CHAPTER 5 BEST PROXIMITY POINTS FOR GENERALIZED CONTRACTION MAPPINGS IN METRIC SPACES

The aim of this chapter is to introduce new mappings which is generalize contraction non-self mappings and prove some best proximity point and common best proximity point theorems for these class in metric spaces.

5.1 Best proximity points for Geraghty's proximal contraction mappings

In this section, we introduce the new class of proximal contractions, so called Geraghty's proximal contraction mappings, and prove best proximity theorems for these classes and also give some examples to illustrate our main Theorems. Let $\mathcal{G}$ be the family of functions from $[0,\infty)$ into $[0,1)$ which satisfies the condition: if $\beta \in \mathcal{G}$, then we have

$$\beta(t_n) \to 1 \Longrightarrow t_n \to 0.$$

Definition 5.1.1. A mapping $T : A \to B$ is called *Geraghty's proximal contraction of the first kind* if, there exists $\beta \in \mathcal{G}$ such that

$$\left.\begin{aligned} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{aligned}\right\} \quad \Longrightarrow \quad d(u, v) \leq \beta(d(x, y))d(x, y)$$

for all $u, v, x, y \in A$.

Definition 5.1.2. A mapping $T : A \to B$ is called *Geraghty's proximal contraction of the second kind* if there exists $\beta \in \mathcal{G}$ such that

$$\left.\begin{aligned} d(u, Tx) = d(A, B)) \\ d(v, Ty) = d(A, B) \end{aligned}\right\} \quad \Longrightarrow \quad d(Tu, Tv) \leq \beta(d(Tx, Ty))d(Tx, Ty)$$

for all $u, v, x, y \in A$.

It is easy to see that, if we take $\beta(t) = k$, where $k \in [0, 1)$, then Geraghty's proximal contraction of the first kind and Geraghty's proximal contraction of the second kind reduce to a proximal contraction of the first kind (Definition 2.6.8) and a proximal contraction of the second kind (Definition 2.6.9), respectively.

Next, we extend the result of Sadiq Basha [54] and Banach's fixed point theorem to the class of nonself-mappings satisfying Geraghty's proximal contraction condition.

Theorem 5.1.3. *Let (X, d) be a complete metric space and A, B be nonempty closed subsets of X such that A_0 and B_0 are nonempty. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ satisfy the following conditions:*

(i) *S and T are Geraghty's proximal contractions of the first kind;*

(ii) *g is an isometry;*

(iii) *the pair (S, T) is a proximal cyclic contraction;*

(iv) *$S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;*

(v) *$A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$.*

Then there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(gx_{n+1}, Sx_n) = d(A, B)$$

converges to the element x. For any fixed $y_0 \in B_0$, the sequence $\{y_n\}$ defined by

$$d(gy_{n+1}, Ty_n) = d(A, B)$$

converges to the element y.

Proof. Let x_0 be a fixed element in A_0. In view of the fact that $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, it follows that there exists an element $x_1 \in A_0$ such that

$$d(gx_1, Sx_0) = d(A, B).$$

Again, since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists an element $x_2 \in A_0$ such that

$$d(gx_2, Sx_1) = d(A, B).$$

By the same method, we can find $x_n \in A_0$ such that

$$d(gx_n, Sx_{n-1}) = d(A, B).$$

So, inductively, one can determine an element $x_{n+1} \in A_0$ such that

$$d(gx_{n+1}, Sx_n) = d(A, B). \tag{5.1.1}$$

Since S is Geraghty's proximal contraction of the first kind, g is an isometry and $\beta \in \mathcal{G}$, it follows that for each $n \geq 1$,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(gx_{n+1}, gx_n) \\ &\leq \beta(d(x_n, x_{n-1}))d(x_n, x_{n-1}) \\ &\leq d(x_n, x_{n-1}), \end{aligned}$$

which implies that the sequence $\{d(x_{n+1}, x_n)\}$ is non-increasing and bounded below. Hence there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$. Suppose that $r > 0$. Observe that

$$\frac{d(x_{n+1}, x_n)}{d(x_n, x_{n-1})} \leq \beta(d(x_n, x_{n-1})),$$

which implies that $\lim_{n\to\infty} \beta(d(x_n, x_{n-1})) = 1$. Since $\beta \in \mathcal{G}$, we have $r = 0$, which is a contradiction and hence

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0. \tag{5.1.2}$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and the subsequences $\{x_{m_k}\}, \{x_{n_k}\}$ of $\{x_n\}$ such that, for any $n_k > m_k \geq k$,

$$r_k := d(x_{m_k}, x_{n_k}) \geq \varepsilon, \quad d(x_{m_k}, x_{n_k-1}) < \varepsilon$$

for any $k \geq 1$. For each $n \geq 1$, let $\alpha_n := d(x_{n+1}, x_n)$. Then we have

$$\begin{aligned} \varepsilon \leq r_k &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \varepsilon + \alpha_{n_k-1} \end{aligned} \tag{5.1.3}$$

and so it follows from (5.1.2) and (5.1.3) that

$$\lim_{k\to\infty} r_k = \varepsilon. \tag{5.1.4}$$

Notice also that

$$\begin{aligned}
\varepsilon &\le r_k \\
&\le d(x_{m_k}, x_{m_k+1}) + d(x_{n_k+1}, x_{n_k}) + d(x_{m_k+1}, x_{n_k+1}) \\
&= \alpha_{m_k} + \alpha_{n_k} + d(x_{m_k+1}, x_{n_k+1}) \\
&\le \alpha_{m_k} + \alpha_{n_k} + \beta(d(x_{m_k}, x_{n_k}))d(x_{m_k}, x_{n_k})
\end{aligned}$$

and so

$$\frac{r_k - \alpha_{m_k} - \alpha_{n_k}}{d(x_{m_k}, x_{n_k})} \le \beta(d(x_{m_k}, x_{n_k})).$$

Taking $k \to \infty$ in the above inequality, by (5.1.2), (5.1.4) and $\beta \in \mathcal{G}$, we get $\varepsilon = 0$, which is a contradiction. So we know that the sequence $\{x_n\}$ is a Cauchy sequence. Hence $\{x_n\}$ converges to some element $x \in A$.

Similarly, in view of the fact that $T(B_0) \subseteq A_0$ and $B_0 \subseteq g(B_0)$, we can conclude that there exists a sequence $\{y_n\}$ such that converges to some element $y \in B$. Since the pair (S, T) is a proximal cyclic contraction and g is an isometry, we have

$$d(x_{n+1}, y_{n+1}) = d(gx_{n+1}, gy_{n+1}) \le kd(x_n, y_n) + (1-k)d(A, B). \tag{5.1.5}$$

Taking $n \to \infty$ in (5.1.5), it follows that

$$d(x, y) = d(A, B) \tag{5.1.6}$$

and so $x \in A_0$ and $y \in B_0$. Since $S(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$, there exist $u \in A$ and $v \in B$ such that

$$d(u, Sx) = d(A, B), \quad d(v, Ty) = d(A, B). \tag{5.1.7}$$

From (5.1.1) and (5.1.7), since S is Geraghty's proximal contraction of the first kind of S, we get

$$d(u, gx_{n+1}) \le \beta(d(x, x_n))d(x, x_n). \tag{5.1.8}$$

Letting $n \to \infty$ in the above inequality, we get $d(u, gx) \leq 0$ and so $u = gx$. Therefore,

$$d(gx, Sx) = d(A, B). \tag{5.1.9}$$

Similarly, we can show that $v = gy$ and so

$$d(gy, Ty) = d(A, B). \tag{5.1.10}$$

From (5.1.6), (5.1.9) and (5.1.10), we get

$$d(x, y) = d(gx, Sx) = d(gy, Ty) = d(A, B).$$

Next, to prove the uniqueness, suppose that there exist $x^* \in A$ and $y^* \in B$ with $x \neq x^*$ and $y \neq y^*$ such that

$$d(gx^*, Sx^*) = d(A, B), \quad d(gy^*, Ty^*) = d(A, B).$$

Since g is an isometry and S is Geraghty's proximal contraction of the first kind, it follows that

$$d(x, x^*) = d(gx, gx^*) \leq \beta(d(x, x^*))d(x, x^*)$$

and hence

$$1 = \frac{d(x, x^*)}{d(x, x^*)} \leq \beta(d(x, x^*)) < 1,$$

which is a contradiction. Thus we have $x = x^*$. Similarly, we can prove that $y = y^*$. This completes the proof. $\square$

If g is the identity mapping in Theorem 5.1.3, then we obtain the following:

Corollary 5.1.4. *Let (X, d) be a complete metric space and A, B be nonempty closed subsets of X. Further, suppose that A_0 and B_0 are nonempty. Let $S : A \to B$, $T : B \to A$ be the mappings satisfying the following conditions:*

(i) *S and T are Geraghty's proximal contractions of the first kind;*

(ii) *$S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;*

(iii) *the pair (S, T) is a proximal cyclic contraction.*

Then there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that

$$d(x, Sx) = d(y, Ty) = d(x, y) = d(A, B).$$

If we take $\beta(t) = k$, where $0 \leq k < 1$, we obtain the following:

Corollary 5.1.5. [55] *Let (X, d) be a complete metric space and A, B be nonempty closed subsets of X. Further, suppose that A_0 and B_0 are nonempty. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ be the mappings satisfying the following conditions:*

(i) *S and T are proximal contractions of the first kind;*

(ii) *g is an isometry;*

(iii) *the pair (S, T) is a proximal cyclic contraction;*

(iv) *$S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;*

(v) *$A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$.*

Then there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(gx_{n+1}, Sx_n) = d(A, B)$$

converges to the element x. For any fixed $y_0 \in B_0$, the sequence $\{y_n\}$ defined by

$$d(gy_{n+1}, Ty_n) = d(A, B)$$

converges to the element y.

If g is the identity mapping in Corollary 5.1.5, we obtain the following:

Corollary 5.1.6. *Let (X, d) be a complete metric space and A, B be nonempty closed subsets of X. Further, suppose that A_0 and B_0 are nonempty. Let $S : A \to B$, $T : B \to A$ be the mappings satisfying the following conditions:*

(i) *S and T are proximal contraction of first kind;*

(ii) *$S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;*

(iii) *the pair (S, T) is a proximal cyclic contraction.*

Then there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that

$$d(x, Sx) = d(y, Ty) = d(x, y) = d(A, B).$$

Next, we establish a best proximity point theorem for nonself-mappings which are Geraghty's proximal contractions of the first kind and the second kind.

