CHAPTER 2 BASIC CONCEPTS AND PRELIMINARIES

In this chapter, we give some definitions, notations, lemmas and some useful results that will be used in the later chapters. Throughout this dissertation, we let $\mathbb{R}$ be the set of all real numbers, $\mathbb{N}$ be the set of all natural numbers, H be a Hilbert space.

2.1 Basic Concepts

Definition 2.1.1. [21] Let X denote any nonempty set that contains each of its elements x and each real number α, a unique element $\alpha \cdot x$, written as αx, which is called a scalar multiple of x. (One could also include complex numbers α as well, but we restrict ourselves here to the real case.) Also, assume that for each two elements $x, y \in X$ there exists a unique element $x + y \in X$ called the sum of x and y. The system $(X, \cdot, +)$ is called a linear space (over $\mathbb{R}$) if the following conditions are satisfied: [Here $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$.]

(1) $x + y = y + x$;

(2) $x + (y + z) = (x + y) + z$;

(3) $\alpha(x + y) = \alpha x + \alpha y$;

(4) $x + y = x + z$ implies $y = z$;

(5) $(\alpha + \beta)x = \alpha x + \beta x$;

(6) $(\alpha\beta)x = \alpha(\beta x)$;

(7) $1x = x$.

Definition 2.1.2. [22] Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}$, satisfying the following condition for all x, y and z in X:

(M1) $d(x, y) = 0 \Longleftrightarrow x = y$;

(M2) $d(x, y) = d(y, x)$;

(M3) $d(x, y) \leq d(x, z) + d(z, y)$.

The function d assigns to each pair (x, y) of element of X a nonnegative real number $d(x, y)$, which does not on the order of the elements; $d(x, y)$ is called the *distance* between x and y. The set X together with a metric, denoted by (X, d), is called a *metric space*. The conditions (M1)-(M3) are usually called the *metric axioms*.

Definition 2.1.3. [22] Let X be a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : X \to \mathbb{R}$ is said to be *a norm on* X if it satisfies the following conditions:

(N1) $\|x\| \geq 0, \forall x \in X$;

(N2) $\|x\| = 0 \Leftrightarrow x = 0$;

(N3) $\|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in X$;

(N4) $\|\alpha x\| = |\alpha|\|x\|, \quad \forall x \in X$ and $\forall \alpha \in \mathbb{K}$.

From this norm we can define a metric, induced by the norm $\|\cdot\|$, by

$$d(x, y) = \|x - y\|, \quad (x, y \in X).$$

A linear space X equipped with the norm $\|\cdot\|$ is called a *normed linear space*.

Definition 2.1.4. [23] Let $(X, \|\cdot\|)$ be a normed space.

(1) A sequence $\{x_n\} \subset X$ is said to *converge strongly* in X if there exists $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. That is, if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_n - x\| < \epsilon, \forall n \geq N$. We often write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ to mean that x is the limit of the sequence $\{x_n\}$.

(2) A sequence $\{x_n\} \subset X$ is said to be *a Cauchy sequence* if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_m - x_n\| < \epsilon, \forall\, m, n \geq N$. That is, $\{x_n\}$ is *a Cauchy sequence* in X if and only if $\|x_m - x_n\| \to 0$ as $m, n \to \infty$.

Definition 2.1.5. [21] Sequence $\{x_n\}_{n=1}^{\infty}$ in a normed linear space X is said to be *a bounded sequence* if there exists $M > 0$ such that $\|x_n\| \leq M, \forall n \in \mathbb{N}$.

Definition 2.1.6. [21] A subset C of a normed linear space X is said to be *convex subset in* X if $\lambda x + (1 - \lambda)y \in C$ for each $x, y \in C$ and for each scalar $\lambda \in [0, 1]$.

Definition 2.1.7. [21] The real-value function of two variables $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is called *inner product* on a real vector space X if for any $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$ the following conditions are satisfied:

(I1) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$;

(I2) $\langle x, y\rangle = \langle y, x\rangle$;

(I3) $\langle x, x\rangle \geq 0$ for each $x \in X$ and $\langle x, x\rangle = 0$ if and only if $x = 0$. A *real inner product space* is a real vector space equipped with an inner product.

Definition 2.1.8. [21] A **Hilbert spaces** is an inner product space which is complete under the norm induced by its inner product.

An inner product on X defines a norm on X given by $\|x\| = \sqrt{\langle x, x\rangle}$.

