.6.15, we obtain that $\lim_{n\to\infty}\|y_n - x_n\| = 0$. Therefore,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(\alpha_n + (1-\alpha_n)\lambda)\|y_n - x_n\| = 0. \tag{3.1.3}$$

On the other hand, we consider

$$\begin{aligned}
\|x_n - SP_C(I-\lambda A)x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - SP_C(I-\lambda A)x_n\| \\
&= \|x_n - x_{n+1}\| + \|(1-\alpha_n)(\lambda SP_C(I-\lambda A)x_n + (1-\lambda)x_n) \\
&\quad - SP_C(I-\lambda A)x_n\| \\
&\leq \|x_n - x_{n+1}\| + (1-\alpha_n)(1-\lambda)\|x_n - SP_C(I-\lambda A)x_n\| \\
&\quad + \alpha_n\|SP_C(I-\lambda A)x_n\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_n - SP_C(I-\lambda A)x_n\| &\leq \frac{1}{1-(1-\alpha_n)(1-\lambda)}\|x_n - x_{n+1}\| \\
&\quad + \frac{1}{1-(1-\alpha_n)(1-\lambda)}\alpha_n\|SP_C(I-\lambda A)x_n\| \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Next, we prove that $\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle \leq 0$.

Since $\{x_n\}$ is bounded. Then, we can take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle = \lim_{i\to\infty}\langle x^* - x_{n_i}, x^*\rangle.$$

Again, since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_i} \rightharpoonup x^{'}$. Consequently,

$$\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle = \langle x^* - x^{'}, x^*\rangle \leq 0.$$

Notice that $\lim_{n\to\infty}\|x_n - SP_C(I-\lambda A)x_n\| = 0$. By the demiclosedness principle of nonexpansive mapping S, we have $x' \in Fix(S)$. Since $x^* = P_{Fix(S)}(0)$. It follows from the properties of projection operator that

$$\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle = \langle x^* - x', x^*\rangle \leq 0. \tag{3.1.4}$$

By (3.1.1), we have

$$\begin{aligned}
\|x_{n+1}-(1-\alpha_n)x^*\|^2 &= \|[(1-\alpha_n)\lambda SP_C(I-\lambda A)x_{n+1}+(1-\lambda)x_n]-(1-\alpha_n)x^*\|^2 \\
&= \|(1-\alpha_n)[\lambda SP_C(I-\lambda A)x_n+(1-\lambda)x_n]-x^*\|^2 \\
&= \|(1-\alpha_n)[\lambda SP_Cx_n-(1-\lambda)x_n]-(1-\lambda+\lambda)x^*\|^2 \\
&\leq (1-\alpha_n)\|\lambda(Sx_n-x^*)+(1-\lambda)(x_n-x^*)\|^2 \\
&\leq (1-\alpha_n)\|\lambda(x_n-x^*)+(1-\lambda)(x_n-x^*)\|^2 \\
&\leq (1-\alpha_n)\|x_n-x^*\|^2. \qquad (3.1.5)
\end{aligned}$$

Observe that

$$\begin{aligned}
\|x_{n+1}-(1-\alpha_n)x^*\|^2 &= \|x_{n+1}-x^*\|^2-2\alpha_n\langle -x^*,x_{n+1}-x^*\rangle+\alpha_n^2\|x^*\|^2 \\
&\geq \|x_{n+1}-x^*\|^2-2\alpha_n\langle x_{n+1}-x^*,x^*\rangle. \qquad (3.1.6)
\end{aligned}$$

Therefore by (3.1.5) and (3.1.6), we get

$$\|x_{n+1}-x^*\|^2 \leq (1-\alpha_n)\|x_n-x^*\|^2+2\alpha_n\langle x_{n+1}-x^*,x^*\rangle. \qquad (3.1.7)$$

By the condition of *(ii)* and the inequality (3.1.4), we can apply Lemma (2.1.23) to (3.1.7) and conclude that $\{x_n\}$ converges strongly to x^* as $n \to \infty$ that is, the minimum - norm fixed point of S. This completes the proof. □