Theorem 5.1.7. *Let (X, d) be a complete metric space and A, B be nonempty closed subsets of X. Further, suppose that A_0 and B_0 are nonempty. Let $S : A \to B$ and $g : A \to A$ be the mappings satisfying the following conditions:*

(i) *S is Geraghty's proximal contractions of the first and second kinds;*

(ii) *g is an isometry;*

(iii) *S preserves isometric distance with respect to g;*

(iv) *$S(A_0) \subseteq B_0$;*

(v) *$A_0 \subseteq g(A_0)$.*

Then there exists a unique point $x \in A$ such that

$$d(gx, Sx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(gx_{n+1}, Sx_n) = d(A, B)$$

converges to the element x.

Proof. Since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, as in the proof of Theorem 5.1.3, we can construct the sequence $\{x_n\}$ in A_0 such that

$$d(gx_{n+1}, Sx_n) = d(A, B) \tag{5.1.11}$$

for each $n \geq 1$. Since g is an isometry and S is Geraghty's proximal contraction of the first kind, we see that

$$d(x_n, x_{n+1}) = d(gx_n, gx_{n+1}) \leq \beta(d(x_n, x_{n-1}))d(x_n, x_{n-1})$$

for all $n \geq 1$. Again, similarly, we can show that the sequence $\{x_n\}$ is a Cauchy sequence and so it converges to some $x \in A$. Since S is Geraghty's proximal contraction of the second kind and preserves the isometric distance with respect to g, we have

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &= d(Sgx_n, Sgx_{n+1}) \\ &\leq \beta(d(Sx_{n-1}, Sx_n))d(Sx_{n-1}, Sx_n) \\ &\leq d(Sx_{n-1}, Sx_n), \end{aligned}$$

which means that the sequence $\{d(Sx_{n+1}, Sx_n)\}$ is non-increasing and bounded below. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = r.$$

Suppose that $r > 0$. Observe that

$$\frac{d(Sx_n, Sx_{n+1})}{d(Sx_{n-1}, Sx_n)} \leq \beta(d(Sx_{n-1}, Sx_n)).$$

Taking $n \to \infty$ in the above inequality, we get $\beta(d(Sx_{n-1}, Sx_n)) \to 1$. Since $\beta \in \mathcal{G}$, we have $r = 0$ which is a contradiction and thus

$$\lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = 0. \tag{5.1.12}$$

Now, we claim that $\{Sx_n\}$ is a Cauchy sequence. Suppose that $\{Sx_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and the subsequences $\{Sx_{m_k}\}, \{Sx_{n_k}\}$ of $\{Sx_n\}$ such that, for any $n_k > m_k \geq k$,

$$r_k := d(Sx_{m_k}, Sx_{n_k}) \geq \varepsilon, \quad d(Sx_{m_k}, Sx_{n_k-1}) < \varepsilon$$

for any $k \geq 1$. For each $n \geq 1$, let $\gamma_n := d(Sx_{n+1}, Sx_n)$. Then we have

$$\begin{aligned} \varepsilon \leq r_k &\leq d(Sx_{m_k}, Sx_{n_k-1}) + d(Sx_{n_k-1}, Sx_{n_k}) \\ &< \varepsilon + \gamma_{n_k-1} \end{aligned} \tag{5.1.13}$$

and so it follows from (5.1.12) and (5.1.13) that

$$\lim_{k\to\infty} r_k = \varepsilon.$$

Notice also that

$$
\begin{aligned}
\varepsilon &\leq r_k \\
&\leq d(Sx_{m_k}, Sx_{m_k+1}) + d(Sx_{n_k+1}, Sx_{n_k}) + d(Sx_{m_k+1}, Sx_{n_k+1}) \\
&= \gamma_{m_k} + \gamma_{n_k} + d(Sx_{m_k+1}, Sx_{n_k+1}) \\
&\leq \gamma_{m_k} + \gamma_{n_k} + \beta(d(Sx_{m_k}, Sx_{n_k}))d(Sx_{m_k}, Sx_{n_k}).
\end{aligned}
$$

So, it follows that

$$1 = \lim_{k\to\infty} \frac{r_k - \gamma_{m_k} - \gamma_{n_k}}{d(Sx_{m_k}, Sx_{n_k})} \leq \lim_{k\to\infty} \beta(d(Sx_{m_k}, Sx_{n_k})) < 1$$

and so $\lim_{k\to\infty} \beta(d(Sx_{m_k}, Sx_{n_k})) = 1$. Since $\beta \in \mathcal{G}$, we have $\lim_{k\to\infty} d(Sx_{m_k}, Sx_{n_k}) = 0$, that is, $\varepsilon = 0$, which is a contradiction. So, we obtain the claim and then it converges to some $y \in B$. Therefore, we can conclude that

$$d(gx, y) = \lim_{n\to\infty} d(gx_{n+1}, Sx_n) = d(A, B),$$

which implies that $gx \in A_0$. Since $A_0 \subseteq g(A_0)$, we have $gx = gz$ for some $z \in A_0$ and then $d(gx, gz) = 0$. By the fact that g is an isometry, we have $d(x, z) = d(gx, gz) = 0$. Hence $x = z$ and so $x \in A_0$. Since $S(A_0) \subseteq B_0$, there exists $u \in A$ such that

$$d(u, Sx) = d(A, B). \tag{5.1.14}$$

Since S is Geraghty's proximal contraction of the first kind, it follows from (5.1.11) and (5.1.14) that

$$d(u, gx_{n+1}) \leq \beta(d(x, x_n))d(x, x_n) \tag{5.1.15}$$

for all $n \geq 1$. Taking $n \to \infty$ in (5.1.15), it follows that the sequence $\{gx_n\}$ converges to a point u. Since g is continuous and $\lim_{n\to\infty} x_n = x$, we have $gx_n \to gx$ as $n \to \infty$. By the uniqueness of the limit, we conclude that $u = gx$. Therefore, it follows that $d(gx, Sx) = d(u, Sx) = d(A, B)$. The proof of uniqueness follow from the proof of Theorem 5.1.3. This completes the proof. $\square$

If g is the identity mapping in Theorem 5.1.7, then we obtain the following:

Corollary 5.1.8. *Let (X, d) be a complete metric space and A, B be nonempty closed subsets of X. Further, suppose that A_0 and B_0 are nonempty. Let $S : A \to B$ be the mappings satisfying the following conditions:*

(i) *S is Geraghty's proximal contraction of the first and second kinds;*

(ii) $S(A_0) \subseteq B_0$.

Then there exists a unique point $x \in A$ such that

$$d(x, Sx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(x_{n+1}, Sx_n) = d(A, B)$$

converges to the best proximity point x of S.

If we take $\beta(t) = k$ in Theorem 5.1.7, where $0 \leq k < 1$, we obtain the following:

Corollary 5.1.9. [55] *Let (X, d) be a complete metric space and let A, B be nonempty closed subsets of X. Further, suppose that A_0 and B_0 are nonempty. Let $S : A \to B$ and $g : A \to A$ be the mappings satisfying the following conditions:*

(i) *S is a proximal contraction of the first and second kinds;*

(ii) *g is an isometry;*

(iii) *S preserves isometric distance with respect to g;*

(iv) $S(A_0) \subseteq B_0$;

(v) $A_0 \subseteq g(A_0)$.

Then there exists a unique point $x \in A$ such that

$$d(gx, Sx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(gx_{n+1}, Sx_n) = d(A, B)$$

converges to the element x.

If g is the identity mapping in Corollary 5.1.9, then we obtain the following:

Corollary 5.1.10. *Let (X, d) be a complete metric space and A, B be nonempty closed subsets of X. Further, suppose that A_0 and B_0 are nonempty. Let $S : A \to B$ be a mapping satisfying the following conditions:*

(i) *S is a proximal contraction of the first and second kinds;*

(ii) $S(A_0) \subseteq B_0$.

Then there exists a unique point $x \in A$ such that

$$d(x, Sx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(x_{n+1}, Sx_n) = d(A, B)$$

converges to the best proximity point x of S.

Next, we give an example to show that Definition 5.1.1 is different form Definition 2.6.8. Moreover, we give an example which supports Theorem 5.1.3. First, we give some proposition for our example.

Proposition 5.1.11. *Let $f : [0, \infty) \to [0, \infty)$ be a function defined by $f(t) = \ln(1 + t)$. Then we have the following inequality:*

$$f(a) - f(b) \leq f(|a - b|) \tag{5.1.16}$$

for all $a, b \in [0, \infty)$.

Proof. If $x = y$, we have done. Suppose that $x > y$. Then since we have

$$\frac{1+x}{1+y} = \frac{1+x+y-y}{1+y} = 1 + \frac{x-y}{1+y} < 1 + |x - y|,$$

it follows that $\ln(1 + x) - \ln(1 + y) < \ln(1 + |x - y|)$. In the case $x < y$, by a similar argument, we can prove that inequality (5.1.16) holds. $\square$

Proposition 5.1.12. *For each $x, y \in \mathbb{R}$, we have that the following inequality holds:*

$$\frac{1}{(1 + |x|)(1 + |y|)} \leq \frac{1}{1 + |x - y|}$$

Proof. Since

$$\begin{aligned}1+|x-y| &\le 1+|x|+|y|\\ &\le 1+|x|+|y|+|x||y|\\ &=(1+|x|)(1+|y|),\end{aligned}$$

so that

$$\frac{1}{(1+|x|)(1+|y|)} \le \frac{1}{1+|x-y|}.$$

□

Example 5.1.13. Consider the complete metric space $\mathbb{R}^2$ with Euclidean metric. Let

$$A=\{(0,x): x\in\mathbb{R}\},\quad B=\{(2,y): y\in\mathbb{R}\}.$$

Then $d(A,B)=2$. Define the mappings $S: A\to B$ as follows:

$$S((0,x))=\big(2,\ln(1+|x|)\big).$$

First, we show that S is Geraghty's proximal contractions the first kind with $\beta\in\mathcal{G}$ defined by

$$\beta(t)=\begin{cases}1, & t=0,\\ \dfrac{\ln(1+t)}{t}, & t>0.\end{cases}$$

Let $(0,x_1),(0,x_2),(0,a_1)$ and $(0,a_2)$ be elements in A satisfying

$$d((0,a_1),S(0,x_1))=d(A,B)=2,\quad d((0,a_2),S(0,x_2))=d(A,B)=2.$$

Then we have $a_i=\ln(1+|x_i|)$ for $i=1,2$. If $x_1=x_2$, we have done. Assume that $x_1\neq x_2$. Then, by Proposition 5.1.11 and the fact that the function $f(x)=\ln(1+t)$

is increasing, we have

$$\begin{aligned}
d((0,a_1),(0,a_2)) &= d((0,\ln(1+|x_1|)),(0,\ln(1+|x_2|))) \\
&= |\ln(1+|x_1|)-\ln(1+|x_2|)| \\
&\leq |\ln(1+||x_1|-|x_2||)| \\
&\leq |\ln(1+|x_1-x_2|)| \\
&= \frac{|\ln(1+|x_1-x_2|)|}{|x_1-x_2|}|x_1-x_2| \\
&= \beta(d((0,x_1),(0,x_2))d((0,x_1),(0,x_2)).
\end{aligned}$$

Thus S is Geraghty's proximal contraction of the first kind.

