CHAPTER 6 CONCLUSIONS

In Chapter 3 of this dissertation, we established the following results.

(1) For any $x \in l(p,\theta)$, there exist $k_0 \in \mathbb{N}$ and $\lambda \in (0,1)$ such that $\varrho(\frac{x^k}{2}) \leq \frac{1-\lambda}{2}\varrho(x^k)$ for all $k \in \mathbb{N}$ with $k \geq k_0$, where

$$x^k = (\overbrace{0, 0, \cdots, 0}^{k-1}, x(k), x(k+1), x(k+2), \cdots).$$

(2) For any $x \in l(p,\theta)$ and $\varepsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that $\varrho(x) \leq 1-\varepsilon$ implies $\|x\| \leq 1-\delta$.

(3) The space $l(p,\theta)$ is a Banach space with respect to the Luxemburg norm.

(4) The space $l(p,\theta)$ has the property (β).

(5) The space $l(p,\theta)$ has the uniform opial property.

(6) The functional ϱ is a convex modular on $ces_{(p)}(q)$.

(7) For all $x \in ces_{(p)}(q)$, the modular ϱ on $ces_{(p)}(q)$ satisfies the following properties:

(i) if $0 < a < 1$, then $a^M\varrho(\frac{x}{a}) \leq \varrho(x)$ and $\varrho(ax) \leq a\varrho(x)$;

(ii) if $a > 1$, then $\varrho(x) \leq a^M\varrho(\frac{x}{a})$;

(iii) if $a \geq 1$, then $\varrho(x) \leq a\varrho(x) \leq \varrho(ax)$.

(8) For any $x \in ces_{(p)}(q)$, we have

(i) if $\|x\| < 1$, then $\varrho(x) \leq \|x\|$;

(ii) if $\|x\| > 1$, then $\varrho(x) \geq \|x\|$;

(iii) $\|x\| = 1$ if and only if $\varrho(x) = 1$;

(iv) $\|x\| < 1$ if and only if $\varrho(x) < 1$;

(v) $\|x\| > 1$ if and only if $\varrho(x) > 1$.

(9) For any $x \in ces_{(p)}(q)$, we have

(i) if $0 < a < 1$ and $\|x\| > a$, then $\varrho(x) > a^M$;

(ii) if $a \geq 1$ and $\|x\| < a$, then $\varrho(x) < a^M$.

(10) Let (x_n) be a sequence in $ces_{(p)}(q)$.

(i) If $\|x_n\| \to 1$ as $n \to \infty$, then $\varrho(x_n) \to 1$ as $n \to \infty$.

(ii) If $\varrho(x_n) \to 0$ as $n \to \infty$, then $\|x_n\| \to 0$ as $n \to \infty$.

(11) Let $x \in ces_{(p)}(q)$ and $\{x_n\} \subseteq ces_{(p)}(q)$. If $\varrho(x_n) \to \varrho(x)$ as $n \to \infty$ and $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \geq 1$, then $x_n \to x$ as $n \to \infty$.

(12) The space $ces_{(p)}(q)$ has the property (H).

(13) The space $ces_{(p)}(q)$ has the uniform opial property.

In Chapter 4 of this dissertation, we established the following results.

(14) Let X_ρ be a $\rho-$*complete* bounded modular space, where ρ satisfies the Δ_2-condition. Let $c, l \in \mathbb{R}^+$, $c > l$ and $T : X_\rho \to X_\rho$ be a mapping such that, for all $x, y \in X_\rho$,

$$\psi(\rho(c(Tx - Ty))) \leq \psi(\rho(l(x-y))) - \phi(\rho(l(x-y))), \tag{6.0.1}$$

where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous and nondecreasing functions with $\psi(t) = \phi(t) = 0$ if and only if $t = 0$.
Then T has a unique fixed point.

(15) Let X_ρ be a $\rho-$*complete* bounded modular space, where ρ satisfies the Δ_2-condition. Let $c, l \in \mathbb{R}^+$, $c > l$ and $T, f : X_\rho \to X_\rho$ be two $\rho-$*compatible* mappings such that $T(X_\rho) \subseteq f(X_\rho)$ and

$$\psi(\rho(c(Tx - Ty))) \leq \psi(\rho(l(fx - fy))) - \phi(\rho(l(fx - fy))) \tag{6.0.2}$$

for all $x, y \in X_\rho$, where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous and nondecreasing functions with $\psi(t) = \phi(t) = 0$ if and only if $t = 0$.
If one of T or f is continuous, then there exists a unique common fixed point of T and f.

