CHAPTER 4 FIXED POINT THEOREMS FOR GENERALIZED CONTRACTION MAPPINGS IN MODULAR SPACES

The aim of this chapter is to prove the existence of fixed point and common fixed point for generalized contractions in modular spaces and also prove fixed points theorems for contraction mapping in modular metric spaces.

4.1 Fixed point theorems for generalized contraction mappings in modular spaces

In this section, we prove the existence theorem of fixed points for a generalized weak contractive mapping which is a generalized contraction mappings in modular spaces.

Proposition 4.1.1. *Let ρ be a modular space on X. If $a, b \in \mathbb{R}^+$ with $a \leq b$, then $\rho(ax) \leq \rho(bx)$.*

Proof. In case $a = b$, clearly. Suppose $b > a$. Then we have $\frac{a}{b} < 1$ and

$$\begin{aligned} \rho(ax) &= \rho(\tfrac{a}{b}bx) \\ &= \rho(\tfrac{a}{b}bx + (1 - \tfrac{a}{b})(0)) \\ &\leq \rho(bx) + \rho(0) \\ &= \rho(bx). \end{aligned}$$

This completes the proof. □

Proposition 4.1.2. *Let X_ρ be a modular space which ρ satisfies the Δ_2-condition and $\{x_n\}$ be a sequence in X_p. If $\rho(c(x_n - x_{n-1})) \to 0$ as $n \to \infty$, then*

$$\rho(\alpha l(x_n - x_{n-1})) \to 0$$

as $n \to \infty$, where $c, l, \alpha \in \mathbb{R}^+$ with $\frac{l}{c} + \frac{1}{\alpha} = 1$.

Proof. Since $\rho(c(x_n - x_{n-1})) \to 0$ as $n \to \infty$, by the Δ_2-condition, we get

$$\rho(2c(x_n - x_{n-1})) \to 0. \tag{4.1.1}$$

Again, by the Δ_2-condition, we get

$$\rho(2^2 c(x_n - x_{n-1})) \to 0. \tag{4.1.2}$$

By the same method, we can conclude that for each integer $m \in \mathbb{N}$,

$$\rho(2^m c(x_n - x_{n-1})) \to 0 \tag{4.1.3}$$

as $n \to \infty$. From the fact that $\frac{l}{c} + \frac{1}{\alpha} = 1$, we get $\alpha l = (\alpha - 1)c \geq c$, and so there exists a positive integer m_α such that

$$(\alpha - 1)c \leq 2^{m_\alpha} c.$$

By Proposition 4.1.1, we get

$$\rho((\alpha - 1)c(x_n - x_{n-1})) \leq \rho(2^{m_\alpha} c(x_n - x_{n-1})). \tag{4.1.4}$$

Using (4.1.3) and (4.1.4), we obtain

$$\lim_{n\to\infty} \rho(\alpha l(x_n - x_{n-1})) = \lim_{n\to\infty} \rho((\alpha - 1)c(x_n - x_{n-1})) = 0. \tag{4.1.5}$$

This completes the proof. □

Theorem 4.1.3. *Let X_ρ be a $\rho-$complete bounded modular space, where ρ satisfies the Δ_2-condition. Let $c, l \in \mathbb{R}^+$, $c > l$ and $T : X_\rho \to X_\rho$ be a mapping such that, for all $x, y \in X_\rho$,*

$$\psi(\rho(c(Tx - Ty))) \;\leq\; \psi(\rho(l(x - y))) - \phi(\rho(l(x - y))), \tag{4.1.6}$$

where $\psi, \phi : [0, \infty) \to [0, \infty)$ are continuous and nondecreasing functions with $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Proof. Let $x_0 \in X_\rho$. Now, we construct the sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for all $n \geq 1$. First, we prove that the sequence $\{\rho(c(Tx_n - Tx_{n+1}))\}$ converges to 0. Note that

$$\begin{aligned} \psi(\rho(c(x_n - x_{n+1}))) &\leq \psi(\rho(l(x_{n-1} - x_n))) - \phi(\rho(l(x_{n-1} - x_n))) \\ &\leq \psi(\rho(l(x_{n-1} - x_n))). \end{aligned} \tag{4.1.7}$$