Next, we prove that S is not a proximal contraction of the first kind. Suppose S is proximal contraction of the first kind, then for each $(0,x^*),(0,y^*),(0,a^*),(0,b^*) \in A$ satisfying

$$d((0,x^*),S(0,a^*)) = d(A,B) = 2 \text{ and } d((0,y^*),S(0,b^*)) = d(A,B) = 2, \quad (5.1.17)$$

there exists $k \in [0,1)$ such that

$$d((0,x^*),(0,y^*)) \leq kd((0,a^*),(0,b^*)).$$

From (5.1.17), we get $x^* = \ln(1+|a^*|)$ and $y^* = \ln(1+|b^*|)$ and so

$$\begin{aligned}
\left|\ln(1+|a^*|)-\ln(1+|b^*|)\right| &= d((0,x^*),(0,y^*)) \\
&\leq kd((0,a^*),(0,b^*)) \\
&= k|a^*-b^*|.
\end{aligned}$$

Letting $b^* = 0$, we get

$$1 = \lim_{|a^*|\to 0^+} \frac{\left|\ln(1+|a^*|)\right|}{|a^*|} \leq k < 1,$$

which is a contradiction. Thus S is not a proximal contraction of the first kind.

Example 5.1.14. Consider the complete metric space $\mathbb{R}^2$ with metric defined by

$$d((x_1,x_2),(y_1,y_2)) = |x_1-y_1| + |x_2-y_2|,$$

for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. Let

$$A = \{(0, x) : x \in \mathbb{R}\}, \quad B = \{(2, y) : y \in \mathbb{R}\}.$$

Define two mappings $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ as follows:

$$S((0, x)) = \left(2, \frac{|x|}{2(1 + |x|)}\right), \quad T((2, y)) = \left(0, \frac{|y|}{2(1 + |y|)}\right), \quad g((x, y)) = (x, -y).$$

Then $d(A, B) = 2$, $A_0 = A$, $B_0 = B$ and the mapping g is an isometry.

Next, we show that S and T are Geraghty's proximal contractions the first kind with $\beta \in \mathcal{G}$ defined by

$$\beta(t) = \frac{1}{1 + t}$$

for all $t \geq 0$. Let $(0, x_1), (0, x_2), (0, a_1)$ and $(0, a_2)$ be elements in A satisfying

$$d((0, a_1), S(0, x_1)) = d(A, B) = 2, \quad d((0, a_2), S(0, x_2)) = d(A, B) = 2.$$

Then we have

$$a_i = \frac{|x_i|}{2(1 + |x_i|)}$$

for $i = 1, 2$. If $x_1 = x_2$, we have done. Assume that $x_1 \neq x_2$, Then, by Proposition 5.1.12, we have

$$\begin{aligned}
d((0, a_1), (0, a_2)) &= d\left(\left(0, \frac{|x_1|}{2(1 + |x_1|)}\right), \left(0, \frac{|x_2|}{2(1 + |x_2|)}\right)\right) \\
&= \left|\frac{|x_1|}{2(1 + |x_1|)} - \frac{|x_2|}{2(1 + |x_2|)}\right| \\
&= \left|\frac{|x_1| - |x_2|}{2(1 + |x_1|)(1 + |x_2|)}\right| \\
&\leq \left|\frac{x_1 - x_2}{(1 + |x_1|)(1 + |x_2|)}\right| \\
&\leq \frac{1}{1 + |x_1 - x_2|}|x_1 - x_2| \\
&= \beta(d((0, x_1), (0, x_2))d((0, x_1), (0, x_2)).
\end{aligned}$$

Thus S is Geraghty's proximal contraction of the first kind. Similarly, we can see that T is Geraghty's proximal contraction of the first kind. Next, we show that the pair (S, T) is a proximal cyclic contraction. Let $(0, u), (0, x) \in A$ and $(2, v), (2, y) \in B$ be such that

$$d((0, u), S(0, x)) = d(A, B) = 2, \quad d((2, v), T(2, y)) = d(A, B) = 2.$$

Then we get

$$u = \frac{|x|}{2(1 + |x|)}, \quad v = \frac{|y|}{2(1 + |y|)}.$$

In case $x = y$, clear. Suppose that $x \neq y$, then we have

$$\begin{aligned}
d((0, u), (2, v)) &= |u - v| + 2 \\
&= \left| \frac{|x|}{2(1 + |x|)} - \frac{|y|}{2(1 + |y|)} \right| + 2 \\
&= \left| \frac{|x| - |y|}{2(1 + |x|)(1 + |y|)} \right| + 2 \\
&\leq \frac{|x - y|}{2(1 + |x|)(1 + |y|)} + 2 \\
&\leq \frac{1}{2}\big|x - y\big| + 2 \\
&\leq k\big(\big|x - y\big| + 2\big) + (1 - k)2 \\
&= kd((0, x), (2, y)) + (1 - k)d(A, B),
\end{aligned}$$

where $k = [\frac{1}{2}, 1)$. Hence the pair (S, T) is a proximal cyclic contraction. Therefore, all the hypotheses of Theorem 5.1.3 are satisfied. Further, it is easy to see that $(0, 0) \in A$ and $(2, 0) \in B$ are the unique elements such that

$$d(g(0, 0), S(0, 0)) = d(g(2, 0), T(2, 0)) = d((0, 0), (2, 0)) = d(A, B).$$

5.2 Best proximity points for generalized proximal C-contraction mappings

In this section, we first introduce the notion of generalized proximal C-contraction mapping which is a generalization of C-contraction for non-self mappings and establish the best proximity point theorems. Recall a generalization of C-contraction given by the following definition:

Definition 5.2.1. [58] Let (X, d) be a metric space. A mapping $T : X \to X$ is called a *weakly C-contraction* if,

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty), d(y, Tx)) \tag{5.2.1}$$

for all $x, y \in X$, where $\psi : [0, \infty)^2 \to [0, \infty)$ is a continuous function such that $\psi(a, b) = 0$ if and only if $a = b = 0$.

Later, Harjani et al. [59] presented some fixed point result for weakly C contraction mapping incomplete matric spaces endowed with partial order. Now, we extend the notion of weakly C-contraction to non-self mapping with partial order, which called generalized proximal C-contraction mapping as follow:

Definition 5.2.2. Let X be a nonempty set such that $(X, \preceq)$ is a partially ordered set and (X, d) be a metric space. Let A and B be nonempty subsets of X. A mapping $T : A \to B$ is said to be a *generalized proximal C-contraction* if satisfies

$$\left.\begin{array}{l} x \preceq y \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \implies d(u, v) \leq \frac{1}{2}(d(x, v) + d(y, u)) - \psi(d(x, v), d(y, u)) \tag{5.2.2}$$

for all $u, v, x, y \in A$, where $\psi : [0, \infty)^2 \to [0, \infty)$ is continuous function such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

For a self-mapping, it is easy to see that (5.2.2) reduces to (5.2.1).

Theorem 5.2.3. *Let X be a nonempty set such that $(X, \preceq)$ is a partially ordered set and (X, d) be a complete metric space. Let A and B be nonempty closed subsets of X such that A_0 and B_0 are nonempty. Let $T : A \to B$ satisfy the following conditions:*

(i) *T is a continuous, proximally order-preserving and generalized proximal C-contraction such that $T(A_0) \subseteq B_0$;*

(ii) *there exist element x_0 and x_1 in A_0 such that $x_0 \preceq x_1$ and*

$$d(x_1, Tx_0) = d(A, B).$$

Then there exists a point $x \in A$ and such that

$$d(x, Tx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(x_{n+1}, Tx_n) = d(A, B)$$

converges to the point x.

Proof. By the hypothesis (ii), there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and

$$d(x_1, Tx_0) = d(A, B).$$

Since $T(A_0) \subseteq B_0$, there exists a point $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B).$$

By the proximally order-preserving of T, we get $x_1 \preceq x_2$. Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that $x_{n-1} \preceq x_n$ and

$$d(x_n, Tx_{n-1}) = d(A, B).$$

Having found the point x_n, one can choose a point $x_{n+1} \in A_0$ such that $x_n \preceq x_{n+1}$ and

$$d(x_{n+1}, Tx_n) = d(A, B). \tag{5.2.3}$$

Since T is a generalized proximal C-contraction, for each $n \geq 1$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{1}{2}(d(x_{n-1}, x_{n+1}) + d(x_n, x_n)) - \psi(d(x_{n-1}, x_{n+1}), d(x_n, x_n)) \\ &= \frac{1}{2}d(x_{n-1}, x_{n+1}) - \psi(d(x_{n-1}, x_{n+1}), 0) \\ &\leq \frac{1}{2}d(x_{n-1}, x_{n+1}) \\ &\leq \frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \end{aligned} \tag{5.2.4}$$

and so it follow that $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$, that is, the sequence $\{d(x_{n+1}, x_n)\}$ is nonincreasing and bounded below. Then there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r. \tag{5.2.5}$$