3.2 An Algorithm for Variational Inequalities and Fixed Point for Pseudo-Contractive Mappings

Theorem 3.2.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudo-contractive mapping and $A : C \to H$ be a continuous monotone mapping such that $\mathfrak{F} := F(T) \cap VI(C,A) \neq \emptyset$. For $x \in H$, define T_{r_n} and F_{r_n} as follows:*

$$T_rx := \{z \in C : \langle y-z, Tz\rangle + \frac{1}{r}\langle y-z,(1+r)z-x\rangle \leq 0, \quad \forall y \in C\}$$

and

$$F_rx := \{z \in C : \langle y-z, Az\rangle + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C\}.$$

Let f be a contraction of H into itself with a contraction constant β and let $B : H \to H$ be a strongly positive linear bounded self-adjoint operator with coefficients $\overline{\beta} > 0$ and let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{cases} y_n = F_{r_n} x_n \\ x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + [(1-\delta_n)I - \alpha_n B]T_{r_n} y_n, \end{cases} \tag{3.2.1}$$

where $\{\alpha_n\}, \{\delta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ such that

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum\limits_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $\lim_{n\to\infty} \delta_n = 0, \sum\limits_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$*;*

(C3) $\liminf_{n\to\infty} r_n > 0, \sum\limits_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $z \in \mathfrak{F}$, which is the unique solution of the variational inequality:

$$\langle (B - \gamma f)z, x - z \rangle \geq 0, \quad \forall x \in \mathfrak{F}. \tag{3.2.2}$$

Equivalently, $z = P_{\mathfrak{F}}(I - B + \gamma f)z$, which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Az, z \rangle - h(z),$$

where h is a potential function for γf (i.e.,$h'(z) = \gamma f(z)$ for $z \in H$).

Remark (1) The variational inequality (3.2.2) has the unique solution; (see [62]).
(2) It follows from condition (C1) that $(1-\delta_n)I - \alpha_n B$ is positive and $\|(1-\delta_n)I - \alpha_n B\| \leq I - \delta_n - \alpha_n \overline{\beta}$ for all $n \geq 1$; (see [66]).

Proof. We processed the proof with following four steps:

Step 1 First, we will prove that the sequence $\{x_n\}$ is bounded.
Let $v \in \mathfrak{F}$ and let $u_n = T_{r_n} y_n$ and $y_n = F_{r_n} x_n$. Then, from Lemmas 2.6.18 and 2.6.20 that

$$\|u_n - v\| = \|T_{r_n} y_n - T_{r_n} v\| \leq \|y_n - v\| = \|F_{r_n} x_n - F_{r_n} v\| \leq \|x_n - v\|. \tag{3.2.3}$$

Moreover, from (3.2.1) and (3.2.2), we compute

$$\begin{aligned}
\|x_{n+1} - v\| &= \|\alpha_n(\gamma(f(x_n) - Bv)) + \delta_n(x_n - v) + [(1-\delta_n)I - \alpha_n B]T_{r_n} - v\| \\
&\leq \alpha_n\|\gamma f(x_n) - Bv)\| + \delta_n\|x_n - v\| + \|(1-\delta_n)I - \alpha_n B\|\|T_{r_n} - v\| \\
&\leq \alpha_n\beta\gamma\|x_n - v\| + \alpha_n\|\gamma f(v) - Bv\| + \delta_n\|x_n - v\| \\
&\quad +(1+\delta_n - \alpha_n\overline{\beta})\|T_{r_n}y_n - v\| \\
&\leq \alpha_n\beta\gamma\|x_n - v\| + \alpha_n\|\gamma f(v) - Bv\| + \delta_n\|x_n - v\| \\
&\quad +(1+\delta_n - \alpha_n\overline{\beta})\|u_n - v\| \\
&\leq \alpha_n\beta\gamma\|x_n - v\| + \alpha_n\|\gamma f(v) - Bv\| + \delta_n\|x_n - v\| \\
&\quad +(1+\delta_n - \alpha_n\overline{\beta})\|x_n - v\| \\
&= \alpha_n\beta\gamma\|x_n - v\| + \alpha_n\|\gamma f(v) - Bv\| + \delta_n\|x_n - v\| + \|x_n - v\| \\
&\quad -\delta_n\|x_n - v\| - \alpha_n\overline{\beta}\|x_n - v\| \\
&= \alpha_n\beta\gamma\|x_n - v\| + \alpha_n\|\gamma f(v) - Bv\| + \|x_n - v\| - \alpha_n\overline{\beta}\|x_n - v\| \\
&\leq (\alpha_n\beta\gamma + 1 - \alpha_n\overline{\beta})\|x_n - v\| + \alpha_n\|\gamma f(v) - Bv\| \\
&= (1 - \alpha_n(\overline{\beta} - \beta\gamma))\|x_n - v\| + \alpha_n\|\gamma f(v) - Bv\| \\
&\leq \max\left\{\|x_n - v\|, \frac{\|\gamma f(v) - Bv\|}{\overline{\beta} - \beta\gamma}\right\}, \quad \forall n \geq 1.
\end{aligned}$$