Since ψ is nondecreasing, by Proposition 4.1.1, we have

$$\begin{aligned}\rho(c(x_n - x_{n+1})) &\leq \rho(l(x_{n-1} - x_n)) \\ &\leq \rho(c(x_{n-1} - x_n)).\end{aligned}$$

This means that the sequence $\{\rho(c(x_n - x_{n+1}))\}$ is decreasing and bounded below. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \rho(c(x_n - x_{n+1})) = r.$$

If $r > 0$ and take $n \to \infty$ in the inequality (4.1.7), then we get

$$\psi(r) \quad \leq \psi(r) - \phi(r) < \quad \psi(r),$$

which is a contradiction. Thus $r = 0$. So, we have

$$\rho(c(x_n - x_{n+1})) \to 0 \tag{4.1.8}$$

as $n \to \infty$.

Next, we prove that the sequence $\{cx_n\}$ is a ρ-Cauchy sequence. Suppose that $\{cx_n\}$ is not ρ-Cauchy sequence. Then there exist $\varepsilon > 0$ and a subsequence $\{x_{m_k}\}, \{x_{n_k}\}$ with $m_k > n_k \geq k$ such that

$$\rho(c(x_{m_k} - x_{n_k})) \geq \varepsilon, \quad \rho(c(x_{m_k-1} - x_{n_k})) < \varepsilon. \tag{4.1.9}$$

Now, let $\alpha \in \mathbb{R}^+$ such that $\frac{l}{c} + \frac{1}{\alpha} = 1$. Then we get

$$\begin{aligned}\psi(\rho(c(x_{m_k} - x_{n_k}))) &\leq \psi(\rho(l(x_{m_k-1} - x_{n_k-1}))) - \phi(\rho(l(x_{m_k-1} - x_{n_k-1}))) \\ &\leq \psi(\rho(l(x_{m_k-1} - x_{n_k-1}))),\end{aligned} \tag{4.1.10}$$

which implies that

$$\rho(c(x_{m_k} - x_{n_k})) \leq \rho(l(x_{m_k-1} - x_{n_k-1})). \tag{4.1.11}$$

Observe that

$$\begin{aligned}\rho(l(x_{m_k-1} - x_{n_k-1})) &= \rho(l(x_{m_k-1} - x_{n_k} + x_{n_k} - x_{n_k-1}) \\ &= \rho(\tfrac{l}{c}c(x_{m_k-1} - x_{n_k}) + \tfrac{1}{\alpha}\alpha l(x_{n_k} - x_{n_k-1})) \\ &\leq \rho(c(x_{m_k-1} - x_{n_k})) + \rho(\alpha l(x_{n_k} - x_{n_k-1})) \\ &< \varepsilon + \rho(\alpha l(x_{n_k} - x_{n_k-1})).\end{aligned} \tag{4.1.12}$$

By (4.1.9), (4.1.11) and (4.1.12), we get

$$\varepsilon \leq \rho(c(x_{m_k} - x_{n_k})) \leq \rho(l(x_{m_k-1} - x_{n_k-1})) < \varepsilon + \rho(\alpha l(x_{n_k} - x_{n_k-1})). \quad (4.1.13)$$

On the other hand, using (4.1.8) and Proposition 4.1.2, we have

$$\lim_{k\to\infty} \rho(\alpha l(x_{n_k} - x_{n_k-1})) = 0. \quad (4.1.14)$$

From (4.1.13) and (4.1.14), we obtain

$$\lim_{k\to\infty} \rho(c(x_{m_k} - x_{n_k})) = \lim_{k\to\infty} \rho(l(x_{m_k-1} - x_{n_k-1})) = \varepsilon. \quad (4.1.15)$$