Taking $n \to \infty$ in (5.2.4), we have

$$r \leq \lim_{n\to\infty} \frac{1}{2} d(x_{n-1}, x_{n+1}) \leq \frac{1}{2}(r + r) = r$$

and so

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 2r. \tag{5.2.6}$$

Again, taking $n \to \infty$ in (5.2.4), using (5.2.5), (5.2.6) and the continuity of ψ, we get

$$r \leq \frac{1}{2}(2r) = r - \psi(2r, 0) \leq r$$

and hence $\psi(2r, 0) = 0$. So, by the property of ψ, we have $r = 0$, which implies that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{5.2.7}$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequence $\{x_{m_k}\}, \{x_{n_k}\}$ of $\{x_n\}$ such that $n_k > m_k \geq k$ with

$$r_k := d(x_{m_k}, x_{n_k}) \geq \varepsilon, \quad d(x_{m_k}, x_{n_k-1}) < \varepsilon \tag{5.2.8}$$

for each $k \geq 1$. For each $n \geq 1$, let $\alpha_n := d(x_{n+1}, x_n)$. So, we have

$$\begin{aligned} \varepsilon \leq r_k &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \varepsilon + \alpha_{n_k-1}, \end{aligned}$$

it follows from (5.2.7) that

$$\lim_{k\to\infty} r_k = \varepsilon. \tag{5.2.9}$$

Notice also that

$$\begin{aligned} r_k &= d(x_{n_k}, x_{m_k}) \\ &\leq d(x_{n_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k}) \\ &= d(x_{n_k}, x_{m_k+1}) + \alpha_{m_k} \\ &\leq d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k+1}) + \alpha_{m_k} \\ &= r_k + \alpha_{m_k} + \alpha_{m_k}. \end{aligned} \tag{5.2.10}$$

Taking $k \to \infty$ in (5.2.10), by (5.2.7) and (5.2.9), we conclude that

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \varepsilon. \tag{5.2.11}$$

Similarly, we can show that

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}) = \varepsilon. \tag{5.2.12}$$

On the other hand, by the construction of $\{x_n\}$, we may assume that $x_{m_k} \preceq x_{n_k}$ such that

$$d(x_{n_k+1}, Tx_{n_k}) = d(A, B) \tag{5.2.13}$$

and

$$d(x_{m_k+1}, Tx_{m_k}) = d(A, B). \tag{5.2.14}$$

By the triangle inequality, (5.2.13), (5.2.14) and the generalized proximal C-contraction of T, we have

$$\begin{aligned}
\varepsilon \leq r_k &\leq d(x_{m_k}, x_{m_k+1}) + d(x_{n_k+1}, x_{n_k}) + d(x_{m_k+1}, x_{n_k+1}) \\
&= \alpha_{m_k} + \alpha_{n_k} + d(x_{m_k+1}, x_{n_k+1}) \\
&\leq \alpha_{m_k} + \alpha_{n_k} + \frac{1}{2}[d(x_{n_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k+1})] \\
&\quad -\psi(d(x_{n_k}, x_{m_k+1}), d(x_{m_k}, x_{n_k+1})).
\end{aligned}$$

Taking $k \to \infty$ in the above inequality, by (5.2.7), (5.2.11), (5.2.12) and the continuity of ψ, we get

$$\varepsilon \leq \frac{1}{2}(\varepsilon + \varepsilon) - \psi(\varepsilon, \varepsilon) \leq \varepsilon.$$

Therefore, $\psi(\varepsilon, \varepsilon) = 0$. By the property of ψ, we have that $\varepsilon = 0$, which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since A is a closed subset of the complete metric space X, there exist $x \in A$ such that

$$\lim_{n\to\infty} x_n = x. \tag{5.2.15}$$

Letting $n \to \infty$ in (5.2.3) , by (5.2.15) and the continuity of T, it follows that

$$d(x, Tx) = d(A, B).$$

This completes the proof. □

Corollary 5.2.4. *Let X be a nonempty set such that $(X, \preceq)$ is a partially ordered set and (X, d) be a complete metric space. Let A and B be nonempty closed subsets of X such that A_0 and B_0 are nonempty. Let $T : A \to B$ satisfy the following conditions:*

(i) *T is a continuous, increasing such that $T(A_0) \subseteq B_0$ and*

$$\left.\begin{aligned} &x \preceq y \\ &d(u, Tx) = d(A, B) \\ &d(v, Ty) = d(A, B) \end{aligned}\right\} \Longrightarrow d(u, v) \leq \alpha(d(x, v) + d(y, u)), \tag{5.2.16}$$

where $\alpha \in (0, \frac{1}{2})$;

(ii) *there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and*

$$d(x_1, Tx_0) = d(A, B).$$

Then there exists a point $x \in A$ and such that

$$d(x, Tx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(x_{n+1}, Tx_n) = d(A, B)$$

converges to the point x.

Proof. Let $\alpha \in (0, \frac{1}{2})$ and the function ψ in Theorem 5.2.3 be defined by

$$\psi(a, b) = (\frac{1}{2} - \alpha)(a + b).$$

Obviously, it follows that $\psi(a, b) = 0$ if and only if $a = b = 0$ and (5.2.2) become to (5.2.16). Hence we obtain the Corollary 5.2.4. □

For a self-mapping, the condition (ii) implies that $x_0 \preceq Tx_0$ and so Theorem 5.2.3 includes the results of Harjani et al. [59] as follows:

Corollary 5.2.5. [59] *Let X be a nonempty set such that $(X, \preceq)$ is a partially ordered set and (X, d) be a complete metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping such that, for all $x, y \in X$,*

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty), d(y, Tx))$$

for $x \preceq y$, where $\psi : [0, \infty)^2 \to [0, \infty)$ is a continuous function such that $\psi(x, y) = 0$ if and only if $x = y = 0$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

Now, we give an example to illustrate Theorem 5.2.3.

Example 5.2.6. Consider the complete metric space $\mathbb{R}^2$ with Euclidean metric, define the partial order $\preceq$ on $\mathbb{R}^2$ in the following way:

$$(x_1, y_1) \preceq (x_2, y_2) \Longleftrightarrow x_1 \leq x_2,\ y_1 \leq y_2$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Let

$$A = \{(x, 0) : x \in \mathbb{R}\}, \quad B = \{(0, y) : y \in \mathbb{R}\ , y \geq 1\}.$$

Then $d(A, B) = 1$, $A_0 = \{(0, 0)\}$ and $B_0 = \{(0, 1)\}$. Define a mapping $T : A \to B$ as follows:

$$T((x, 0)) = (0, 1 + |x|)$$

for all $(x, 0) \in A$. Clearly, T is continuous and $T(A_0) \subseteq B_0$. If $x_1 \preceq x_2$ and

$$d(u_1, Tx_1) = d(A, B) = 1, \quad d(u_2, Tx_2) = d(A, B) = 1$$

for some $u_1, u_2, x_1, x_2 \in A$, then we have

$$u_1 = u_2 = (0, 0), \quad x_1 = x_2 = (0, 0).$$

Therefore, T is a generalized proximal C-contraction with $\psi : [0, \infty)^2 \to [0, \infty)$ defined by

$$\psi(a, b) = \frac{1}{4}(a + b).$$

Further, observe that $(0, 0) \in A$ such that

$$d((0, 0), T(0, 0)) = d(A, B) = 1.$$

In Theorem 5.2.7, we don't need the condition that T is continuous. Now, we improve the condition in Theorem 5.2.3 to prove the new best proximity point theorem as follows:

Theorem 5.2.7. *Let X be a nonempty set such that $(X, \preceq)$ is a partially ordered set and (X, d) be a complete metric space. Let A and B be nonempty closed subsets of X such that A_0 and B_0 are nonempty. Let $T : A \to B$ satisfy the following conditions:*

(i) *T is a proximally order-preserving and generalized proximal C-contraction such that $T(A_0) \subseteq B_0$;*

(ii) *there exist element $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and*

$$d(x_1, Tx_0) = d(A, B);$$

(iii) *if $\{x_n\}$ is an increasing sequence in A converges to x, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then there exists a point $x \in A$ and such that

$$d(x, Tx) = d(A, B).$$

Proof. As in the proof of Theorem 5.2.3, we have

$$d(x_{n+1}, Tx_n) = d(A, B) \tag{5.2.17}$$

for all $n \geq 0$. Moreover, $\{x_n\}$ is a Cauchy sequence and so it converges to some point $x \in A$. Observe that, for each $n \in\geq 1$,

$$\begin{aligned} d(A, B) = d(x_{n+1}, Tx_n) &\leq d(x_{n+1}, x) + d(x, Tx_n) \\ &\leq d(x, x_{n+1}) + d(x, x_{n+1}) + d(x_{n+1}, Tx_n) \\ &\leq d(x, x_{n+1}) + d(x, x_{n+1}) + d(A, B). \end{aligned}$$

Taking $n \to \infty$ in the above inequality, we obtain $\lim_{n\to\infty} d(x, Tx_n) = d(A, B)$ and hence $x \in A_0$. Since $T(A_0) \subseteq B_0$, there exists $v \in A$ such that

$$d(v, Tx) = d(A, B). \tag{5.2.18}$$

Next, we prove that $x = v$. By the condition (c), we have $x_n \preceq x$ for all $n \geq 1$. Using (5.2.17), (5.2.18) and the generalized proximal C-contraction of T, we have

$$d(x_{n+1}, v) \leq \frac{1}{2}[d(x_n, v) + d(x, x_{n+1})] - \psi(d(x_n, v), d(x, x_{n+1})). \tag{5.2.19}$$

Letting $n \to \infty$ in (5.2.19), we get

$$d(x, v) \leq \frac{1}{2}d(x, v) - \psi(d(x, v), 0),$$

which implies that $d(x, v) = 0$, that is, $x = v$. If we replace v by x in (5.2.18), we have

$$d(x, Tx) = d(A, B).$$

This completes the proof. $\square$

Corollary 5.2.8. *Let X be a nonempty set such that $(X, \preceq)$ is a partially ordered set and (X, d) be a complete metric space. Let A and B be nonempty closed subsets of X such that A_0 and B_0 are nonempty. Let $T : A \to B$ satisfy the following conditions:*

(i) *T is an increasing mapping such that $T(A_0) \subseteq B_0$ and*

$$\left.\begin{array}{l} x \preceq y \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \implies d(u, v) \leq \alpha(d(x, v) + d(y, u)), \tag{5.2.20}$$

where $\alpha \in (0, \frac{1}{2})$;