Therefore, by the simple introduction, we have

$$\|x_n - v\| = \max\left\{\|x_1 - v\|, \frac{\|\gamma f(v) - Bv\|}{\overline{\beta} - \beta\gamma}\right\}, \quad \forall n \geq 1$$

which show that $\{x_n\}$ is bounded, so $\{y_n\}, \{u_n\}$ and $\{f(x_n)\}$ are bounded.

Step 2 We will show that $\|x_{n+1} - x_n\| \to 0$ and $\|u_n - y_n\| \to 0$ as $n \to \infty$. Notice that each T_{r_n} and F_{r_n} are firmly nonexpansive. Hence, we have

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \|T_{r_n}y_{n+1} - T_{r_n}y_n\| \leq \|y_{n+1} - y_n\| \\
&= \|F_{r_n}x_{n+1} - F_{r_n}x_n\| \leq \|x_{n+1} - x_n\|.
\end{aligned}$$

From (3.2.1), we note that

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|\alpha_n\gamma f(x_n)+\delta_n x_n+[(1-\delta_n)I-\alpha_n B]T_{r_n}y_n \\
&\quad -\alpha_{n-1}\gamma f(x_{n-1})-\delta_{n-1}x_{n-1}-[(1-\delta_{n-1})I-\alpha_{n-1}B]T_{r_n}y_{n-1}\| \\
&= \|\alpha_n\gamma f(x_n)+\delta_n x_n+(I-\delta_n-\alpha_n B)u_n \\
&\quad -\alpha_{n-1}\gamma f(x_{n-1})-\delta_{n-1}x_{n-1}-(I-\delta_{n-1}-\alpha_{n-1}B)u_{n-1}\| \\
&\leq \|\alpha_n\gamma f(x_n)-\alpha_n\gamma f(x_{n-1})+\alpha_n\gamma f(x_{n-1}) \\
&\quad +\delta_n x_n-\delta_{n-1}x_{n-1}-\alpha_{n-1}\gamma f(x_{n-1})+(I-\delta_n-\alpha_n B)u_n \\
&\quad -(I-\delta_n-\alpha_n B)u_{n-1}+(I-\delta_n-\alpha_n B)u_{n-1} \\
&\quad -(I-\delta_{n-1}-\alpha_{n-1}B)u_{n-1}\| \\
&\leq \|\alpha_n\gamma f(x_n)-\alpha_n\gamma f(x_{n-1})\|+\|\alpha_n\gamma f(x_{n-1}) \\
&\quad -\alpha_{n-1}\gamma f(x_{n-1})\|+\|\delta_n x_n-\delta_{n-1}x_{n-1}\| \\
&\quad +\|(I-\delta_n-\alpha_n B)u_n-(I-\delta_n-\alpha_n B)u_{n-1}\| \\
&\quad +\|(I-\delta_n-\alpha_n B)u_{n-1}-(I-\delta_{n-1}-\alpha_{n-1}B)u_{n-1}\| \\
&= \alpha_n\gamma\|f(x_n)-f(x_{n-1})\|+|\alpha_n-\alpha_{n-1}|\|\gamma f(x_{n-1})\| \\
&\quad +\delta_n x_n-\delta_{n-1}x_{n-1}+\delta_n x_{n-1}-\delta_n x_{n-1} \\
&\quad +(I-\delta_n-\alpha_n B)\|u_n-u_{n-1}\| \\
&\quad +\|(I-\delta_n-\alpha_n B-I+\delta_{n-1}+\alpha_{n-1}B)u_{n-1}\|
\end{aligned}$$