Letting $k \to \infty$ in (4.1.10), by the property of ψ and (4.1.15), we get

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon) < \psi(\varepsilon),$$

which is a contradiction. Therefore, $\{cx_n\}$ is a ρ-Cauchy sequence. Since X_ρ is $\rho-$ *complete*, there exists a point $u \in X_\rho$ such that $\rho(c(x_n - u)) \to 0$ as $n \to \infty$. Consequently, $\rho(l(x_n - u)) \to 0$ as $n \to \infty$.

Next, we prove that u is a unique fixed point of T. Putting $x = x_{n-1}$ and $y = u$ in (4.1.6), we obtain

$$\psi(\rho(c(x_n - Tu))) \leq \psi(\rho(l(x_{n-1} - u))) - \phi(\rho(l(x_{n-1} - u))). \quad (4.1.16)$$

Taking $n \to \infty$ in the inequality (4.1.16), we have

$$\psi(\rho(c(u - Tu))) \leq \psi(0) - \phi(0) = 0,$$

which implies that $\rho(c(Tu - u)) = 0$ and $Tu = u$. Suppose that there exists $v \in X_\rho$ such that $Tv = v$ and $v \neq u$, we have

$$\begin{aligned} \psi(\rho(c(u - v))) &= \psi(\rho(c(Tu - Tv))) \\ &\leq \psi(\rho(l(u - v))) - \phi(\rho(l(u - v))) \\ &< \psi(\rho(l(u - v))) \\ &\leq \psi(\rho(c(u - v))), \end{aligned}$$

which is a contradiction. Hence $u = v$. This completes the proof. □

Corollary 4.1.4. *Let X_ρ be a $\rho-$complete bounded modular space, where ρ satisfies the Δ_2- condition. Let $c, l \in \mathbb{R}^+$, $c > l$ and $T : X_\rho \to X_\rho$ be a mapping such that, for all $x, y \in X_\rho$,*

$$\rho(c(Tx - Ty)) \leq \rho(l(x - y)) - \phi(\rho(l(x - y))),$$

where $\phi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Proof. Take $\psi(t) = t$, we obtain Corollary 4.1.4. □

Next, we prove some existence theorems of common fixed points for a generalized weak contractive mapping which is a generalized contraction mappings in modular spaces.

Theorem 4.1.5. *Let X_ρ be a $\rho-$complete bounded modular space, where ρ satisfies the Δ_2-condition. Let $c, l \in \mathbb{R}^+$, $c > l$ and $T, f : X_\rho \to X_\rho$ be two $\rho-$compatible mappings such that $T(X_\rho) \subseteq f(X_\rho)$ and*

$$\psi(\rho(c(Tx - Ty))) \leq \psi(\rho(l(fx - fy))) - \phi(\rho(l(fx - fy))) \tag{4.1.17}$$

for all $x, y \in X_\rho$, where $\psi, \phi : [0,\infty) \to [0,\infty)$ are continuous and nondecreasing functions with $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. If one of T or f is continuous, then there exists a unique common fixed point of T and f.

Proof. Let $x \in X_\rho$ and generate inductively the sequence $\{Tx_n\}$ as follow: $Tx_n = fx_{n+1}$ for all $n \geq 1$ First, we prove that the sequence $\{\rho(c(Tx_n - Tx_{n-1}))\}$ converges to 0. By (4.1.17), we have

$$\begin{aligned}\psi(\rho(c(Tx_n - Tx_{n-1}))) &\leq \psi(\rho(l(fx_n - fx_{n-1}))) - \phi(\rho(l(fx_n - fx_{n-1})))\\ &\leq \psi(\rho(l(fx_n - fx_{n-1}))).\end{aligned} \tag{4.1.18}$$