(ii) *there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and*

$$d(x_1, Tx_0) = d(A, B);$$

(iii) *if $\{x_n\}$ is an increasing sequence in A converges to a point $x \in X$, then $x_n \preceq x$ for all $n \geq 1$.*

Then there exists a point $x \in A$ and such that

$$d(x, Tx) = d(A, B).$$

Corollary 5.2.9. [59] *Let X be a nonempty set such that $(X, \preceq)$ is a partially ordered set and (X, d) be a complete metric space. Assume that, if $\{x_n\} \subseteq X$ is a nondecreasing sequence such that $x_n \to x$ in X, then $x_n \preceq x$ for all $n \geq 1$. Let $T : X \to X$ be a nondecreasing mapping such that*

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty), d(y, Tx))$$

for $x \preceq y$, where $\psi : [0,\infty)^2 \to [0,\infty)$ is a continuous function such that $\psi(x,y) = 0$ if and only if $x = y = 0$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

Now, we recall the condition defined by Nieto and Rodríguez-López [60] for the uniqueness of the best proximity point in Theorems 5.2.3 and Theorems 5.2.7.

$$\text{For all } x, y \in X, \quad \text{there exists } z \in X \quad \text{which is comparable to } x \quad \text{and } y. \tag{5.2.21}$$

Theorem 5.2.10. *Let X be a nonempty set such that $(X, \preceq)$ is a partially ordered set and (X, d) be a complete metric space. Let A and B be nonempty closed subsets of X and A_0 and B_0 are nonempty such that A_0 satisfies the condition (5.2.21). Let $T : A \to B$ satisfy the following conditions:*

(i) *T is a continuous, proximally order-preserving and generalized proximal C-contraction such that $T(A_0) \subseteq B_0$;*

(ii) *there exist element $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and*

$$d(x_1, Tx_0) = d(A, B).$$

Then there exists a unique point $x \in A$ and such that

$$d(x, Tx) = d(A, B).$$

Proof. We shall only proof the part of uniqueness part. Suppose that there exist x and x^* in A which are best proximity point, that is

$$d(x, Tx) = d(A, B), \quad d(x^*, Tx^*) = d(A, B).$$

Case I: x is comparable to x^*, that is, $x \preceq x^*$ (or $x^* \preceq x$). By the generalized proximal C-contraction of T, we have

$$d(x, x^*) \ \leq \frac{1}{2}[d(x, x^*) + d(x^*, x)] - \psi(d(x, x^*), d(x^*, x)) \ \leq \ d(x^*, x),$$

which implies that $\psi(d(x, x^*), d(x^*, x)) = 0$. Using the property of ψ, we get $d(x^*, x) = 0$ and hence $x = x^*$.

Case II: x is not comparable to x^*. Since A_0 satisfies the condition (5.2.21), there exist $z \in A_0$ such that z comparable to x and x^*, that is, $x \preceq z$ (or $z \preceq x$) and $x^* \preceq z$ (or $z \preceq x^*$). Suppose that $x \preceq z$ and $x^* \preceq z$. Since $T(A_0) \subseteq B_0$, there exists a point $v_0 \in A_0$ such that

$$d(v_0, Tz) = d(A, B).$$

By proximally order-preserving, we get $x \preceq v_0$ and $x^* \preceq v_0$. Since $T(A_0) \subseteq B_0$, there exists a point $v_1 \in A_0$ such that

$$d(v_1, Tv_0) = d(A, B).$$

Again, by proximally order-preserving, we get $x \preceq v_1$ and $x^* \preceq v_1$. One can proceed further in a similar fashion to find $v_n \in A_0$ with $v_{n+1} \in A_0$ such that

$$d(v_{n+1}, Tv_n) = d(A, B).$$

Hence $x \preceq v_n$ and $x^* \preceq v_n$ for all $n \geq 1$. By the generalized proximal C-contraction of T, we have

$$d(v_{n+1}, x) \leq \frac{1}{2}[d(v_n, x) + d(x, v_{n+1})] - \psi(d(v_n, x), d(x, v_{n+1})), \tag{5.2.22}$$

$$d(v_{n+1}, x^*) \leq \frac{1}{2}[d(v_n, x^*) + d(x^*, v_{n+1})] - \psi(d(v_n, x^*), d(x^*, v_{n+1})). \tag{5.2.23}$$

It follow from (5.2.22), we get $d(v_{n+1}, x) \leq d(v_n, x)$. This mean that the sequence $\{d(v_n, x)\}$ is nonincreasing and converges to some nonnegative real number r. Letting $n \to \infty$ in (5.2.22), we have

$$r \leq \frac{1}{2}(r + r) - \psi(r, r) \leq r, \tag{5.2.24}$$

which implies that $\psi(r, r) = 0$, that is $r = 0$ and thus $\lim_{n\to\infty} d(v_n, x) = 0$. Therefore, $v_n \to x$ as $n \to \infty$. Similarly, we can show that $v_n \to x^*$ as $n \to \infty$. By the uniqueness of limit, we conclude that $x = x^*$. This completes the proof. □

Theorem 5.2.11. *Let X be a nonempty set such that $(X, \preceq)$ is a partially ordered set and (X, d) be a complete metric space. Let A and B be nonempty closed subsets of X and A_0 and B_0 are nonempty such that A_0 satisfies the condition (5.2.21). Let $T : A \to B$ satisfy the following conditions:*

(i) *T is an proximally order-preserving and generalized proximal C-contraction such that $T(A_0) \subseteq B_0$;*

(ii) *there exist element $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$;*

(iii) *if $\{x_n\}$ is an increasing sequence in A converges to x, then $x_n \preceq x$ for all $n \geq 1$.*

Then there exists a unique point $x \in A$ and such that

$$d(x, Tx) = d(A, B).$$

Proof. Combining the proofs of Theorem 5.2.7 and Theorem 5.2.11, we have the conclusion.

□

5.3 Best proximity point theorems for generalized cyclic contractions mappings

In this section, we prove the existence of a best proximity point for a generalized cyclic contraction mapping. First, we recall the notion and result in [61] as follows:

Definition 5.3.1. [61] Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) is said to be satisfy the *property* (UC) if and only if $\{x_n\}$ and $\{\acute{x}_n\}$ are the sequences in A and $\{y_n\}$ is a sequence in B such that

$$\lim_{n\to\infty} d(x_n, y_n) = d(A, B) \text{ and } \lim_{n\to\infty} d(\acute{x}_n, y_n) = d(A, B),$$

then $\lim_{n\to\infty} d(x_n, \acute{x}_n) = 0$.

Lemma 5.3.2. [61] *Let A and B be subsets of a metric space (X,d). Assume that (A, B) has the property (UC). Let $\{x_n\}$ and $\{y_n\}$ be the sequences in A and B, respectively, such that either of the following holds:*

$$\lim_{m\to\infty} \sup_{n\geq m} d(x_m, y_n) = d(A, B)$$

or

$$\lim_{n\to\infty} \sup_{m\geq n} d(x_m, y_n) = d(A, B).$$

Then $\{x_n\}$ is a Cauchy sequence.

Theorem 5.3.3. *Let A and B be nonempty closed subsets of a partially ordered metric space $(X, \preceq)$ and d be a metric on X. Let $T : A \cup B \to A \cup B$ be a cyclic mapping such that T and T^2 are nondecreasing on A such that*

$$d(T\acute{x}, T^2x) \leq \alpha d(\acute{x}, Tx) + \beta d(x, T\acute{x}) + (1 - \alpha - \beta)d(A, B)$$

and

$$d(T\acute{y}, T^2y) \leq \alpha d(\acute{y}, Ty) + \beta d(y, T\acute{y}) + (1 - \alpha - \beta)d(A, B)$$

for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and for all $(x, \acute{x}) \in A \times A$, $(y, \acute{y}) \in B \times B$ with $x \preceq \acute{x}$, $y \preceq \acute{y}$. Assume that there exits $x_0 \in A$ with $x_0 \preceq T^2x_0$ and define $x_{n+1} = Tx_n$ for all $n \geq 1$. If $T|_A$ is continuous and $\{x_{2n}\}$ has convergent subsequence in A, then T has best proximity point $p \in A$.

Proof. Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ converging to some $p \in A$. By the continuity of T, we get $x_{2n_k+1} = Tx_{2n_k} \to Tp$ as $n \to \infty$. Since T and T^2 are nondecreasing on A and $x_0 \preceq T^2x_0$, it follows that $\{T^{2n}x_0\}$ and $\{T^{2n-1}x_0\}$ are nondecreasing. Indeed,

$$\begin{aligned}
d(A, B) &\leq d(x_{2n_k}, x_{2n_k+1}) \\
&\leq \alpha d(Tx_{2n_k-2}, T^2x_{2n_k-2}) + \beta d(Tx_{2n_k-2}, T^2x_{2n_k-2}) + (1 - \alpha - \beta)d(A, B) \\
&\leq \alpha^2 d(x_{2n_k-2}, Tx_{2n_k-2}) + 2\alpha\beta d(x_{2n_k-2}, Tx_{2n_k-2}) \\
&\quad +\beta^2 d(x_{2n_k-2}, Tx_{2n_k-2}) + (1 - \alpha^2 - 2\alpha\beta - \beta^2)d(A, B) \\
&\leq \alpha^3 d(x_{2n_k-3}, Tx_{2n_k-3}) + 3\alpha^2\beta d(x_{2n_k-3}, Tx_{2n_k-3}) \\
&\quad +3\alpha\beta^2 d(x_{2n_k-3}, Tx_{2n_k-3}) + \beta^3 d(x_{2n_k-3}, Tx_{2n_k-3}) \\
&\quad +(1 - \alpha^3 - 3\alpha^2\beta - 3\alpha\beta^2 - \beta^3)d(A, B) \\
&\vdots \\
&\leq \alpha^{2n_k} d(x_0, Tx_0) + \binom{2n_k}{1}\alpha^{2n_k-1}\beta d(x_0, Tx_0) + \cdots + \beta^{2n_k} d(x_0, Tx_0) \\
&\quad (1 - \alpha^{2n_k} - \binom{2n_k}{1}\alpha^{2n_k-1}\beta - \cdots - \binom{2n_k}{2n_k - 1}\alpha\beta^{2n_k-1} - \beta^{2n_k})d(A, B).
\end{aligned}$$

Taking $k \to \infty$ in the above equality, we obtain

$$d(p, Tp) = \lim_{n\to\infty} d(x_{2n_k}, Tx_{2n_k}) = d(A, B).$$

$\square$

Corollary 5.3.4. *Let A and B be nonempty closed subsets of a partially ordered metric space $(X, \preceq)$ and d be a metric on X. Let $T : A \cup B \to A \cup B$ be a cyclic mapping such that T and T^2 are nondecreasing on A such that*

$$d(T\acute{x}, T^2x) \leq \alpha d(\acute{x}, Tx) + (1-\alpha)d(A, B)$$

and

$$d(T\acute{y}, T^2y) \leq \alpha d(\acute{y}, Ty) + (1-\alpha)d(A, B)$$

for some $\alpha \in [0, 1)$ and for all $(x, \acute{x}) \in A \times A$, $(y, \acute{y}) \in B \times B$ with $x \preceq \acute{x}$, $y \preceq \acute{y}$. Assume that there exits $x_0 \in A$ with $x_0 \preceq T^2x_0$ and define $x_{n+1} = Tx_n$ for all $n \geq 1$. If $T|_A$ is continuous and $\{x_{2n}\}$ has convergent subsequence in A, then T has best proximity point $p \in A$.