Lemma 2.1.9. [23]**(The Schwarz inequality)**
If x and y are any two vector in an inner product space X, then

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Lemma 2.1.10. [23] Let H be a real Hilbert space. Then the following inequalities hold:

(i) $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$,

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$,

(iii)$\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$,

Definition 2.1.11. [21] The metric projection (or *nearest point*) from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in C$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

Lemma 2.1.12. [23] Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$. Then

(i) $z = P_C x \Longleftrightarrow \langle z - x, y - x\rangle \geq 0, \quad \forall y \in C$,

(ii) $\|P_C x - P_C y\| \leq \|x - y\|, \quad \forall x, y \in H$,

(iii) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H$,

(iv) $\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall x \in H, y \in C$,

(v) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H, y \in C$.

Definition 2.1.13. [21] A normed space $(X, \|\cdot\|)$ is called strictly convex if for all $x, y \in X$, $x \neq y$, $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1 - \lambda)y\| < 1$, $\forall \lambda \in (0, 1)$.

Definition 2.1.14. [21] A sequence $\{x_n\}$ in a normed spaces is said to *converge weakly* to some vector x if $\lim_{n\to\infty} f(x_n) = f(x)$ holds for every continuous linear functional f. We often write $x_n \rightharpoonup x$ to mean that $\{x_n\}$ converges weakly to x.

Lemma 2.1.15. [23] Let $\{x_n\}$ be a sequence of a normed space $(X, \|\cdot\|)$, $x \in X$ and let $x_n \to x$ if and only if, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exist a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converging to x.

Definition 2.1.16. [23] A normed space X is called *complete* if every Cauchy sequence in X converges to an element in X.

Lemma 2.1.17. [24] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Definition 2.1.18. [21] Let F and X be linear spaces over the field $\mathbb{K}$.

(1) A mapping $T : F \to X$ is called *a linear operator* if $T(x+y) = Tx + Ty$ and $T(\alpha x) = \alpha Tx, \forall x, y \in F$, and $\forall \alpha \in \mathbb{K}$.

(2) A mapping $T : F \to \mathbb{K}$ is called *a linear functional on F* if T is a linear operator.

Definition 2.1.19. [23] Let F and X be normed spaces over the field $\mathbb{K}$ and $T : X \to F$ a linear operator. T is said to be *bounded* on X if there exists a real number $M > 0$ such that $\|T(x)\| \leq M\|x\|, \forall x \in X$.

Remark 2.1.20. [23] In a Hilbert space H, weak convergence is defined by $\lim_{n\to\infty}\langle x_n, y\rangle = \langle x, y\rangle$ for all $y \in H$. The notation $x_n \rightharpoonup x$ is sometimes used to denote this kind of convergence.

Remark 2.1.21. If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.

Lemma 2.1.22. [23] Let X be an inner product space and $\{x_n\}$ be a bounded sequence of H such that $x_n \rightharpoonup x$. Then following inequality holds:

$$\|x\| \leq \lim_{n\to\infty} \inf \|x_n\|.$$

Lemma 2.1.23. [10] Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n,\ n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

2.2 Basic Concepts in Hilbert Spaces

Let C be a closed convex subset of a real Hilbert space H with inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. We have the following are hold:

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \tag{2.2.1}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.2.2}$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$.

Lemma 2.2.1. [25] Let $(E, \langle .,. \rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2.$$

Lemma 2.2.2. [26] A Hilbert space H satisfies the **Opial condition** that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$, holds for every $y \in H$ with $y \neq x$.

Definition 2.2.3. [27] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called *lower semicontinuous* if for any $a \in \mathbb{R}$, the set $\{x \in C : f(x) \leq a\}$ is closed.

Lemma 2.2.4. [21]**(Demi-closedness Principle)** Assume that S is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H. If S has a fixed point, the $I - S$ is demi-closed: that is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ (for short, $x_n \rightharpoonup x \in C$), and the sequence $\{(I - S)x_n\}$ converges strongly to some y (for short, $(I - S)x_n \to y$), it follows that $(I - S)x = y$. Here I is the identity operator of H.

Lemma 2.2.5. [29] A Hilbert space H satisfies the **Kadec-Klee property** that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$.

2.3 The Classical of Fixed Point Theory

Definition 2.3.1. [21] Let H be a Hilbert space and let C a nonempty bounded convex subset of H. A mapping $T : C \to C$ is called *nonexpansive* on C if

$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \forall x, y \in C.$$

Definition 2.3.2. [22] An element $x \in C$ is said to be a *fixed point* of a mapping $T : C \to C$. The set of all fixed points of T is denoted by $F(T) = \{x \in C : Tx = x\}$.

Lemma 2.3.3. [23] Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let S be a nonexpansive mapping of C into itself. Then, $F(T) \neq \emptyset$.