Since ψ is nondecreasing, by Proposition 4.1.1 with $c > l$,

$$\begin{aligned}\rho(c(Tx_n - Tx_{n-1})) &\leq \rho(l(fx_n - fx_{n-1}))\\ &= \rho(l(Tx_{n-1} - Tx_{n-2}))\\ &\leq \rho(c(Tx_{n-1} - Tx_{n-2})).\end{aligned} \tag{4.1.19}$$

This means that the sequence $\{\rho(c(Tx_n - Tx_{n-1}))\}$ is nonincreasing and bounded below. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \rho(c(Tx_n - Tx_{n-1})) = r. \tag{4.1.20}$$

If $r > 0$ and take $n \to \infty$ in the inequality (4.1.19), we get

$$\lim_{n\to\infty} \rho(l(fx_n - fx_{n-1})) = r. \tag{4.1.21}$$

Since

$$\psi(\rho(c(Tx_n - Tx_{n-1}))) \leq \psi(\rho(l(fx_n - fx_{n-1}))) - \phi(\rho(l(fx_n - fx_{n-1}))) \quad (4.1.22)$$

it follows from (4.1.20), (4.1.21) and (4.1.22) that $\psi(r) \leq \psi(r) - \phi(r) < \psi(r)$, which is a contradiction and so $r = 0$. That is,

$$\lim_{n\to\infty} \rho(c(Tx_n - Tx_{n-1})) = 0. \quad (4.1.23)$$

Next, we prove that the sequence $\{cTx_n\}$ is a ρ-Cauchy sequence. Suppose that $\{cTx_n\}$ is not a ρ-Cauchy sequence. Then there exist $\varepsilon > 0$ and subsequences $\{Tx_{m_k}\}, \{Tx_{n_k}\}$ of $\{cTx_n\}$ with $m_k > n_k \geq k$ such that

$$\rho(c(Tx_{m_k} - Tx_{n_k})) \geq \varepsilon, \quad \rho(c(Tx_{m_k-1} - Tx_{n_k})) < \varepsilon. \quad (4.1.24)$$

Now, let $\alpha \in \mathbb{R}^+$ such that $\frac{l}{c} + \frac{1}{\alpha} = 1$. Then we have

$$\begin{aligned}
\psi(\rho(c(Tx_{m_k} - Tx_{n_k}))) &\leq \psi(\rho(l(fx_{m_k} - fx_{n_k}))) - \phi(\rho(l(fx_{m_k} - fx_{n_k}))) \\
&\leq \psi(\rho(l(fx_{m_k} - fx_{n_k}))) \\
&= \psi(\rho(l(Tx_{m_k-1} - Tx_{n_k-1})))
\end{aligned} \quad (4.1.25)$$

and hence

$$\rho(c(Tx_{m_k} - Tx_{n_k})) \leq \rho(l(Tx_{m_k-1} - Tx_{n_k-1})).$$

Since

$$\begin{aligned}
\rho(l(Tx_{m_k-1} - Tx_{n_k-1})) &= \rho(l(Tx_{m_k-1} - Tx_{n_k} + Tx_{n_k} - Tx_{n_k-1})) \\
&= \rho(\tfrac{l}{c}c(Tx_{m_k-1} - Tx_{n_k}) + \tfrac{1}{\alpha}\alpha l(Tx_{n_k} - Tx_{n_k-1}))) \\
&\leq \rho(c(Tx_{m_k-1} - Tx_{n_k})) + \rho(\alpha l(Tx_{n_k} - Tx_{n_k-1})) \\
&< \varepsilon + \rho(\alpha l(Tx_{n_k} - Tx_{n_k-1})),
\end{aligned} \quad (4.1.26)$$

it follow from (4.1.23) and Proposition 4.1.2 that

$$\lim_{k\to\infty} \rho(\alpha l(Tx_{n_k} - Tx_{n_k-1})) = 0.$$

Therefore, we have

$$\lim_{k\to\infty} \rho(c(Tx_{m_k} - Tx_{n_k})) = \lim_{k\to\infty} \rho(l(Tx_{m_k-1} - Tx_{n_k-1})) = \varepsilon. \quad (4.1.27)$$