$$\begin{aligned}
&= \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma\|f(x_{n-1})\| \qquad (3.2.4)\\
&\quad +\delta_n\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\|\\
&\quad +|I - \delta_n - \alpha_n B|\|u_n - u_{n-1}\|\\
&\quad +|\delta_{n-1} - \delta_n + \alpha_{n-1}B + \alpha_n B|\|u_{n-1}\|\\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma\|f(x_{n-1})\|\\
&\quad +\delta_n\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\|\\
&\quad +|I - \delta_n - \alpha_n B|\|x_n - x_{n-1}\|\\
&\quad +|\delta_{n-1} - \delta_n|\|u_{n-1}\| + |\alpha_{n-1}B + \alpha_n B|\|u_{n-1}\|\\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma\|f(x_{n-1})\|\\
&\quad +\delta_n\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\|\\
&\quad +|I - \delta_n - \alpha_n B|\|x_n - x_{n-1}\| + |\delta_{n-1} - \delta_n|\|x_{n-1}\|\\
&\quad -|\alpha_{n-1} - \alpha_n|B\|x_{n-1}\|\\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma\|f(x_{n-1})\|\\
&\quad +\delta_n\|x_n - x_{n-1}\| + |I - \delta_n - \alpha_n B|\|x_n - x_{n-1}\| - |\alpha_{n-1} - \alpha_n|B\|x_{n-1}\| \qquad (3.2.5)\\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma\|f(x_{n-1})\| + |\delta_n + I - \delta_n\\
&\quad -\alpha_n B|\|x_n - x_{n-1}\| - |\alpha_{n-1} - \alpha_n|B\|x_{n-1}\|\\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma\|f(x_{n-1})\|\\
&\quad +|I - \alpha_n B|\|x_n - x_{n-1}\| - |\alpha_n - \alpha_{n-1}|B\|x_{n-1}\|\\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|\gamma f(x_{n-1}) - Bx_{n-1}\| + |I - \alpha_n B|\|x_n - x_{n-1}\|\\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|\gamma f(x_{n-1}) - Bx_{n-1}\| + |I - \alpha_n B|\|y_n - y_{n-1}\|\\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|K + |I - \alpha_n B|\|y_n - y_{n-1}\|,
\end{aligned}$$

where $K = \|\gamma f(x_{n-1}) - Bx_{n-1}\| = 2\sup\{\|f(x_n)\| + \|u_n\| : n \in N\}$.

Moreover, since $y_n = F_{r_n}x_n$ and $y_{n+1} = F_{r_{n+1}}x_{n+1}$, we get

$$\langle y - y_n, Ay_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0, \quad \forall y \in C \qquad (3.2.6)$$

and

$$\langle y - y_{n+1}, Ay_{n+1}\rangle + \frac{1}{r_{n+1}}\langle y - y_{n+1}, y_{n+1} - x_{n+1}\rangle \geq 0, \quad \forall y \in C. \qquad (3.2.7)$$

Putting $y = y_{n+1}$ in (3.2.6) and $y = y_n$ in (3.2.7), we obtain

$$\langle y_{n+1} - y_n, Ay_n \rangle + \frac{1}{r_n} \langle y_{n+1} - y_n, y_n - x_n \rangle \geq 0 \tag{3.2.8}$$

and

$$\langle y_n - y_{n+1}, Ay_{n+1} \rangle + \frac{1}{r_{n+1}} \langle y_n - y_{n+1}, y_{n+1} - x_{n+1} \rangle \geq 0. \tag{3.2.9}$$

Adding (3.2.8) and (3.2.9), we have

$$\langle y_{n+1} - y_n, Ay_n - Ay_{n+1} \rangle + \left\langle y_{n+1} - y_n, \frac{y_n - x_n}{r_n} - \frac{y_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0$$

which implies that

$$-\langle y_{n+1} - y_n, Ay_{n+1} - Ay_n \rangle + \left\langle y_{n+1} - y_n, \frac{y_n - x_n}{r_n} - \frac{y_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0.$$

Using the fact that A is monotone, we get

$$\left\langle y_{n+1} - y_n, \frac{y_n - x_n}{r_n} - \frac{y_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0,$$

and hence

$$\left\langle y_{n+1} - y_n, y_n - y_{n+1} + y_{n+1} - y_n - \frac{r_n}{r_{n+1}}(y_{n+1} - x_{n+1}) \right\rangle \geq 0.$$