Definition 2.3.4. [23] Let H be a Hilbert space and let C a nonempty bounded convex subset of H. A mapping $f : C \to C$ is called *a contraction* on C if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leqslant \alpha\|x - y\|, \quad \forall x, y \in C.$$

2.4 Some Nonlinear Mappings in Hilbert Spaces

Let C be a closed convex subset of a real Hilbert space H with inner product and norm are denoted by $\langle ., . \rangle$ and $\|.\|$, respectively. Let $T : C \to C$ a nonlinear mapping.

Definition 2.4.1. [23] The *metric projection (nearest point)* P_C from a Hilbert space H to a closed convex subset C of H is defined as follows: given $x \in H$, $P_C x$ is the only point in C with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\| \quad \text{for all } y \in C.$$

It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - P_C x, P_C x - z\rangle \ge 0, \quad \forall z \in C; \tag{2.4.1}$$

$$\|(x-y) - (P_C x - P_C y)\|^2 \ge \|x-y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.4.2}$$

Theorem 2.4.2. [22] **(Banach Contraction Mapping Principle)** Let (X, d) be a complete metric space and $f : X \to X$ be a contraction. Then f has a unique fixed point, i.e. there exists a unique $x^* \in X$ such that $Tx^* = x^*$.

Definition 2.4.3. [23] A mapping $S : C \to C$ is called *strictly pseudo-contractive* if there exists a constant $0 \le k < 1$ such that

$$\|Sx - Sy\|^2 \le \|x-y\|^2 + k\|(I-S)x - (I-S)y\|^2, \quad \forall x, y \in C.$$

Remark 2.4.4. If $k = 0$, then S is nonexpansive.

In this case, we say that $S : C \to C$ is a k-strictly pseudo-contraction.

Putting $B = I - S$. Then, we have

$$\|(I-B)x - (I-B)y\|^2 \le \|x-y\|^2 + k\|Bx - By\|^2, \quad \forall x, y \in C.$$

Observe that

$$\|(I-B)x - (I-B)y\|^2 = \|x-y\|^2 + \|Bx - By\|^2 - 2\langle x - y, Bx - By\rangle, \quad \forall x, y \in C.$$

Hence, we obtain

$$\langle x - y, Bx - By\rangle \ge \frac{1-k}{2}\|Bx - By\|^2, \quad \forall x, y \in C.$$

Then, B is $\frac{1-k}{2}$-inverse-strongly monotone mapping.

Lemma 2.4.5. [30] Assume that C is a closed convex subset of Hilbert space H, and let $S : C \to C$ be a self-mapping of C

(i) If S is a k-strict pseudo-contraction, then S satisfies the Lipscchitz condition

$$\|Sx - Sy\| \leq \frac{1+k}{1-k}\|x - y\| \qquad \forall x, y \in C.$$

(ii) If S is a k-strict pseudo-contraction, then the mapping $I - S$ is demiclosed (at 0). That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I - S)x_n \to 0$, $then (I - S)\tilde{x} = 0$.

(iii) If S is a k-strict pseudo-contraction, then the fixed point set $F(S)$ of S is closed and convex so that the projection $P_{F(S)}$ is well defined.

Lemma 2.4.6. [31] Let C be a nonempty closed convex subset of a real Hilbert space H and let $S : C \to C$ be a k-strict pseudo-contraction mapping with a fixed point. Then $F(S)$ is closed and convex. Define $S_k : C \to C$ by $S_k = kx + (1 - k)Sx$ for each $x \in C$. Then S_k is nonexpansive such that $F(S_k) = F(S)$.

Definition 2.4.7. [21] Let A be a strongly positive on H if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \qquad \forall x \in H. \tag{2.4.3}$$

A typical problem is that of minimizing a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in F(S)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{2.4.4}$$

where A is a nonexpansive mapping and b is a given point in H.

Optimization problem (for short, OP) as the following

$$\min_{x \in F} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{2.4.5}$$

where $F = \cap_{n=1}^{\infty} C_n,\ C_1, C_2, \ldots$ are infinitely closed convex subsets of H such that $\cap_{n=1}^{\infty} C_n \neq \emptyset$, $u \in H$, $\mu \geq 0$ is a real number, A is a strongly positive linear bounded operator on H and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Lemma 2.4.8. [32] Let C be a nonempty closed convex subset of a real Hilbert space H, and $g : C \to \mathbb{R} \cup \{\infty\}$ be a proper lower-semicontinuous differentiable convex function. If z is a solution to the minimization problem

$$g(z) = \inf_{x \in C} g(x),$$

then

$$\Big\langle g'(x), x - z \Big\rangle \geq 0, \quad x \in C.$$

In particular, if z solves problem OP, then

$$\Big\langle u + \big[\gamma f - (I + \mu A)\big] z, x - z \Big\rangle \leq 0.$$

Lemma 2.4.9. [33] Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

Definition 2.4.10. [21] A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v \rangle \geq 0, \quad \forall u, v \in C. \tag{2.4.6}$$

Definition 2.4.11. [21] A is called *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2, \quad \forall u, v \in C. \tag{2.4.7}$$

Lemma 2.4.12. [21] Let $A : H \to H$ be a α-inverse-strongly monotone mapping. If $\lambda \leq 2\alpha$, for any $\lambda > 0$ and $\alpha > 0$ then $I - \lambda A$ is a nonexpansive mapping from H into itself.