Taking $k \to \infty$ in (4.1.25), it follow from (4.1.27) and the continuity of ψ that

$$\psi(\varepsilon) \quad \leq \psi(\varepsilon) - \phi(\varepsilon) \quad < \psi(\varepsilon) \tag{4.1.28}$$

which is a contradiction. Hence $\{cTx_n\}$ is a ρ-Cauchy sequence and, by the Δ_2-condition, $\{Tx_n\}$ is a ρ-Cauchy sequence. Since X_ρ is ρ-complete, there exists a point $u \in X_\rho$ such that $\rho(Tx_n - u) \to 0$ as $n \to \infty$, that is, $Tx_n \to u$, further $fx_n \to u$ as $n \to \infty$. If T is continuous, then $T^2x_n \to Tu$ and $Tfx_n \to Tu$ as $n \to \infty$. Since f and T are ρ -compatible, $\rho((fTx_n - Tfx_n)) \to 0$ as $n \to \infty$ and so $fTx_n \to Tu$ as $n \to \infty$.

Next, we prove that u is a fixed point of T. Suppose that $Tu \neq u$. Since

$$\begin{aligned} \psi(\rho(c(T^2x_n - Tx_n))) &= \psi(\rho(c(T(Tx_n) - Tx_n))). \\ &\leq \psi(\rho(l(fTx_n - fx_n))) - \phi(\rho(l(fTx_n - fx_n))), \end{aligned} \tag{4.1.29}$$

taking $n \to \infty$ in the inequality (4.1.29) and using Proposition 4.1.1 with $c > l$, we have

$$\begin{aligned} \psi(\rho(c(Tu - u))) &\leq \psi(\rho(l(Tu - u))) - \phi(\rho(l(Tu - u))) \\ &< \psi(\rho(l(Tu - u))) \\ &\leq \psi(\rho(c(Tu - u))), \end{aligned}$$

which is a contradiction and hence $Tu = u$. Since $T(X_\rho) \subseteq f(X_\rho)$, there exists a point $u_1 \in X_\rho$ such that $u = Tu = fu_1$. From

$$\psi(\rho(c(T^2x_n - Tu_1))) \leq \psi(\rho(l(fTx_n - fu_1))) - \phi(\rho(l(fTx_n - fu_1))),$$

letting $n \to \infty$ yields

$$\psi(\rho(c(Tu - Tu_1))) \leq \psi(\rho(l(Tu - fu_1))) - \phi(\rho(l(Tu - fu_1))).$$

Therefore, we have

$$\begin{aligned} \psi(\rho(c(u - Tu_1))) &\leq \psi(\rho(l(u - fu_1))) - \phi(\rho(l(u - fu_1))) \\ &\leq \psi(\rho(l(u - fu_1))) \\ &= \psi(\rho(l(u - u))) \\ &= 0, \end{aligned}$$

which implies that $u = Tu_1 = fu_1$. Since f and T are ρ -compatible, we get $fu = fTu_1 = Tfu_1 = Tu = u$. If f is continuous, then, by a similar argument, one can prove $Tu = fu = u$.