(16) Let X_ω be a complete modular metric space and $T : X_\omega \to X_\omega$ be a contraction mapping. Assume that there exists $x_0 \in X$ such that $\omega_\lambda(x_0, Tx_0) < \infty$ for all $\lambda > 0$. Then T has a fixed point in $x_* \in X_\omega$ and the sequence $\{T^n x_0\}$ converges to x_*. Moreover, if, $z \in F(X_\omega)$, where $F(X_\omega)$ is a set of fixed point of T such that $\omega_\lambda(x_*, z) < \infty$ for all $\lambda > 0$, then $x_* = z$.

(17) Let X_ω be a complete modular metric space and $T : X_\omega \to X_\omega$ be a contraction mapping. Suppose that $x^* \in X_\omega$ is a fixed point of T, $\{\varepsilon_n\}$ is a sequence of positive numbers for which $\lim_{n\to\infty} \varepsilon_n = 0$ and $\{y_n\} \subseteq X_\omega$ satisfies

$$\omega_\lambda(y_{n+1}, Ty_n) \leq \varepsilon_n$$

for all $\lambda > 0$. Then $\lim_{n\to\infty} y_n = x^*$.

(18) Let X_ω be a complete modular metric space and, for all $x^* \in X_\omega$, we define

$$B_\omega(x^*, \gamma) := \{x \in X_\omega | \omega_\lambda(x, x^*) \leq \gamma \text{ for all } \lambda > 0\}.$$

If $T : B_\omega(x^*, \gamma) \to X_\omega$ is a contraction mapping with

$$\omega_{\frac{\lambda}{2}}(Tx^*, x^*) \leq (1 - k)\gamma \tag{6.0.3}$$

for all $\lambda > 0$, where $0 \leq k < 1$, then T has a fixed point in $B_\omega(x^*, \gamma)$.

(19) Let X_ω be a complete modular metric space and T be a self-mapping on X_ω satisfying the following:

$$\omega_\lambda(Tx, Ty) \leq k(\omega_{2\lambda}(Tx, x) + \omega_{2\lambda}(Ty, y)) \tag{6.0.4}$$

for all $x, y \in X_\omega$, where $k \in \left[0, \frac{1}{2}\right)$. Assume that there exists $x_0 \in X$ such that $\omega_\lambda(x_0, Tx_0) < \infty$ for all $\lambda > 0$. Then T has a fixed point in $x_* \in X_\omega$ and the sequence $\{T^n x\}$ converges to x_*. Moreover, if, $z \in F(X_\omega)$, where $F(X_\omega)$ is a set of fixed point of T such that $\omega_\lambda(x_*, z) < \infty$ for all $\lambda > 0$, then $x_* = z$.

In Chapter 5 of this dissertation, we established the existence of best proximity points.

(20) Let (X, d) be a complete metric space and A, B be nonempty closed subsets of X such that A_0 and B_0 are nonempty. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ be a nonself-mapping satisfying the following conditions:

(i) S and T are Geraghty's proximal contractions of the first kind;

(ii) g is an isometry;

(iii) the pair (S, T) is a proximal cyclic contraction;

(iv) $S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;

(v) $A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$.

Then there exist a unique point $x \in A$ and $y \in B$ such that

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(gx_{n+1}, Sx_n) = d(A, B)$$

converges to the element x. For any fixed $y_0 \in B_0$, the sequence $\{y_n\}$ defined by

$$d(gy_{n+1}, Ty_n) = d(A, B)$$

converges to the element y.

(21) Let (X, d) be a complete metric space and A, B be nonempty closed subsets of X. Further, suppose that A_0 and B_0 are nonempty. Let $S : A \to B$ and $g : A \to A$ be the mappings satisfying the following conditions:

(i) S is Geraghty's proximal contractions of the first and second kinds;

(ii) g is an isometry;

(iii) S preserves isometric distance with respect to g;

(iv) $S(A_0) \subseteq B_0$;

(v) $A_0 \subseteq g(A_0)$.

Then there exists a unique point $x \in A$ such that

$$d(gx, Sx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(gx_{n+1}, Sx_n) = d(A, B)$$

converges to the element x.

(22) Let X be a nonempty set such that $(X, \preceq)$ be a partially ordered set and (X, d) be a complete metric space. Let A and B be nonempty closed subsets of X such that A_0 and B_0 are nonempty. Let $T : A \to B$ satisfy the following conditions:

(i) T is a continuous, proximally order-preserving and generalized proximal C-contraction such that $T(A_0) \subseteq B_0$;

(ii) there exist element x_0 and x_1 in A_0 such that $x_0 \preceq x_1$ and

$$d(x_1, Tx_0) = d(A, B).$$

Then there exists a point $x \in A$ and such that

$$d(x, Tx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(x_{n+1}, Tx_n) = d(A, B)$$

converges to the point x.

