CHAPTER 1 INTRODUCTION

1.1 Background

The fixed point theory is the most important tool to solve problems in many branches of sciences and applied sciences. Moreover, the construction of fixed point iteration processes for nonlinear mappings takes an important role to find solutions of many problems in applied areas as it application. In case, the problem in science had been transformed into a mathematical model such as equality, inequality, equality system and inequality system.There are two questions following.

(1). Does the answer of model exist ?

(2). How can find the answer ?

Fixed-point iteration processes for nonlinear mappings in Hilbert spaces including Mann and Ishikawa iterations process have been studied extensively by many authors to approximate fixed points of various classes of operators such as nonexpansive mappings, contraction mappings, strict pseudo-contraction mappings, resolvent operators and to solve variational inequalities in Hilbert spaces for the detail please see [1],[2] and the references therein.

The Equilibrium theory represents an important area of mathematical sciences such as optimization, operation research, game theory, financial mathematics and mechanics. Equilibrium problems include variational inequalities, optimization problems, Nash equilibria problems, saddle point problems, fixed point problems, and complementarity problems as special cases., see [3]. Related to the equilibrium problems, we also have the problem of finding the fixed points of the nonexpansive mappings which is the subject of current interest in functional analysis. It is natural to construct a unified approach for these problems. In this direction, several authors have introduced some iterative schemes for finding a common element of a set of the solutions of the equilibrium problems, the set solutions of the variational inequality problems for nonlinear mappings and a set of the fixed points of an infinite (a finite) family of nonexpansive mappings.

The theory of variational inequality represents, in fact, a very natural generalization of the theory of boundary value problems and allows us to consider new problems arising from many fields of applied sciences such as applied mathematics, mechanics, physics, engineering, the theory of convex programming and the theory of control.

Variational inequalities was introduced by Stampacchia [4] in the early sixties and have had a great impact and influence in the development of almost all branches of pure and applied sciences. It is well-known that the variational inequalities are equivalent to the fixed point problems. This alternative equivalent formulation has been used to suggest and analyze in variational inequalities. In particular, the solution of the variational inequalities can be computed to use for the iterative projection methods. It is well-known that the convergence of a projection method requires the operator to be strongly monotone and Lipschitz continuous. These conditions are very stict and rule out its application in several important problems.

Let C be a nonempty set and $T : C \to C$.

In 1953, Mann [5] introduced the Mann's iteration which is defined by the initial guess x_0 and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.1.1}$$

where $\{\alpha_n\} \in [0, 1]$. In an infinite-dimensional Hilbert space, Mann iteration can yield only weak convergence.

In 1967, Halpern [6] defined the iteration with starting from x_0, for fixed u and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.1.2}$$

which is satisfied the conditions $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty}\alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.

In 1974, The Ishikawa's iteration process is defined by Ishikawa [7] as the following:

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ty_n, \quad n \geq 0, \end{cases} \tag{1.1.3}$$

where the initial guess x_0 is taken in C arbitrarily and the sequences $\{\alpha_n\}, \{\beta_n\} \in [0, 1]$.

In 2003, Nakajo and Takahashi [8] proposed the modification of the following Mann iteration method (1.1.1).

$$\begin{cases} x_0 \in C \quad \text{is arbitrary}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2 \ldots, \end{cases} \tag{1.1.4}$$

where P_C is metric projection on the set C.

In 2004, Takahashi and Toyoda [9] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)TP_C(x_n - \lambda_n A x_n), \quad n \geq 0. \tag{1.1.5}$$

where $x_0 \in C, \{\alpha_n\}$ is a sequence in $(0,1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$, and obtained weak convergence theorem in a Hilbert space H.

In 2004, Xu [10] studied the iteration process $\{x_n\}$ called viscosity approximation method as shown in the following:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \;\; \text{for } n \geq 1,$$

where $\{\alpha_n\} \subset (0,1)$ and $f : C \to C$ is a contraction mapping.

Let C be a nonempty closed and convex subset of a real Hilbert space H. The classical variational inequality problem is to find a $u \in C$ such that $\langle v - u, Bu \rangle \geq 0$ for all $v \in C$. The set of solutions of the variational inequality is denoted by $VI(C, B)$. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ $x, y \in C$. we denote by $F(T)$ the set of fixed points of T; that is $F(T) = \{x \in C : x = Tx\}$

In 2005, Iiduka and Takahashi [11] proposed a new iterative scheme as following:

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x + (1-\alpha_n)TP_C(x_n - \lambda_n B x_n), \;\; \forall n \geq 0, \end{cases} \tag{1.1.6}$$

where B is β-inverse-strongly monotone, $\{\alpha_n\}$ is a sequence in (0, 1), and $\{\lambda_n\}$ is a sequence in $(0, 2\beta)$. They showed that if $F(T) \cap VI(C, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.1.6) converges strongly to some $z \in F(T) \cap VI(C, B)$.