Finally, suppose that there exists $v \in X_\rho$ such that $Tv = v = fv$ and $v \neq u$. Then we have

$$\begin{aligned}\psi(\rho(c(u-v))) &= \psi(\rho(c(Tu-Tv))) \\ &\leq \psi(\rho(l(fu-fv))) - \phi(\rho(l(fu-fv))) \\ &< \psi(\rho(l(u-v))) \\ &\leq \psi(\rho(c(u-v))),\end{aligned} \tag{4.1.30}$$

which is a contradiction. Hence $u = v$. This completes the proof. □

Corollary 4.1.6. *Let X_ρ be a $\rho-$complete bounded modular space, where ρ satisfies the Δ_2- condition. Let $c, l \in \mathbb{R}^+$, $c > l$ and $T, f : X_\rho \to X_\rho$ be $\rho -$ compatible mappings such that $T(X_\rho) \subseteq f(X_\rho)$ and satisfying the inequality*

$$\rho(c(Tx - Ty)) \leq \rho(l(fx - fy)) - \phi(\rho(l(fx - fy))) \tag{4.1.31}$$

for all $x, y \in X_\rho$, where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function with $\phi(t) = 0$ if and only if $t = 0$. If one of T or f is continuous, then there exists a unique common fixed point of T and f.

Proof. Take $\psi(t) = t$, we obtain Corollary 4.1.6. □

4.2 Fixed point theorems for contraction mappings in modular metric spaces

In this section, we prove new existence theorems of fixed points for contraction mappings in modular metric spaces.

Definition 4.2.1. Let ω be a metric modular on X, X_ω be a modular metric space induced by ω and $T : X_\omega \to X_\omega$ be an arbitrary mapping. A mapping T is called a *contraction* if, for all $x, y \in X_\omega$ and $\lambda > 0$, there exists $0 \leq k < 1$ such that

$$\omega_\lambda(Tx, Ty) \leq k\omega_\lambda(x, y). \tag{4.2.1}$$

Theorem 4.2.2. *Let X_ω be a complete modular metric space and $T : X_\omega \to X_\omega$ be a contraction mapping. Assume that there exists $x_0 \in X$ such that $\omega_\lambda(x_0, Tx_0) < \infty$ for all $\lambda > 0$. Then T has a fixed point in $x_* \in X_\omega$ and the sequence $\{T^n x_0\}$ converges to x_*. Moreover, if, $z \in F(X_\omega)$, where $F(X_\omega)$ is a set of fixed point of T such that $\omega_\lambda(x_*, z) < \infty$ for all $\lambda > 0$, then $x_* = z$.*

Proof. Let x_0 be an element in X_ω such that $\omega_\lambda(x_0, Tx_0) < \infty$ for all $\lambda > 0$ and we write $x_1 = Tx_0$, $x_2 = Tx_1 = T^2 x_0$ and, in general, $x_n = Tx_{n-1} = T^n x_0$ for all $n \geq 1$. Observe that

$$\omega_\lambda(T^n x_0, T^{n+1} x_0) \leq k\omega_\lambda(T^{n-1} x_0, T^n x_0) \leq \cdots \leq k^n \omega_\lambda(x_0, Tx_0) < \infty$$

for all $n \geq 1$. Assume that n and m are two positive integers with $m > n$. Then we have

$$\begin{aligned} \omega_\lambda(T^n x_0, T^m x_0) &\leq \omega_{\frac{\lambda}{m-n}}(T^n x_0, T^{n+1} x_0) + \omega_{\frac{\lambda}{m-n}}(T^{n+1} x_0, T^{n+2} x_0) \\ &\quad + \cdots + \omega_{\frac{\lambda}{m-n}}(T^{m-1} x_0, T^m x_0) \\ &\leq (k^n + k^{n+1} + \cdots + k^{m-1})\omega_{\frac{\lambda}{m-n}}(x_0, Tx_0) \\ &\leq (k^n + k^{n+1} + \cdots)\omega_{\frac{\lambda}{m}}(x_0, Tx_0) \\ &= \tfrac{k^n}{1-k}\omega_\lambda(x_0, Tx_0). \end{aligned}$$

Since $\omega_\lambda(x_0, Tx_0) < \infty$, we deduce that, for any $\epsilon > 0$, $\omega_\lambda(T^n x_0, T^m x_0) < \epsilon$ for all $m > n > N$ with sufficiently large. Thus $\{T^n x_0\}$ is a Cauchy sequence and hence it converges to some $x_* \in X_\omega$ by the completeness of X_ω. Observe further that

$$\omega_\lambda(x_*, Tx_*) \leq \omega_{\frac{\lambda}{2}}(x_*, T^n x_0) + k\omega_{\frac{\lambda}{2}}(T^{n-1} x_0, x_*).$$

Letting $n \to \infty$, we have $\omega_\lambda(x_*, Tx_*) = 0$ for all $\lambda > 0$. Therefore, x_* is a fixed point of f.