Proof. Let $u, v \in H$ and $\lambda > 0$,

$$\begin{aligned} \|(I - \lambda A)u - (I - \lambda A)v\|^2 &= \|(u - v) - \lambda(Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda\langle u - v, Au - Av \rangle + \lambda^2 \|Au - Av\|^2 \\ &\leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2. \end{aligned}$$

□

Definition 2.4.13. [21] A mapping $A : C \to C$ is called *L-Lipschitz-continuous* if there exists a positive real number L such that

$$\|Au - Av\| \leq L\|u - v\|, \quad \forall u, v \in C. \tag{2.4.8}$$

Remark 2.4.14. It is easy to see that if A is an α-inverse-strongly monotone mapping of C into H, then A is $\frac{1}{\alpha}$–Lipschitz continuous.

Definition 2.4.15. [21] Let $\eta : C \times C \to H$ and $B : C \to H$ be two mappings. B is said to be:

(1) *monotone* if

$$\langle Bx - By, \eta(x, y)\rangle \geq 0, \quad \forall x, y \in C;$$

(2) σ*-strongly monotone* if there exists a positive real number σ such that

$$\langle Bx - By, \eta(x, y)\rangle \geq \sigma\|x - y\|^2, \quad \forall x, y \in C;$$

(3) L*-Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|\eta(x, y)\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

Lemma 2.4.16. [31] Let $V : C \to H$ be a k-strict pseudo-contraction, then
(1) the fixed point set $F(V)$ of V is closed convex so that the projection $P_{F(V)}$ is well defined;
(2) define a mapping $T : C \to H$ by

$$Tx = tx + (1 - t)Vx, \ \forall x \in C. \tag{2.4.9}$$

If $t \in [k, 1)$, then T is a nonexpansive mapping such that $F(V) = F(T)$.

Definition 2.4.17. [34] For the infinite family of nonexpansive mapping of $T_1, T_2, ...$, we define the mapping W_n of C into itself as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1})I, \\ \vdots \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1})I, \\ \vdots \\ U_{n,2} = \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I, \end{cases} \tag{2.4.10}$$

where $T_1, T_2, ...$ be an infinite family of nonexpansive mappings of C into itself and $\lambda_1, \lambda_2, ...$ be real numbers such that $0 \leq \lambda_n \leq 1$ for every $n \in \mathbb{N}$.

Lemma 2.4.18. [34] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, ...$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, let $\mu_1, \mu_2, ...$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then,

(1) W_n is nonexpansive and $F(W_n) = \cap_{i=1}^{n} F(T_i)$, $\forall n \geq 1$;

(2) for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists;

(3) a mapping $W : C \to C$ defined by

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \forall x \in C \tag{2.4.11}$$

is a nonexpansive mapping satisfying $F(W) = \cap_{i=1}^{\infty} F(T_i)$ and it is called the W-mapping generated by $T_1, T_2, ...$ and $\mu_1, \mu_2, ...$

Lemma 2.4.19. [35] Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_i : C \to C\}$ be a countable family of nonexpansive mappings with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$, $\{\mu_i\}$ be a real sequence such that $0 < \mu_i \leq b < 1, \forall i \geq 1$. If D is any bounded subset of C, then

$$\lim_{n\to\infty} \sup_{x\in D} \|Wx - W_n x\| = 0.$$

2.5 Variational Inequalities in Hilbert Spaces

Let C be a closed convex subset of a real Hilbert space H with inner product and norm are denoted by $\langle ., . \rangle$ and $\|.\|$, respectively.

Definition 2.5.1. [21] Let $B : C \to H$ be a nonlinear mapping. The *variational inequality problem* is to find $x \in C$ such that

$$\langle Bx, y - x \rangle \geq 0, \quad \forall y \in C. \tag{2.5.1}$$

We denote by $VI(C, B)$ the set of solutions of the variational inequality problem, that is,

$$VI(C, B) = \{x \in C : \langle Bx, y - x \rangle \geq 0, \quad \forall y \in C\}. \tag{2.5.2}$$

Lemma 2.5.2. [23] Let H be Hilbert space, let C be a nonempty closed convex subset of H and let B be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,

$$u \in VI(C, B) \Longleftrightarrow u = P_C(u - \lambda Bu),$$

where P_C is the metric projection of H onto C.