We observe that

$$\begin{aligned}
& \|y_{n+1} - y_n\|^2 \\
\leq \; & \left\langle y_{n+1} - y_n, x_{n+1} - x_n (1 - \frac{r_n}{r_{n+1}})(y_{n+1} - x_{n+1}) \right\rangle \\
\leq \; & \|y_{n+1} - y_n\| \left\{ \|x_{n+1} - x_n\| + |(1 - \frac{r_n}{r_{n+1}})| \|(y_{n+1} - x_{n+1})\| \right\}. \qquad (3.2.10)
\end{aligned}$$

Without loss of generality, let k be a real number such that $r_n > k > 0$ for all $n \in N$. Then, we have

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|y_{n+1} - x_{n+1}\| \\
&\leq \|x_{n+1} - x_n\| + \frac{1}{k} |r_{n+1} - r_n| M, \qquad (3.2.11)
\end{aligned}$$

where $M = \sup\{\|y_n - x_n\| : n \in N\}$. Furthermore, from (3.2.4) and (3.2.11), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \alpha_n \gamma \beta \|x_n - x_{n-1}\| + \|\alpha_n - \alpha_{n-1}\| K + (1 - \alpha_n)(\|x_n - x_{n-1}\| \\
&\quad + \frac{1}{k} |r_n - r_{i-1}| M) \\
&= (1 - \alpha_n + \alpha_n \gamma \beta) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| K + \frac{1}{k} |r_n - r_{n-1}| M \\
&= (1 - \alpha_n (1 - \gamma \beta)) \|x_n - x_{n-1}\| + K |\alpha_n - \alpha_{n-1}| + \frac{M}{k} |r_n - r_{n-1}|.
\end{aligned}$$

Using Lemma 2.1.23, and by the conditions (C1) and (C3), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Consequently, from (3.2.11), we obtain

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{3.2.12}$$

Since $u_n = T_{r_n} y_n$ and $u_{n+1} = T_{r_{n+1}} y_{n+1}$, we have

$$\langle y - u_n, Tu_n\rangle - \frac{1}{r_n}\langle y - u_n, (1 - r_n)u_n - y_n\rangle \leq 0, \quad \forall y \in C \tag{3.2.13}$$

and

$$\langle y - u_{n+1}, Tu_{n+1}\rangle - \frac{1}{r_{n+1}}\langle y - u_{n+1}, (1 - r_{n+1})u_{n+1} - y_{n+1}\rangle \leq 0, \quad \forall y \in C. \tag{3.2.14}$$

Putting $y := u_{n+1}$ in (3.2.13) and $y := u_n$ in (3.2.14), we get

$$\langle u_{n+1} - u_n, Tu_n\rangle - \frac{1}{r_n}\langle u_{n+1} - u_n, (1 - r_n)u_n - y_n\rangle \leq 0, \tag{3.2.15}$$

and

$$\langle u_n - u_{n+1}, Tu_{n+1}\rangle - \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, (1 - r_{n+1})u_{n+1} - y_{n+1}\rangle \leq 0. \tag{3.2.16}$$

Adding (3.2.15) and (3.2.16), we have

$$\langle u_{n+1} - u_n, Tu_n - Tu_{n+1}\rangle - \left\langle u_{n+1} - u_n, \frac{(1 - r_n)u_n - y_n}{r_n} - \frac{(1 - r_{n+1})u_{n+1} - y_{n+1}}{r_{n+1}}\right\rangle \leq 0.$$

Using the fact that T is pseudo-contractive, we get

$$\left\langle u_{n+1} - u_n, \frac{u_n - y_n}{r_n} - \frac{u_{n+1} - y_{n+1}}{r_{n+1}}\right\rangle \geq 0$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - y_n - \frac{r_n}{r_{n+1}}(u_{n+1} - y_{n+1})\right\rangle \geq 0.$$

Thus, using the methods in (3.2.10) and (3.2.11), we can obtain

$$\|u_{n+1} - u_n\| \leq \|y_{n+1} - y_n\| + \frac{1}{r_{n+1}}|r_{n+1} + r_n|M_1, \tag{3.2.17}$$

where $M_1 = \sup\{\|u_n - y_n\| : n \in N\}$. Therefore, from (3.2.12) and property of $\{r_n\}$, we get