In 2007, Moudafi [12] intoduced the following Krasnoselski-Mann algorithm:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(\beta_n Sx_n + (1-\beta_n)Tx_n), \tag{1.1.7}$$

where $S, T : C \to C$ are two nonexpansive mappings, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$.

In 2007, Tada and Takahashi [13] introduced the following iterative scheme by the hybrid method in a Hilbert space H. Let $x_0 = x \in H$ and

$$\begin{cases} u_n \in C \text{ such that} \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ w_n = (1-\alpha_n)x_n + \alpha_n Su_n, \\ C_n = \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.1.8}$$

for every $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$.

In 2009, Cianciaruso et al. [14] introduced a two step algorithm to solve the following problem:

$$\langle x^* - Sx^*, x - x^*\rangle \geq 0, \quad \forall x \in F(T), \tag{1.1.9}$$

and defined the following algorithm:

$$\begin{cases} y_n = \beta_n Sx_n + (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Ty_n, \end{cases} \tag{1.1.10}$$

where $f : C \to C$ is a contraction mapping, S and $T : C \to C$ are two nonexpansive mappings, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$.

In 2010, Yao et al. [15] modified the two step algorithm (1.1.10) to extend range of f from C to H by using the metric projection of H onto C. They introduced the following iterative scheme:

$$\begin{cases} y_n = \beta_n Sx_n + (1-\beta_n)x_n, \\ x_{n+1} = P_C[\alpha_n f(x_n) + (1-\alpha_n)Ty_n], \end{cases} \tag{1.1.11}$$

where $f : C \to H$ is a contraction mapping, S and $T : C \to C$ are two nonexpansive mappings, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$.

In 2011, Yao and Xu [16] independently introduced two iterative methods for finding the minimum-norm fixed point of nonexpansive mapping which is defined on closed convex subset C of H. The proposed algorithms are based on the well-known Browder's iterative method [17] and Halpern's iterative method [18].

Recently, In 2011 Gu et al.[19] introduced the following iterative algorithm:

$$\begin{cases} y_n = P_C[\beta_n S x_n + (1-\beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum\limits_{i=1}^{n}(\alpha_{i-1} - \alpha_i)T_i y_n], \quad \forall n \geq 1, \end{cases} \tag{1.1.12}$$

where $f : C \to H$ is a contraction mapping, $S : C \to H$ is a nonexpansive mapping, $\{T_i\}_{i=1}^{\infty} : C \to C$ is a countable family of nonexpansive mappings, $\alpha_0 = 1$, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$.

Very recently, Phan Tu Vuong et. al. [20], considered the sequences $\{x_n\}, \{y_n\}, \{z_n\}$, and $\{t_n\}$ generated by $x_0 \in C$ and

$$\begin{cases} y_n = \arg\min_{y \in C}\{\lambda_n F(x_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ z_n = \arg\min_{y \in C}\{\lambda_n F(y_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ t_n = \alpha_n x_n + (1-\alpha_n)[\beta_n z_n + (1-\beta_n)S z_n], \\ x_{n+1} = t_n, \end{cases} \tag{1.1.13}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1[$, $\{\beta_n\} \subset]0, 1[$, and $\{\lambda_n\} \subset]0, 1]$.

Motivated by above, we introduce and modify new iterative schemes for approximate the common solution of fixed point problem, the variational inequalities, the system of variational inclusions and the system of equilibrium problems.

1.2 Objectives

There are three main objectives:

1. To study and extend the classical fixed point theory, the variational inequalities and the system of equilibrium problems of nonlinear mappings.

2. To prove new convergence theorems for finding the common solution of fixed point problems, systems of variational inequalities and systems of equilibrium problems for nonlinear mappings.

3. To apply our results to solve some problems of classical variational inequality problems and optimization problems.

1.3 The Summary of the Study

The summary of the dissertation is concluded as follows:

In Chapter 1, we give a brief introduction and review the background of this dissertation.

In Chapter 2, we give the necessary tools and concern some well-known definitions and preliminaries which will be useful in the later chapters.

In Chapter 3, we introduce a new iterative scheme for solving variational inequality problems, fixed points for nonexpensive and pseudocontraction mappings.

In Chapter 4, we introduce a new iterative scheme for finding a common solution of the set of fixed point, the system of mixed equilibrium problems and the variational inclusion.

In Chapter 5, we introduce a new iterative method for finding a common fixed point of a countable family of strictly pseudocontractive mappings, and solutions of the equilibrium problems, the variational inequality problems and the fixed point problems for a strict pseudocontraction mapping.

Finally, we conclude this research in Chapter 6.