Lemma 2.5.3. [23] Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $\xi > 0$ and let $B : C \to H$ be ξ-inverse strongly monotone. Then, $VI(C, B) \neq \emptyset$.

Definition 2.5.4. [37] Let $A, B : C \to H$ be two mappings. We consider the following problem for finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \ \forall x \in C, \\ &\langle \mu Bx^* + y^* - x^*, x - y^* \rangle \geq 0, \ \forall x \in C, \end{aligned} \tag{2.5.3}$$

which is called a *general system of variational inequalities* where $\lambda \geq 0$ and $\mu \geq 0$ are two constants. The set of solution of (2.5.3) is denoted by $GVI(C, A, B)$.

Remark 2.5.5. If $A = B$, then the problem (2.5.3) is reduced into the *new system of variational inequalities* for finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \ \forall x \in C, \\ &\langle \mu Ax^* + y^* - x^*, x - y^* \rangle \geq 0, \ \forall x \in C, \end{aligned} \tag{2.5.4}$$

which is defined by Verma [37] (see also Verma [38]).

Remark 2.5.6. If $x^* = y^*$, then the problem (2.5.4) is reduced into the *classical variational inequalities* for finding $x^* \in C$ such that

$$\langle Ax^*, v - x^* \rangle \geq 0 \tag{2.5.5}$$

for all $v \in C$, which was originally introduced and studied by Stampacchia [4]. The set of solution of (2.5.5) is denoted by $VI(C, A)$.

Definition 2.5.7. [21] Let $A : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ be a set-valued mapping. We consider the following *variational inclusion problem*, which is to find a point $u \in H$ such that

$$\theta \in A(u) + M(u), \tag{2.5.6}$$

where θ is the zero vector in H. The set of solutions of problem (2.5.6) is denoted by $VI(A, M)$.

Remark 2.5.8. If $M = \partial\delta_C$, where C is a nonempty closed convex subset of H and $\delta_C : H \to [0, \infty]$ is the indicator function of C, i.e.,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$$

Then the variational inclusion problem (2.5.6) is equivalent to the classical variational inequality (2.5.5).

Definition 2.5.9. [21] A set-valued mapping $M : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Mx$ and $g \in My$ imply $\langle x - y, f - g\rangle \geq 0$.

Definition 2.5.10. [31] A monotone mapping $M : H \to 2^H$ is *maximal* if the graph of $G(M)$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(M)$ implies $f \in Mx$.

Definition 2.5.11. [39] Let the set-valued mapping $M : H \to 2^H$ be a maximal monotone. We define the *resolvent operator* $J_{M,\lambda}$ associate with M and λ as follows:

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad u \in H, \tag{2.5.7}$$

where λ is a positive number. It is worth mentioning that the resolvent operator $J_{M,\lambda}$ is single-valued, nonexpansive and 1-inverse strongly monotone.

Remark 2.5.12. ([41],[42]) Let A be an inverse-strongly monotone mapping of C into H and let $N_C v$ be the *normal cone* to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$$

and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$.

2.6 Equilibrium Problems

Definition 2.6.1. [40] Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The *equilibrium problem* for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{2.6.1}$$

The set of solutions of (2.6.1) is denoted by $EP(F)$, that is,

$$EP(F) = \{x \in C : F(x, y) \geq 0, \quad \forall y \in C\}.$$

Given a mapping $B : C \to H$, let $F(x, y) = \langle Bx, y - x\rangle$ for all $x, y \in C$. Then, $z \in EP(F)$ if and only if $\langle Bz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality.

Let $\Im = \{F_k\}_{k\in\Lambda}$ be a family of bifunctions from $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The system of equilibrium problems for $\Im = \{F_k\}_{k\in\Lambda}$ is to determine common equilibrium points for $\Im = \{F_k\}_{k\in\Lambda}$ such that

$$F_k(x, y) \geq 0, \quad \forall k \in \Lambda, \quad \forall y \in C, \tag{2.6.2}$$

where Λ is an arbitrary index set. The set of solutions of (2.6.2) is denoted by $SEP(\Im)$, that is,

$$SEP(\Im) = \left\{ \ x \in C : F_k(x, y) \geq 0, \quad \forall k \in \Lambda, \quad \forall y \in C \ \right\}. \tag{2.6.3}$$

If Λ is a singleton, then the problem (2.6.2) is reduced to the problem (2.6.1).