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0.$$

Furthermore, since $x_n = \alpha_{n-1}\gamma f(x_{n+1}) + \delta_{n-1}x_{n-1} + [(1-\delta_{n-1})I - \alpha_{n-1}B]T_{r_n}y_{n-1}$, we have

$$\begin{aligned}
\|x_n - u_n\| &\leq \|x_n - u_{n-1}\| + \|u_{n-1} - u_n\| \\
&= \|\alpha_{n-1}\gamma f(x_{n-1}) + \delta_{n-1}x_{n-1} + [(1-\delta_{n-1})I - \alpha_{n-1}B]T_{r_n}y_{n-1} - u_{n-1}\| \\
&\quad + \|u_{n-1} - u_n\| \\
&= \alpha_{n-1}\gamma f(x_{n-1}) + \delta_{n-1}x_{n-1} + (I - \delta_{n-1} - \alpha_{n-1}B)u_{n-1} - u_{n-1}\| \\
&\quad + \|u_{n-1} - u_n\| \\
&= \alpha_{n-1}\gamma f(x_{n-1}) + \delta_{n-1}x_{n-1} + u_{n-1} - \delta_{n-1}u_{n-1} - \alpha_{n-1}Bu_{n-1} - u_{n-1}\| \\
&\quad + \|u_{n-1} - u_n\| \\
&\leq \alpha_{n-1}\gamma f(x_{n-1}) - \alpha_{n-1}Bu_{n-1} + \delta_{n-1}x_{n-1} - \delta_{n-1}u_{n-1}\| + \|u_{n-1} - u_n\| \\
&\leq \alpha_{n-1}\|\gamma f(x_{n-1}) - Bu_{n-1}\| + \delta_{n-1}\|x_{n-1} - u_{n-1}\| + \|u_{n-1} - u_n\|.
\end{aligned}$$

Thus, by (C1) and (C2), we obtain

$$\|x_n - u_n\| \to 0, n \to \infty. \tag{3.2.18}$$

For $v \in \mathfrak{F}$, using Lemma 2.6.18, we obtain

$$\begin{aligned}
\|y_n - v\|^2 &= \|F_{r_n}y_n - F_{r_nv}\|^2 \\
&\leq \langle F_{r_n}x_n - F_{r_n}v, x_n - v\rangle \\
&\leq \langle y_n - v, x_n - v\rangle \\
&= \frac{1}{2}(\|y_n - v\|^2 + \|x_n - v\|^2 - \|x_n - y_n\|^2)
\end{aligned}$$

and

$$\|y_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - y_n\|^2. \tag{3.2.19}$$

Therefore, from (3.2.1), the convexity of $\|\cdot\|^2$, (3.2.2) and (3.2.19), we get

$$\begin{aligned}
\|x_{n+1} - v\|^2 &= \|\alpha_n\gamma f(x_n) + \delta_n x_n + [(1-\delta_n)I - \alpha_n B]T_{r_n}y_n - v\|^2 \\
&= \|(1-\delta_n)(T_{r_n}y_n - v) + \delta_n(x_n - v) + \alpha_n(\gamma f(x_n) - BT_{r_n}y_n)\|^2 \\
&\leq \|(1-\delta_n)(T_{r_n}y_n - v) + \delta_n(x_n - v)\|^2 + 2\alpha_n\langle\gamma f(x_n) - BT_{r_n}y_n, x_{n+1} - v\rangle \\
&\leq (1-\delta_n)\|(y_n - v)\|^2 + \delta_n\|(x_n - v)\|^2 + 2\alpha_n L^2
\end{aligned}$$

where L is constant such that $L = \sup\|\gamma f(x_n) - BT_{r_n}y_n, x_{n+1} - v\|$ and hence

$$(1-\delta_n)\|(y_n - v)\|^2 \leq \delta_n\|(x_n - v)\|^2 - \|(x_{n+1} - v)\|^2 + 2\alpha_n L^2. \tag{3.2.20}$$

So, we have $\|y_n - v\| \to 0$ as $n \to \infty$. Consequently, from (3.2.17) and (3.2.19), we obtain

$$\|u_n - y_n\| \leq \|u_n - x_n\| + \|x_n - y_n\| \to 0 \text{ as } n \to \infty.$$

Step 3 We will show that

$$\limsup_{n\to\infty} \langle (\gamma f - B)z, x_n - z\rangle \leq 0. \tag{3.2.21}$$