Corollary 5.3.5. *Let A and B be nonempty closed subsets of a partially ordered metric space $(X, \preceq)$ and d be a metric on X. Let $T : A \cup B \to A \cup B$ be a cyclic mapping such that T and T^2 are nondecreasing on A such that*

$$d(T\acute{x}, T^2x) \leq \beta d(x, T\acute{x}) + (1-\beta)d(A, B)$$

and

$$d(T\acute{y}, T^2y) \leq \beta d(y, T\acute{y}) + (1-\beta)d(A, B)$$

for some $\beta \in [0, 1)$ and for all $(x, \acute{x}) \in A \times A$, $(y, \acute{y}) \in B \times B$ with $x \preceq \acute{x}$, $y \preceq \acute{y}$. Assume that there exits $x_0 \in A$ with $x_0 \preceq T^2x_0$ and define $x_{n+1} = Tx_n$ for all $n \geq 1$. If $T|_A$ is continuous and $\{x_{2n}\}$ has convergent subsequence in A, then T has best proximity point $p \in A$.

Lemma 5.3.6. *Let A and B be nonempty subsets of a partially ordered metric space $(X, \preceq)$ and d be a metric on X. Let $T : A \cup B \to A \cup B$ be a cyclic mapping such that*

$$d(T\acute{x}, T^2x) \leq \alpha d(\acute{x}, Tx) + \beta d(x, T\acute{x}) + (1-\alpha-\beta)d(A, B) \tag{5.3.1}$$

and

$$d(T\acute{y}, T^2y) \leq \alpha d(\acute{y}, Ty) + \beta d(y, T\acute{y}) + (1-\alpha-\beta)d(A, B) \tag{5.3.2}$$

for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and for all $(x, \acute{x}) \in A \times A$, $(y, \acute{y}) \in B \times B$ with $x \preceq \acute{x}$, $y \preceq \acute{y}$. Then

$$d^*(T\acute{x}, T^2x) \leq \alpha d^*(\acute{x}, Tx) + \beta d^*(x, T\acute{x})$$

and

$$d^*(T\acute{y}, T^2y) \leq \alpha d^*(\acute{y}, Ty) + \beta d^*(y, T\acute{y}),$$

where $d^*(a,b) = d(a,b) - d(A,B)$ *for* $(a,b) \in A \times B$.

Proof. By the definition of d^* and (5.3.1), we have

$$\begin{aligned} d^*(T\acute{x}, T^2x) &= d(T\acute{x}, T^2x) - d(A,B) \\ &\leq \alpha d(\acute{x}, Tx) + \beta d(x, T\acute{x}) + (1-\alpha-\beta)d(A,B) - d(A,B) \\ &= \alpha d(\acute{x}, Tx) + \beta d(x, T\acute{x}) - \alpha d(A,B) - \beta d(A,B) \\ &= \alpha\big(d(\acute{x}, Tx) - d(A,B)\big) + \beta\big(d(x, T\acute{x}) - d(A,B)\big) \\ &= \alpha d^*(\acute{x}, Tx) + \beta d^*(x, T\acute{x}). \end{aligned} \tag{5.3.3}$$

Similarly, we see that $d^*(T\acute{y}, T^2y) \leq \alpha d^*(\acute{y}, Ty) + \beta d^*(y, T\acute{y})$. □

Theorem 5.3.7. *Let* $(X, \preceq)$ *be a partially ordered set and* d *be a metric on* X. *Let* A *and* B *be two nonempty subsets of* X *such that* (A,B) *satisfies the property* (UC) *and* A *is complete. Let* $T : A \cup B \to A \cup B$ *be a cyclic mapping such that* T *and* T^2 *are nondecreasing on* A. *Suppose that*

$$d(T\acute{x}, T^2x) \leq \alpha d(\acute{x}, Tx) + \beta d(x, T\acute{x}) + (1-\alpha-\beta)d(A,B)$$

and

$$d(T\acute{y}, T^2y) \leq \alpha d(\acute{y}, Ty) + \beta d(y, T\acute{y}) + (1-\alpha-\beta)d(A,B)$$

for some $\alpha, \beta \in [0,1)$ *with* $\alpha + \beta < 1$ *and for all* $(x, \acute{x}) \in A \times A$, $(y, \acute{y}) \in B \times B$ *with* $x \preceq \acute{x}$, $y \preceq \acute{y}$. *If* $T|_A$ *is continuous and that there exits* $x_0 \in A$ *such that* $x_0 \preceq T^2x_0$ *and* $x_{n+1} = Tx_n$ *for all* $n \geq 1$, *then* T *has a best proximity point* $p \in A$ *and* $x_{2n} \to p$.

Proof. Since T and T^2 are nondecreasing on A and $x_0 \preceq T^2x_0$, it follows that $\{T^{2n}x_0\}$ and $\{T^{2n-1}x_0\}$ are nondecreasing. Let $n \geq 1$ with $n \geq m$.

By Lemma 5.3.6, we have

$$
\begin{aligned}
& d^*(T^{2m}x_0, T^{2n+1}x_0) \\
&= d^*(T(T^{2n}x_0), T^2(T^{2m-2}x_0)) \\
&\le \alpha d^*(T(T^{2n-1}x_0), T^2(T^{2m-3}x_0)) + \beta d^*(T(T^{2n}x_0), T^2(T^{2m-4}x_0)) \\
&\le \alpha^2 d^*(T(T^{2n-2}x_0), T^2(T^{2m-4}x_0)) \\
&\quad +2\alpha\beta d^*(T(T^{2n-1}x_0), T^2(T^{2m-5}x_0)) + \beta^2 d^*(T(T^{2n}x_0), T^2(T^{2m-6}x_0)) \\
&\le \alpha^3 d^*(T^{2m-3}x_0, T^{2n-2}x_0) + 3\alpha^2\beta d^*(T^{2m-4}x_0, T^{2n-1}x_0) \\
&\quad +3\alpha\beta^2 d^*(T^{2m-5}x_0, T^{2n}x_0) + \beta^3 d^*(T^{2m-6}x_0, T^{2n+1}x_0) \\
&\vdots \\
&\le \beta^m d^*(x_0, T^{2n+1}x_0) + \binom{m}{1}\alpha\beta^{m-1} d^*(Tx_0, T^{2n}x_0) \\
&\quad +\binom{m}{2}\alpha^2\beta^{m-2} d^*(T^2x_0, T^{2n-2}x_0) + \cdots \\
&\quad +\alpha^m d^*(T^{2m-m}x_0, T^{(2n+1)-(m)}x_0).
\end{aligned}
$$

Since $\alpha, \beta \in [0,1)$, it follows from the above inequality that

$$\lim_{m\to\infty} \sup_{n\ge m} d^*(T^{2m}x_0, T^{2n+1}x_0) = 0. \tag{5.3.4}$$

Since (A, B) satisfies the property (UC), it follows from Lemma 5.3.2 that $\{x_{2n}\}$ is a Cauchy sequence and since A is complete, there exists $p \in A$ such that

$$T^{2n}x_0 = x_{2n} \to p.$$

By the continuity of T on A, we get $T^{2n+1}x_0 = T(T^{2n}x_0) \to Tp$ as $n \to \infty$. Since $\{T^{2n}x_0\}$ and $\{T^{2n-1}x_0\}$ are nondecreasing, we have

$$
\begin{aligned}
& d(A, B) \\
\le\ & d(T(T^{2n-1}x_0), T^2(T^{2n-1}x_0)) \\
\le\ & \alpha d(T^{2n-1}x_0, T^{2n}x_0) + \beta d(T^{2n-1}x_0, T^{2n}x_0) + (1-\alpha-\beta)d(A, B) \\
\le\ & \alpha^2 d(T^{2n-2}x_0, T(T^{2n-2}x_0)) + 2\alpha\beta d(T^{2n-2}x_0, T(T^{2n-2}x_0)) \\
& +\beta^2 d(T^{2n-2}x_0, T(T^{2n-2}x_0)) + (1-\alpha^2-2\alpha\beta-\beta^2)d(A, B)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \alpha^3 d(T^{2n-3}x_0, T(T^{2n-3}x_0)) + 3\alpha^2\beta d(T^{2n-3}x_0, T(T^{2n-3}x_0)) \\
&\quad +3\alpha\beta^2 d(T^{2n-3}x_0, T(T^{2n-3}x_0)) + \beta^3 d(T^{2n-3}x_0, T(T^{2n-3}x_0)) \\
&\quad +(1-\alpha^3-3\alpha^2\beta-3\alpha\beta^2-\beta^3)d(A,B) \\
&\vdots \\
&\leq \alpha^{2n} d(x_0, Tx_0) + \binom{2n}{1}\alpha^{2n-1}\beta d(x_0, Tx_0) + \cdots \\
&\quad +\binom{2n}{2n-1}\alpha\beta^{2n-1}d(x_0, Tx_0) + \beta^{2n}d(x_0, Tx_0) \\
&\quad +\Big(1-\alpha^{2n}-\binom{2n}{1}\alpha^{2n-1}\beta-\cdots-\binom{2n}{2n-1}\alpha\beta^{2n-1}-\beta^{2n}\Big)d(A,B).
\end{aligned}
$$

Since $\alpha, \beta \in [0,1)$, letting $n \to \infty$ in the above inequality, we obtain

$$d(p, Tp) = \lim_{n\to\infty} d((T^{2n}x_0, T^{2n+1}x_0) = d(A,B).$$

This completes the proof. $\square$

Corollary 5.3.8. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. Let A and B be two nonempty subsets of X satisfies the property (UC), and A is complete. Let $T : A \cup B \to A \cup B$ be a cyclic mapping such that T and T^2 are nondecreasing on A. Suppose that*

$$d(T\acute{x}, T^2x) \leq \alpha d(\acute{x}, Tx) + (1-\alpha)d(A,B)$$

and

$$d(T\acute{y}, T^2y) \leq \alpha d(\acute{y}, Ty) + (1-\alpha)d(A,B)$$

for some $\alpha \in [0,1)$ and for all $(x, \acute{x}) \in A \times A$, $(y, \acute{y}) \in B \times B$ with $x \preceq \acute{x}$, $y \preceq \acute{y}$. If $T|_A$ is continuous and that there exits $x_0 \in A$ such that $x_0 \preceq T^2x_0$ and $x_{n+1} = Tx_n$ for all $n \geq 1$, then T has best proximity point $p \in A$ and $x_{2n} \to p$.