Definition 2.6.2. [3] For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;

(A3) F is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$, (A5) for each $y \in C, x \mapsto F(x,y)$ is weakly upper semicontinuous;

(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0; \tag{2.6.4}$$

(B2) C is a bounded set.

Definition 2.6.3. [43] Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, $\Psi : C \to H$ be a nonlinear mapping and $\varphi : C \to \mathbb{R}$ be a real-valued function. The *generalized mixed equilibrium problem* is to find $x \in C$ such that

$$F(x,y) + \langle \Psi x, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{2.6.5}$$

The set of solutions of (2.6.5) is denoted by $GMEP(F, \varphi, \Psi)$, that is

$$GMEP(F,\varphi,\Psi) = \{x \in C : F(x,y) + \langle \Psi x, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C\}.$$

Remark 2.6.4. [48] If $\Psi \equiv 0$, then the problem (2.6.5) is reduced into the *mixed equilibrium problem* for finding $x \in C$ such that

$$F(x,y) + \varphi(y) \geq \varphi(x), \forall y \in C. \tag{2.6.6}$$

The set of solutions of (2.6.6) is denoted by $MEP(F, \varphi)$. We see that x is a solution of problem (2.6.6) implies that $x \in \text{dom } \varphi = \{x \in C | \varphi(x) < +\infty\}$.

Remark 2.6.5. If $\varphi \equiv 0$, the problem (2.6.5) is reduced into the *generalized equilibrium problem* is to find $x \in C$ such that

$$F(x,y) + \langle \varphi x, y - x\rangle \geq 0, \quad \forall y \in C. \tag{2.6.7}$$

The set of solution of (2.6.7) is denoted by $GEP(F, B)$, that is,

$$GEP(F,\varphi) = \left\{x \in C : F(x,y) + \langle \varphi x, y - x\rangle \geq 0, \quad \forall y \in C\right\}.$$

In the case of $\varphi \equiv 0$, then the problem (2.6.7) is reduced to the problem (2.6.1). In the case of $F \equiv 0$, the problem (2.6.7) is reduced to the classical variational inequality problem (2.5.1).

Remark 2.6.6. [48] If $\varphi \equiv 0$, then the mixed equilibrium problem (2.6.6) is reduced into the *equilibrium problem* (2.6.1)

Remark 2.6.7. If $F \equiv 0$ and $\Psi \equiv 0$, then the problem (2.6.5) is reduced into the *minimize problem* for finding $x \in C$ such that

$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{2.6.8}$$

The set of solutions of (2.6.8) is denoted by $Argmin(\varphi)$.

Lemma 2.6.8. [30] Let C be a nonempty closed convex subset of H and let f be a contraction of H into itself with $\alpha \in (0,1)$, and A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$. Then , for $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$,

$$\Big\langle x - y, (A - \gamma f)x - (A - \gamma f)y \Big\rangle \geq (\bar{\gamma} - \alpha\gamma)\|x - y\|^2, \quad x, y \in H.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\bar{\gamma} - \alpha\gamma$.

Lemma 2.6.9. [30] Assume A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

Lemma 2.6.10. [45] Let C be a closed convex subset of H. Let $\{x_n\}$ be a bounded sequence in H. Assume that

(i) The weak ω-limit set $\omega_w(x_n) \subset C$,

(ii) For each $z \in C$, $\lim_{n\to\infty} \|x_n - z\|$ exists. Then, $\{x_n\}$ is weakly convergent to a point in C.

Lemma 2.6.11. [49] Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C.$$

Lemma 2.6.12. [50] Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $x \in H$. Then, the following hold:

(1) T_r is single-valued;

(2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,
$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ is closed and convex.

Lemma 2.6.13. [55] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction mapping satisfies (A1)-(A4) and let $\varphi : C \to \mathbb{R}$ is convex and lower semicontinuous such that $C \cap dom\varphi \neq \emptyset$. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, then there exists $u \in C$ such that

$$F(u, y) + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0.$$

Define a mapping $K_r : H \to C$ as follows:

$$K_r(x) = \Big\{u \in C : F(u, y) + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0, \forall y \in C\Big\}$$

for all $x \in H$. Then, the following hold:

(1) K_r is single-valued;

(2) K_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|K_r x - K_r y\|^2 \leq \langle K_r x - K_r y, x - y\rangle$;

(3) $F(K_r) = MEP(F)$;

(4) $MEP(F)$ is closed and convex.