Let $Q = P_{\mathfrak{F}}$, and since, $Q(I - B + \gamma f)$ is contraction on H into C (see also [65]:p.18) and H is complete. Thus, by Banach Contraction Principle, then there exist a unique element z of H such that $z = Q(I - B + \gamma f)z$.
We choose subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (\gamma f - B)z, x_n - z\rangle = \lim_{n\to\infty} \langle \gamma f z - Bz, x_{n_i} - z\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a sequence $\{x_{n_{ij}}\}$ of $\{x_{n_i}\}$ and $y \in C$ such that $\{x_{n_{ij}}\} \rightharpoonup y$. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup y$. Since C is closed and convex it is weakly closed and hence $y \in C$. Since $x_n - y_n \to 0$ as $n \to \infty$ we have that $y_{n_i} \rightharpoonup y$. Now, we show that $y \in \mathfrak{F}$ Since $y_n = F_{r_n}$, Lemma 2.6.18 and using (3.2.6), we get

$$\langle y - y_n, Ay_n\rangle + \left\langle y - y_n, \frac{y_n - x_n}{r_n}\right\rangle \geq 0, \quad \forall y \in C, \tag{3.2.22}$$

and

$$\langle y - y_{n_i}, Ay_{n_i}\rangle + \left\langle y - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq 0, \quad \forall y \in C. \tag{3.2.23}$$

Set $v_t = tv + (1-t)y$ for all $t \in (0,1]$ and $v \in C$. Consequently, we get $v_i \in C$. From (3.2.19), it follows that

$$\begin{aligned} \langle v_t - y_{n_i}\rangle &\geq \langle v_t - y_{n_i}, Av_t\rangle - \langle v_t - y_{n_i}, Av_t\rangle - \left\langle v_t - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_n}\right\rangle \\ &= \langle v_t - y_{n_i}, Av_t - Ay_{n_i}\rangle - \left\langle v_t - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_n}\right\rangle, \end{aligned}$$

from the fact that $y_{n_i} - x_{n_i} \to 0$ as $i \to \infty$ we obtain that $\frac{u_{n_i} - x_{n_i}}{r_n} \to 0$ as $i \to \infty$. Since A is monotone, we also have that $\langle v_t - y_{n_i}, Av_t - Ay_{n_i}\rangle \geq 0$. Thus, if follows that

$$0 \leq \lim_{i\to\infty} \langle v_t - y_{n_i}, Av_t\rangle = \langle v_t - w, Av_t\rangle,$$

and hence $\langle v-y, Av_t\rangle \geq 0, \quad \forall v \in C$.

If $t \to 0$, the continuity of A gives that

$$\langle v-y, Ay\rangle \geq 0, \quad \forall v \in C.$$

This implies that $y \in VI(C, A)$.

Furthermore, since $u_n = T_{r_n} y_n$, Lemma 2.6.18 and using (3.2.9), we get

$$\langle y-u_{n_i}, Tu_{n_i}\rangle - \frac{1}{r_n}\langle y-u_{n_i}, (r_{n_i}+1)u_{n_i} - y_{n+1}\rangle \leq 0, \quad \forall y \in C. \tag{3.2.24}$$

Put $z_t = t(v) + (1-t)y$ for all $t \in (0,1]$ and $v \in C$. Then, $z_t \in C$ and from (3.2.20) and pseudo-contractivity of T, we get

$$\begin{aligned} \|u_{n_i} - z_t, Tz_t\| &= \langle u_{n_i} - z_t, Tz_t\rangle + \langle z_t - u_{n_i}, Tu_i\rangle - \frac{1}{r_n}\langle z_t - u_{n_i}, (1+r_{n_i})u_{n_i} - y_{n_i}\rangle \\ &= -\langle z_t - u_{n_i}, Tz_t\rangle - \frac{1}{r_{n_i}}\langle z_t - u_{n_i}, u_{n_i} - y_{n_i}\rangle - \langle z_t - u_{n_i}, u_{n_i}\rangle \\ &\geq \|z_t - u_{n_i}\|^2 - \frac{1}{r_{n_i}}\langle z_t - u_{n_i}, u_{n_i} - y_{n_i}\rangle - \langle z_t - u_{n_i}, u_{n_i}\rangle \\ &= -\langle z_t - u_{n_i}, z_t\rangle - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - y_{n_i}}{r_{n_i}}\right\rangle. \end{aligned}$$

Thus, since $u_n - y_n \to 0$, as $n \to \infty$ we obtain that $\frac{u_{n_i} - y_{n_i}}{r_{n_i}} \to 0$ as $i \to \infty$.