Corollary 5.3.9. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. Let A and B be two nonempty subsets of X such that (A, B) satisfies the property (UC), and A is complete. Let $T : A \cup B \to A \cup B$ be a cyclic mapping such that T and T^2 are nondecreasing on A. Suppose that*

$$d(T\acute{x}, T^2x) \leq \beta d(x, T\acute{x}) + (1-\beta)d(A,B)$$

and

$$d(T\acute{y}, T^2y) \leq \beta d(y, T\acute{y}) + (1-\beta)d(A,B)$$

for some $\beta \in [0,1)$ *and for all* $(x,\acute{x}) \in A \times A$, $(y,\acute{y}) \in B \times B$ *with* $x \preceq \acute{x}$, $y \preceq \acute{y}$. *If* T *is continuous and that there exits* $x_0 \in A$ *such that* $x_0 \preceq T^2x_0$ *and* $x_{n+1} = Tx_n$ *for all* $n \geq 1$, *then* T *has best proximity point* $p \in A$ *and* $x_{2n} \to p$.

Now, we give an example to illustrate the Theorems 5.3.7.

Example 4.1 Consider $X = \mathbb{R}^2$ with the usual metric and define the partial order $\preceq$ on $\mathbb{R}^2$ in the following way:

$$(x_1, y_1) \preceq (x_2, y_2) \Longleftrightarrow x_1 \leq x_2,\ y_1 \leq y_2$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Set $A = \{(1, a) : a \geq 0\}$, $B = \{(-1, b) : b \geq 0\}$ and define a mapping $T : A \cup B \to A \cup B$ by

$$T(1, a) = (-1, \tfrac{a}{2}),\ T(-1, b) = (1, \tfrac{b}{2})$$

for all $a, b \geq 0$. Then $d(A, B) = 2$. We show that A and B satisfies the property (UC). Let $\{(1, a_n)\}, \{(1, a'_n)\}$ be two sequences in A and $\{(1, b_n)\}$ be a sequence in B such that

$$\lim_{n\to\infty} d((1, a_n), (-1, b_n)) = 2,\ \lim_{n\to\infty} d((1, a'_n), (-1, b_n)) = 2.$$

Thus $\lim_{n\to\infty} |a_n - b_n| = 0$ and $\lim_{n\to\infty} |a'_n - b_n| = 0$. Since

$$\lim_{n\to\infty} |a_n - a'_n| \ \leq \lim_{n\to\infty} |a_n - b_n| + \lim_{n\to\infty} |b_n - a'_n| \ = 0,$$

$$\lim_{n\to\infty} d((1, a_n), (1, a'_n)) = 0 \Longleftrightarrow \lim_{n\to\infty} |a_n - a'_n| = 0.$$

Hence A and B satisfies the property (UC). Simple computations show that T satisfies the conditions of Theorem 5.3.7 for $\alpha = 2/3$, $\beta = 1/4$. Since $x_0 := (1, 0) \in A$, if define $x_{n+1} = Tx_n$ for all $n \geq 1$, then $x_0 \preceq T^2x_0$ and $\{x_{2n}\}$, $\{x_{2n-1}\}$ are nondecreasing. Therefore, T has a best proximity point. Clearly, this point is x_0 itself. $\square$

5.4 Common best proximity points for proximity commuting mappings

In this section, we prove new common best proximity point theorems for a proximity commuting mapping in a complete metric space. Moreover, we also give an illustrative example for support our main Theorem.

Theorem 5.4.1. *Let A and B be nonempty closed subsets of a complete metric space X such that A is approximatively compact with respect to B. Also, assume that A_0 and B_0 are nonempty. Let $S : A \to B$, $T : A \to B$ be nonself-mappings satisfying the following conditions:*

(i) *For each x and y are elements in A,*

$$d(Sx, Sy) \leq d(Tx, Ty) - \varphi(d(Tx, Ty)),$$

where, $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$;

(ii) *T is continuous;*

(iii) *S and T commute proximally;*

(iv) *S and T can be swapped proximally;*

(v) *$S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$.*

Then there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B),\ d(x, Sx) = d(A, B).$$

Moreover, if x^* is another common best proximity point of the mappings S and T, then

$$d(x, x^*) \leq 2d(A, B).$$

Proof. Let x_0 a fixed element in A_0. In view of the fact that $S(A_0) \subseteq T(A_0)$, it follows that there exists an element $x_1 \in A_0$ such that $Sx_0 = Tx_1$. Again, since

$S(A_0) \subseteq T(A_0)$, there exists an element $x_2 \in A_0$ such that $Sx_1 = Tx_2$. By the similar fashion, we can find $x_n \in A_0$ such that

$$Sx_{n-1} = Tx_n \tag{5.4.1}$$

for all $n \geq 1$. It follows that

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &\leq d(Tx_n, Tx_{n+1}) - \varphi(d(Tx_n, Tx_{n+1})) \\ &= d(Sx_{n-1}, Sx_n) - \varphi(d(Sx_{n-1}, Sx_n)) \\ &\leq d(Sx_{n-1}, Sx_n), \end{aligned} \tag{5.4.2}$$

which mean that the sequence $\{d(Sx_{n-1}, Sx_n)\}$ is non-increasing and bounded below. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(Sx_{n-1}, Sx_n) = r. \tag{5.4.3}$$

If $r > 0$, then

$$d(Sx_n, Sx_{n+1}) \leq d(Sx_{n-1}, Sx_n) - \varphi(d(Sx_{n-1}, Sx_n)). \tag{5.4.4}$$

Taking $n \to \infty$ in (5.4.4), by the continuities of φ, we get $r \leq r - \varphi(r) < r$, which is a contradiction and hence $r = 0$. Therefore,

$$\lim_{n\to\infty} d(Sx_{n-1}, Sx_n) = 0. \tag{5.4.5}$$

Next, we will prove that $\{Sx_n\}$ is a Cauchy sequence. We have two cases.

Case I: Suppose that there exits a positive integer n such that $Sx_n = Sx_{n+1}$. Observe that

$$\begin{aligned} d(Sx_{n+1}, Sx_{n+2}) &\leq d(Tx_{n+1}, Tx_{n+2}) - \varphi(d(Tx_{n+1}, Tx_{n+2})) \\ &= d(Sx_n, Sx_{n+1}) - \varphi(d(Sx_n, Sx_{n+1})) \\ &= 0, \end{aligned}$$

which implies that $Sx_{n+1} = Sx_{n+2}$. So, for all $m > n$, we conclude that $Sx_m = Sx_n$. Hence $\{Sx_n\}$ is a Cauchy sequence in B.

Case II: The successive terms of $\{Sx_n\}$ are different. Suppose that $\{Sx_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequences $\{Sx_{m_k}\}, \{Sx_{n_k}\}$ of $\{Sx_n\}$ with $n_k > m_k \geq k$ such that

$$d(Sx_{m_k}, Sx_{n_k}) \geq \varepsilon, \quad d(Sx_{m_k}, Sx_{n_k - 1}) < \varepsilon. \tag{5.4.6}$$

By using (5.4.6) and the triangular inequality, we get

$$\begin{aligned} \varepsilon &\leq d(Sx_{m_k}, Sx_{n_k}) \\ &\leq d(Sx_{m_k}, Sx_{n_k-1}) + d(Sx_{n_k-1}, Sx_{n_k}) \\ &< \varepsilon + d(Sx_{n_k-1}, Sx_{n_k}). \end{aligned} \tag{5.4.7}$$

Using (5.4.7) and (5.4.5), we have

$$d(Sx_{m_k}, Sx_{n_k}) \to \varepsilon \tag{5.4.8}$$

as $k \to \infty$. Again, by the triangular inequality, we get

$$\begin{aligned} & d(Sx_{m_k}, Sx_{n_k}) \\ \leq\ & d(Sx_{m_k}, Sx_{m_k+1}) + d(Sx_{m_k+1}, Sx_{n_k+1}) + d(Sx_{n_k+1}, Sx_{n_k}) \end{aligned} \tag{5.4.9}$$

and

$$\begin{aligned} & d(Sx_{m_k+1}, Sx_{n_k+1}) \\ \leq\ & d(Sx_{m_k+1}, Sx_{m_k}) + d(Sx_{m_k}, Sx_{n_k}) + d(Sx_{n_k}, Sx_{n_k+1}). \end{aligned} \tag{5.4.10}$$