Let $\varphi : C \to \mathbb{R}$ be a real-valued function and let $\{F_k : C \times C \to \mathbb{R}, \quad k = 1, 2, \ldots, N\}$ be a finite family of equilibrium functions, i.e., $F_k(u, u) = 0$ for each $u \in C$. The *system of mixed equilibrium problems* (for short, SMEP) for function $(F_1, F_2, \ldots, F_N, \varphi)$ which is to find $z \in C$ such that

$$\begin{cases} F_1(z, y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C, \\ F_2(z, y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C, \\ \quad \vdots \\ F_N(z, y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C. \end{cases} \tag{2.6.9}$$

The set of solutions of (2.6.9) is denoted by $\cap_{k=1}^{N} MEP(F_k, \varphi)$, where $MEP(F_k, \varphi)$ is the set of solutions of the *mixed equilibrium problem*. If $\varphi \equiv 0$, and $N = 1$, then the problem (2.6.9) reduces to the *equilibrium problem*.

Lemma 2.6.14. ([40],[53]) For solving the system of mixed equilibrium problems (2.6.9), let us assume that function $F_k : C \times C \to \mathbb{R}, k = 1, 2, \ldots, N$ satisfies the following conditions:

(H1) F_k is monotone, i.e., $F_k(x, y) + F_k(y, x) \leq 0, \quad \forall x, y \in C$;

(H2) for each fixed $y \in C$, $x \mapsto F_k(x, y)$ is convex and upper semicontinuous;

(H3) for each $x \in C, y \mapsto F_k(x, y)$ is convex.

Lemma 2.6.15. [16] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a *Banach space* E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf \beta_n \leq \limsup \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1 - \beta_n)x_n$ for all $n \geq 0$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{2.6.10}$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.6.16. Let C be a convex subset of a Hilbert space H. Let $x \in H$ and $x_0 \in C$. Then $x_0 = P_C x$ if and only if

$$\langle z - x_0, x_0 - x\rangle \leq 0, \quad \forall z \in C.$$

Lemma 2.6.17. [53] Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a continuous accretive mapping. Then, for $r > 0$ and $x \in H$, there exist $z \in C$ such that

$$\langle y - z, Az\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Moreover, by a similar argument of the proof of lemma 2.8 and 2.9 of [64].

Lemma 2.6.18. [53] Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a continuous accretive mapping. For $r > 0$ and $x \in H$, define a mapping $F_r : H \to C$ as follows :

$$F_r x := \{z \in C : \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}$$

for all $x \in H$. Then the following hold:

(1) F_r is single - valued;

(2) F_r is a firmly nonexpansive type mapping, i.e., for all $x, y \in H$,

$$\|F_r x - F_r y\|^2 \leq \langle F_r x - F_r y, x - y\rangle;$$

(3) $F(F_r) = VI(C, A)$;

(4) $VI(C, A)$ is closed and convex.

Lemma 2.6.19. [53] Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to H$ be a continuous pseudo-contractive mappings. Then, for $r > 0$ and $x \in H$, there exist $z \in C$ such that

$$\langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C.$$

Lemma 2.6.20. [53] Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudo-contractive mappings. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows :

$$T_r x := \{z \in C : \langle y - z, Tz\rangle + \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C\}$$

for all $x \in H$. Then the following hold:

(1) T_r is single - valued;

(2) T_r is a firmly nonexpansive type mapping, i.e., for all $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = F(T)$;

(4) F(T) is closed and convex.

Lemma 2.6.21. [11] The function $u \in C$ is a solution of the variational inequality if and only if $u \in C$ satisfies the relation $u = P_C(u - \lambda Bu)$ for all $\lambda > 0$.

Lemma 2.6.22. [62] Let $M : H \to 2^H$ be a maximal monotone mapping and let $B : H \to H$ be a monotone and Lipshitz continuous mapping. Then the mapping $L = M + B : H \to 2^H$ is a maximal monotone mapping.

Lemma 2.6.23. [26] Each Hilbert space H satisfies Opial's condition, that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$, hold for each $y \in H$ with $y \neq x$.

Lemma 2.6.24. [57] Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is,

$$x_n \rightharpoonup x \textit{ and } \|x_n - Tx_n\| \to 0$$

implies $x = Tx$.

Lemma 2.6.25. [56] Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i\in N}$ be an infinite family of nonexpansive mappings of C into itself such that $\cap_{i\in N} F(T_i) \neq \emptyset$ and let $\{\lambda_i\}$ be an real sequence such that $0 \leq \lambda_i \leq b < 1$ for every $i \in N$. Then $F(W) = \cap_{i\in N} F(T_i) \neq \emptyset$.

Lemma 2.6.26. [56] Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}$ be an infinite family of nonexpansive mappings of C into itself and let $\{\lambda_i\}$ be a real sequence such that $0 \leq \lambda_i \leq b < 1$ for every $i \in N$. Then, for every $x \in C$ and $k \in N$, the $\lim_{n\longrightarrow\infty} U_{n,k}$ exist.