Therefore, as $i \to \infty$, it follows that

$$\langle y - z_t, Tz_t\rangle \geq \langle y - z_t, z_t\rangle$$

and hence

$$-\langle v-y, Tz_t\rangle \geq -\langle v-y, z_t\rangle, \quad \forall v \in C.$$

Taking $t \to 0$ and since T is continuous we obtain

$$-\langle v-y, Ty\rangle \geq -\langle v-y, y\rangle, \quad \forall v \in C.$$

Now, we get $v = Ty$. Then we obtain that $y = Ty$ and hence $y \in F(T)$. Therefore, $y \in F(T) \cap VI(C, A)$ and since $z = P_{\mathfrak{F}}(I - B + \gamma f)z$, Lemma 2.6.16 implies that

$$\begin{aligned} \limsup_{n\to\infty}\langle(\gamma f - B)z, x_n - z\rangle &= \lim_{i\to\infty}\langle(I - B + \gamma f)z - z, x_{n_i} - z\rangle \\ &= \langle(\gamma f - B)z, y - z\rangle \leq 0. \end{aligned} \tag{3.2.25}$$

Step 4 Finally, we will show that $x_n \to z$ as $n \to \infty$, where $z = P_{\mathfrak{F}}(I - B + rf)z$.

From (3.2.1) and (3.2.2) we observe that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \langle \alpha_n\gamma f(x_n)+\delta_n x_n+[(1-\delta_n)I-\alpha_n B]T_{r_n}y_n-z, x_{n+1}-z\rangle \\
&= \alpha_n\langle \gamma f(x_n)-Bz, x_{n+1}-z\rangle+\delta_n\langle x_n-z, x_{n+1}-z\rangle \\
&\quad +\langle [(1-\delta_n)I-\alpha_n B](T_{r_n}-z), x_{n+1}-z\rangle \\
&\leq \alpha_n\gamma\langle f(x_n)-f(z), x_{n+1}-z\rangle+\alpha_n\langle \gamma f(z)-Bz, x_{n+1}-z\rangle \\
&\quad +\delta_n\|x_n-z\|\|x_{n+1}-z\|+(1-\delta_n-\alpha_n\overline{\beta})\|z_n-z\|\|x_{n+1}-z\| \\
&\leq \alpha_n\gamma K\|x_n-z\|\|x_{n+1}-z\|+\alpha_n\langle \gamma f(z)-Bz, x_{n+1}-z\rangle \\
&\quad +\delta_n\|x_n-z\|\|x_{n+1}-z\|+(1-\delta_n-\alpha_n\overline{\beta})\|z_n-z\|\|x_{n+1}-z\| \\
&= \alpha_n\gamma K\|x_n-z\|\|x_{n+1}-z\|+\alpha_n\langle \gamma f(z)-Bz, x_{n+1}-z\rangle \\
&\quad +(1-\alpha_n\overline{\beta})\|x_n-z\|\|x_{n+1}-z\| \\
&\leq \frac{\gamma k}{2}\alpha_n(\|x_n-z\|^2+\|x_{n+1}-z\|^2)+\alpha_n\langle \gamma f(z)-Bz, x_{n+1}-z\rangle \\
&\quad +(1-\alpha_n\overline{\beta})(\|x_n-z\|\|x_{n+1}-z\|) \\
&\leq \frac{\gamma k}{2}\alpha_n(\|x_n-z\|^2+\|x_{n+1}-z\|^2)+\alpha_n\langle \gamma f(z)-Bz, x_{n+1}-z\rangle \\
&\quad +\frac{(1-\alpha_n\overline{\beta})}{2}(\|x_n-z\|^2+\|x_{n+1}-z\|^2) \\
&\leq \frac{1-\alpha_n(\overline{\beta}-k\gamma)}{2}\|x_n-z\|^2+\frac{1}{2}\|x_{n+1}-z\|^2 \\
&\quad +\alpha_n\langle \gamma f(x)-Bz, x_{n+1}-z\rangle,
\end{aligned}$$

which implies that

$$\|x_{n+1}+z\|^2 \leq [1-\alpha_n(\overline{\beta}-k\gamma)]\|x_n-z\|^2+2\alpha_n\langle \gamma f(z)-Bz, x_{n+1}-z\rangle.$$

By the condition (*C1*), (3.2.25) and using Lemma 2.1.23, we see that $\lim_{n\to\infty}\|x_n-z\|=0$. This complete the proof. □