(23) Let X be a nonempty set such that $(X, \preceq)$ is a partially ordered set and (X, d) be a complete metric space. Let A and B be nonempty closed subsets of X such that A_0 and B_0 are nonempty. Let $T : A \to B$ satisfy the following conditions:

(i) T is a proximally order-preserving and generalized proximal C-contraction such that $T(A_0) \subseteq B_0$;

(ii) there exist element $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and

$$d(x_1, Tx_0) = d(A, B);$$

(iii) if $\{x_n\}$ is an increasing sequence in A converges to x, then $x_n \preceq x$ for all $n \geq 1$.

Then there exists a point $x \in A$ and such that

$$d(x, Tx) = d(A, B).$$

(24) Let X be a nonempty set such that $(X, \preceq)$ be a partially ordered set and (X, d) be a complete metric space. Let A and B be nonempty closed subsets of X and A_0 and B_0 are nonempty such that A_0 satisfies the condition (5.2.21). Let $T : A \to B$ satisfy the following conditions:

(i) T is a continuous, proximally order-preserving and generalized proximal C-contraction such that $T(A_0) \subseteq B_0$;

(ii) there exist element $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and

$$d(x_1, Tx_0) = d(A, B).$$

Then there exists a unique point $x \in A$ and such that

$$d(x, Tx) = d(A, B).$$

(25) Let X be a nonempty set such that $(X, \preceq)$ be a partially ordered set and (X, d) be a complete metric space. Let A and B be nonempty closed subsets of X and A_0 and B_0 are nonempty such that A_0 satisfies the condition (5.2.21). Let $T : A \to B$ satisfy the following conditions:

(i) T is an proximally order-preserving and generalized proximal C-contraction such that $T(A_0) \subseteq B_0$;

(ii) there exist element $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and

$$d(x_1, Tx_0) = d(A, B);$$

(iii) if $\{x_n\}$ is an increasing sequence in A converges to x, then $x_n \preceq x$ for all $n \geq 1$.

Then there exists a unique point $x \in A$ and such that

$$d(x, Tx) = d(A, B).$$

(26) Let A and B be nonempty closed subsets of a partially ordered metric space $(X, \preceq)$ and d be a metric on X. Let $T : A \cup B \to A \cup B$ be a cyclic mapping such that T and T^2 are nondecreasing on A such that

$$d(T\acute{x}, T^2x) \leq \alpha d(\acute{x}, Tx) + \beta d(x, T\acute{x}) + (1 - \alpha - \beta)d(A, B)$$

and

$$d(T\acute{y}, T^2y) \leq \alpha d(\acute{y}, Ty) + \beta d(y, T\acute{y}) + (1 - \alpha - \beta)d(A, B),$$

for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and for all $(x, \acute{x}) \in A \times A$, $(y, \acute{y}) \in B \times B$ with $x \preceq \acute{x}$, $y \preceq \acute{y}$. Assume that there exists $x_0 \in A$ with $x_0 \preceq T^2x_0$ and define $x_{n+1} = Tx_n$ for all $n \geq 1$. If $T|_A$ is continuous and $\{x_{2n}\}$ has a convergent subsequence in A, then T has a best proximity point $p \in A$.

(27) Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. Let A and B be two nonempty subsets of X such that (A, B) satisfies the property (UC), and A is complete. Let $T : A \cup B \to A \cup B$ be a cyclic mapping such that T and T^2 are nondecreasing on A. Suppose that

$$d(T\acute{x}, T^2x) \leq \alpha d(\acute{x}, Tx) + \beta d(x, T\acute{x}) + (1 - \alpha - \beta)d(A, B)$$

and

$$d(T\acute{y}, T^2y) \leq \alpha d(\acute{y}, Ty) + \beta d(y, T\acute{y}) + (1 - \alpha - \beta)d(A, B)$$

for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and for all $(x, \acute{x}) \in A \times A$, $(y, \acute{y}) \in B \times B$ with $x \preceq \acute{x}$, $y \preceq \acute{y}$. If $T|_A$ is continuous and that there exists $x_0 \in A$ such that $x_0 \preceq T^2x_0$, and $x_{n+1} = Tx_n$ for all $n \geq 1$, then T has a best proximity point $p \in A$ and $x_{2n} \to p$.

(28) Let A and B be nonempty closed subsets of a complete metric space X such that A is approximatively compact with respect to B. Also, assume that A_0 and B_0 are nonempty. Let $S : A \to B$, $T : A \to B$ be the nonself-mapping satisfying the following conditions:

(i) For each $x, y \in A$,

$$d(Sx, Sy) \leq d(Tx, Ty) - \varphi(d(Tx, Ty)),$$

where, $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$;

(ii) T is continuous;

(iii) S and T commute proximally;

(iv) S and T can be swapped proximally;

(v) $S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$.

Then there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B),\ d(x, Sx) = d(A, B).$$

Moreover, if x^* is another common best proximity point of the mappings S and T, then

$$d(x, x^*) \leq 2d(A, B).$$