In view of the previous lemma, we define

$$Wx := \lim_{n\longrightarrow\infty} U_{n,1}x = \lim_{n\longrightarrow\infty} W_n x.$$

Lemma 2.6.27. [57] Let H be a Hilbert space, C is a closed convex subset of H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$; in particular, if $y = 0$ then $x \in F(T)$.

Lemma 2.6.28. [45] Let C be a nonempty closed convex subset of a real Hilbert space H. If $T : C \to C$ is a k-strict pseudo-contraction, then the mapping $I - T$ is demiclosed at 0. That is, if $\{x_n\}$ is a sequence in C weakly converging to x and $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)x = 0$.

Lemma 2.6.29. [58] Let H be a Hilbert space, C be a closed convex subset of H, $f : C \to H$ be a contraction with coefficient $0 < \rho < 1$ and $T : C \to C$ be a nonexpansive mapping. Then, for $0 < \gamma < \bar{\gamma}/\rho$, for $x, y \in C$,

1. the mapping $(I - f)$ is strongly monotone with coefficient $(1 - \rho)$ that is

$$\langle x - y, (I - f)x - (I - f)y \rangle \geq (1 - \rho)\|x - y\|^2,$$

2. the mapping $(I - T)$ is monotone, that is

$$\langle x - y, (I - T)x - (I - T)y \rangle \geq 0.$$

Lemma 2.6.30. [45] Let C be a closed convex subset of H. Let $\{x_n\}$ be a bounded sequence in H. Assume that

(1) The weak ω-limit set $\omega_w(x_n) \subset C$.

(2) For each $z \in C$, $\lim_{n\to\infty} \|x_n - z\|$ exists.

Then $\{x_n\}$ is weakly convergent to a point in C.

Lemma 2.6.31. [31] Let H be a real Hilbert space, C be a closed and convex subset of H, and T be a k-strict pseudo-contraction mapping on C, then $F(T)$ is closed convex, so that the projection $P_{F(T)}$ is well defined.

Lemma 2.6.32. [31] Let H be a Hilbert space, C be a closed and convex subset of H, and $T : C \to H$ be a k-strict pseudo-contraction mapping. Define a mapping $V : C \to H$ by $Vx = \lambda x + (1 - \lambda)Tx$ for all $x \in C$. Then, as $\lambda \in [k, 1)$, V is a nonexpansive mapping such that $F(V) = F(T)$.

Lemma 2.6.33. [19] Let H be a Hilbert space and C be a nonempty closed and convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Then

$$\|Tx - x\|^2 \leq 2\langle x - Tx, x - x' \rangle, \quad \forall x' \in F(T), \forall x \in C.$$

Lemma 2.6.34. [20] For solving the Ky Fan inequality or equilibrium problem, let us assume that the following conditions are satisfied on the bifunction $F : C \times C \to \mathbb{R}$.

(A1) $F(x, x) = 0$ for every $x \in C$;

(A2) F is pseudomonotone on C, i.e., $F(x, y) \geq 0 \Rightarrow F(y, x) \leq 0, \quad \forall x, y \in C$;

(A3) F is jointly weakly continuous on $C \times C$ in the sense that, if $x, y \in C$ and $\{x_n\}$ and $\{y_n\}$ are two sequences in C converging weakly to x and y, respectively, then $F(x_n, y_n) \to F(x, y)$;

(A4) $F(x, \cdot)$ is convex, lower semicontinuous, and subdifferentiable on C for very $x \in C$;

(A5) F satisfies the Lipschitz-type condition, there exist positive integer c_1 and c_2, such that for every $x, y, z \in C$,

$$F(x, y) + F(y, z) \geq F(x, z) - c_1\|y - x\|^2 - c_2\|z - y\|^2.$$

If F satisfies the properties (A1)-(A4), then the set $EP(F)$ of solutions to the Ky Fan inequality is closed and convex.

Proposition 2.6.35. ([60], Lemma 3.1) For every $x^* \in EP(F)$, and every $n \in \mathbb{N}$, one has

1. $\langle x_n - y_n, y - y_n \rangle \leq \lambda_n F(x_n, y) - \lambda_n F(x_n, y_n), \quad \forall y \in C;$

2. $\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - 2\lambda_n c_1)\|y_n - x_n\|^2 - (1 - 2\lambda_n c_2)\|z_n - y_n\|^2.$

Proposition 2.6.36. [61] Let K be a nonempty closed and convex subset of H. Let $u \in H$ and let $\{x_n\}$ be a sequence in H. If any weak limit point of $\{x_n\}$ belongs to K, and $\|x_n - u\| \leq \|u - P_K u\|$ for all $n \in \mathbb{N}$, then $x_n \to P_K u$.