From (5.4.5), (5.4.8), (5.4.9) and (5.4.10), we obtain

$$d(Sx_{m_k+1}, Sx_{n_k+1}) \to \varepsilon \tag{5.4.11}$$

as $k \to \infty$. In view of the fact that

$$\begin{aligned} d(Sx_{m_k+1}, Sx_{n_k+1}) &\leq d(Tx_{m_k+1}, Tx_{n_k+1}) - \varphi(d(Tx_{m_k+1}, Tx_{n_k+1})) \\ &= d(Sx_{m_k}, Sx_{n_k}) - \varphi(d(Sx_{m_k}, Sx_{n_k})), \end{aligned} \tag{5.4.12}$$

letting, $k \to \infty$ in (5.4.12), we obtain

$$\varepsilon \leq \varepsilon - \varphi(\varepsilon),$$

which is a contradiction by the property of φ. Then we deduce that $\{Sx_n\}$ is a Cauchy sequence in B. Since B is a closed subset of a complete metric space X, then there exists $y \in B$ such that $Sx_n \to y$ as $n \to \infty$. Consequently, it follows that the sequence $\{Tx_n\}$ also converges to y. From $S(A_0) \subseteq B_0$, there exists an element $u_n \in A$ such that

$$d(Sx_n, u_n) = d(A, B) \tag{5.4.13}$$

for all $n \geq 1$. So, it follows from (5.4.1) and (5.4.13) that

$$d(Tx_n, u_{n-1}) = d(Sx_{n-1}, u_{n-1}) = d(A, B) \tag{5.4.14}$$

for all $n \geq 1$. By (5.4.13), (5.4.14) and the fact that the mappings S and T are commuting proximally, we obtain

$$Tu_n = Su_{n-1} \tag{5.4.15}$$

for all $n \geq 1$. Moreover, we have

$$\begin{aligned} d(y, A) &\leq d(y, u_n) \\ &\leq d(y, Sx_n) + d(Sx_n, u_n) \\ &= d(y, Sx_n) + d(A, B) \\ &\leq d(y, Sx_n) + d(y, A). \end{aligned} \tag{5.4.16}$$

Therefore, $d(y, u_n) \to d(y, A)$ as $n \to \infty$. Since A is approximatively compact with respect to B, there exists a subsequence $\{u_{n_k}\}$ of the sequence $\{u_n\}$ such that $\{u_{n_k}\}$ converges to some element $u \in A$. Further, since $d(y, u_{n_k-1}) \to d(y, A)$ and A is approximatively compact with respect to B, there exists a subsequence $\{u_{n_{k_j}-1}\}$ of the sequence $\{u_{n_k-1}\}$ such that $\{u_{n_{k_j}-1}\}$ converges to some element $v \in A$. By the continuity of the mappings S and T, we have

$$Tu = \lim_{j\to\infty} Tu_{n_{k_j}} = \lim_{j\to\infty} Su_{n_{k_j}-1} = Sv \tag{5.4.17}$$

and

$$\begin{aligned} d(y, u) &= \lim_{k\to\infty} d(Sx_{n_k}, u_{n_k}) = d(A, B), \\ d(y, v) &= \lim_{j\to\infty} d(Tx_{n_{k_j}}, u_{n_{k_j}-1}) = d(A, B). \end{aligned} \tag{5.4.18}$$

Since S and T can be swapped proximally, we get

$$Tv = Su. \tag{5.4.19}$$

Next, we prove that $Su = Sv$. Suppose the contrary. Then, by (5.4.17), (5.4.18), (5.4.19) and the property of φ, we have

$$\begin{aligned} d(Su, Sv) &\leq d(Tu, Tv) - \varphi(d(Tu, Tv)) \\ &= d(Sv, Su) - \varphi(d(Sv, Su)) \\ &< d(Sv, Su), \end{aligned}$$

which is a contradiction. Thus $Su = Sv$ and also $Tu = Su$. Since $S(A_0)$ is contained in B_0, there exists an element $x \in A$ such that

$$d(x, Tu) = d(A, B),\ d(x, Su) = d(A, B).$$

Since S and T are commuting proximally, we have $Sx = Tx$. Consequently, we have

$$\begin{aligned} d(Su, Sx) &\leq d(Tu, Tx) - \varphi(d(Tu, Tx)) \\ &= d(Su, Sx) - \varphi(d(Su, Sx)). \end{aligned} \tag{5.4.20}$$

In (5.4.20), if $Su \neq Sx$, then

$$d(Su, Sx) \leq d(Su, Sx) - \varphi(d(Su, Sx)) < d(Su, Sx),$$

which is impossible. So, we have $Su = Sx$ and hence $Tu = Tx$. It follows that

$$d(x, Tx) \quad = d(x, Tu) \quad = d(A, B)$$

and

$$d(x, Sx) \quad = d(x, Su) \quad = d(A, B).$$

Therefore, x is a common best proximity point of S and T.

Suppose that x^* is another common best proximity point of the mappings S and T. Then we have

$$d(x^*, Tx^*) \quad = \quad d(A, B)$$

and

$$d(x^*, Sx^*) \quad = \quad d(A, B).$$

Since S and T are commuting proximally, we have $Sx = Tx$ and $Sx^* = Tx^*$. Consequently, we have

$$\begin{aligned} d(Sx^*, Sx) &\leq d(Tx^*, Tx) - \varphi(d(Tx^*, Tx)) \\ &= d(Sx^*, Sx) - \varphi(d(Sx^*, Sx)). \end{aligned} \tag{5.4.21}$$

In (5.4.21), if $Sx^* \neq Sx$, then we have

$$d(Sx^*, Sx) \leq d(Sx^*, Sx) - \varphi(d(Sx^*, Sx)) < d(Sx^*, Sx),$$

which is impossible. So, we have $Sx = Sx^*$. Moreover, it follows that

$$\begin{aligned} d(x, x^*) &\leq d(x, Sx) + d(Sx, Sx^*) + d(Sx^*, x^*) \\ &= d(A, B) + d(A, B) \\ &= 2d(A, B). \end{aligned}$$

This completes the proof. $\square$

If take $\varphi(t) = (1-\alpha)t$, where $0 \leq \alpha < 1$ in Theorem 5.4.1, we obtain following :

Corollary 5.4.2. [54, Theorem 3.1] *Let A and B be nonempty closed subsets of a complete metric space X such that A is approximatively compact with respect to B. Also, assume that A_0 and B_0 are nonempty. Let $S : A \to B$, $T : A \to B$ be the nonself -mapping satisfying the following conditions.*

(i) *Thereexists a non-negative real number $\alpha < 1$ such that*

$$d(Sx_1, Sx_2) \leq \alpha d(Tx_1, Tx_2)$$

for all $x_1, x_2 \in A$;

(ii) *T is continuous;*

(iii) *S and T commute proximally;*

(iv) *S and T can be swapped proximally;*

(v) *$S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$.*

Then there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B), \quad d(x, Sx) = d(A, B).$$

Further, if x^* is another common best proximity point of the mappings S and T, then

$$d(x, x^*) \leq 2d(A, B).$$

For a self-mapping, Theorem 5.4.1 contains the following common fixed point theorems of Jungck [52] for commuting self-mappings, which in turn generalizes Banach's contraction principle.

Corollary 5.4.3. [52] *Let (X, d) be a complete metric space. Let S and T be self-mappings on X satisfying the following conditions:*

(i) *T is continuous;*

(ii) *$S(X) \subseteq T(X)$;*

(iii) *S and T commute.*

Suppose that there exists $\alpha \in [0, 1)$ such that

$$d(Sx, Sy) \leq \alpha d(Tx, Ty)$$

for all $x, y \in X$. Then there exists a unique common fixed point of S and T.

Now, we give an example to illustrate Theorem 5.4.1.

Example 5.4.4. Consider the complete metric space $\mathbb{R}^2$ with Euclidean metric. Let

$$A = \{(x, 1) \,:\, 0 \leq x \leq 1\}$$

and

$$B = \{(x, -1) : 0 \leq x \leq 1\}.$$

Define two mappings $S : A \to B$, $T : A \to B$ as follows:

$$S(x, 1) = \left(x - \frac{x^2}{2}, -1\right)$$

and

$$T((x, 1)) = \big(x, -1\big).$$

It is easy to see that $d(A, B) = 2$, $A_0 = A$ and $B_0 = B$. Further, S and T are continuous and A is approximatively compact with respect to B.

First, we show that S and T satisfy the condition (i) of of Theorem 5.4.1 with a function $\varphi : [0, \infty) \to [0, \infty)$ defined by $\varphi(t) = \dfrac{t^2}{2}$ for all $t \in [0, \infty)$. Let $(x, 1), (y, 1) \in A$. Without a loss generality, we can take that $x > y$. Then we have

$$\begin{aligned}
d(S(x, 1), S(y, 1)) &= \left|\left(x - \frac{x^2}{2}\right) - \left(y - \frac{y^2}{2}\right)\right| \\
&= \big(x - y\big) - \frac{1}{2}\big(x^2 - y^2\big) \\
&= \big(x - y\big) - \frac{1}{2}\big((x - y)(x + y)\big) \\
&\leq \big(x - y\big) - \frac{1}{2}\big(x - y\big)^2 \\
&= d(T(x, 1), T(y, 1)) - \varphi(d(T(x, 1), T(y, 1))).
\end{aligned}$$

Next, we show that S and T are commuting proximally. Let $(u,1),(v,1),(x,1) \in A$ are satisfying

$$d((u,1),S(x,1)) = d(A,B) = 2, \quad d((v,1),T(x,1)) = d(A,B) = 2.$$

It follows that

$$u = x - \frac{x^2}{2},\ v = x$$

and hence

$$S(v,1) \;=\; \left(v - \frac{v^2}{2}, -1\right) \;=\; \left(x - \frac{x^2}{2}, -1\right) = (u,-1) = T(u,1).$$

Finally, we show that S and T can be swapped proximally. If it is true that

$$d((u,1),(y,-1)) = d((v,1),(y,-1)) = d(A,B) = 2, \quad S(u,1) = T(v,1)$$

for some $(u,1),(v,1) \in A$ and $(y,-1) \in B$. Then we get $u = v = 0$ and thus

$$S(v,1) = T(u,1).$$

Therefore, all the hypothesis of Theorem 5.4.1 are satisfied.

Furthermore, $(0,1) \in A$ is a common best proximity point of S and T, because

$$d((0,1),S(0,1)) = d((0,1),(0,-1)) = d((0,1),T((0,1)) = d(A,B).$$

On the other hand, suppose that there exists $k \in [0,1)$ such that

$$d(S(x,1),S(y,1)) \leq kd(T(x,1),T(y,1)),$$

that is,

$$\left|\left(x - \frac{x^2}{2}\right) - \left(y - \frac{y^2}{2}\right)\right| \leq k|x-y|.$$

Putting $y = 0$ and $x > 0$, it follow that

$$1 = \lim_{x \to 0^+}\left|\left(1 - \frac{x}{2}\right)\right| \leq k < 1,$$

which is a contradiction. Therefore, the results of Sadiq Basha in [54] can not be applied to this example and our main result Theorem 5.4.1.