Let z be another fixed points of T such that $\omega_\lambda(x_*, z) < \infty$ for all $\lambda > 0$, then we get

$$\omega_\lambda(x_*, z) \; = \omega_\lambda(Tx_*, Tz) \leq \; k\omega_\lambda(x_*, z)$$

for all $\lambda > 0$. Since $0 \leq k < 1$, we get $\omega_\lambda(x, z) = 0$ for all $\lambda > 0$, which implies that $x_* = z$. This completes the proof. $\square$

Theorem 4.2.3. *Let X_ω be a complete modular metric space and $T : X_\omega \to X_\omega$ be a contraction mapping. Suppose that $x^* \in X_\omega$ is a fixed point of T, $\{\varepsilon_n\}$ is a sequence of positive numbers for which $\lim_{n\to\infty} \varepsilon_n = 0$ and $\{y_n\} \subseteq X_\omega$ satisfies*

$$\omega_\lambda(y_{n+1}, Ty_n) \leq \varepsilon_n$$

for all $\lambda > 0$. Then $\lim_{n\to\infty} y_n = x^*$.

Proof. Let $y_0 = x \in X_\omega$. Then we observe that, for all $m \geq 1$,

$$\begin{aligned}
\omega_\lambda(T^{m+1}x, y_{m+1}) &= \omega_{\frac{\lambda \cdot m}{m}}(T^{m+1}x, y_{m+1}) \\
&\leq \omega_{\frac{\lambda\cdot(m-1)}{m}}(T^{m+1}x, Ty_m) + \omega_{\frac{\lambda}{m}}(Ty_m, y_{m+1}) \\
&\leq k\omega_{\frac{\lambda\cdot(m-1)}{m}}(T^m x, y_m) + \varepsilon_m \\
&\leq k\omega_{\frac{\lambda\cdot(m-2)}{m}}(T^m x, Ty_{m-1}) + kw_{\frac{\lambda}{m}}(Ty_{m-1}x, y_m) + \varepsilon_m \qquad (4.2.2)\\
&\leq k^2\omega_{\frac{\lambda\cdot(m-2)}{m}}(T^{m-1}x, y_{m-1}) + k\varepsilon_{m-1} + \varepsilon_m \\
&\vdots \\
&\leq \sum_{i=0}^{m} k^{m-i}\varepsilon_i
\end{aligned}$$

for all $\lambda > 0$. Thus we get

$$\begin{aligned}
\omega_\lambda(y_{m+1}, x^*) &\leq \omega_{\frac{\lambda}{2}}(y_{m+1}, T^{m+1}x) + \omega_{\frac{\lambda}{2}}(T^{m+1}x, x^*) \\
&\leq \sum_{i=0}^{m} k^{m-i}\varepsilon_i + \omega_{\frac{\lambda}{2}}(T^{m+1}x, x^*).
\end{aligned} \qquad (4.2.3)$$

Next, we claim that $\lim_{m\to\infty} \omega_\lambda(y_{m+1}, x^*) = 0$ for all $\lambda > 0$. Now, let $\varepsilon > 0$. Since $\lim_{n\to\infty} \varepsilon_n = 0$, there exists a positive integer N such that, for all $m \geq N$, $\varepsilon_m \leq \varepsilon$. Thus we have

$$\begin{aligned}
\sum_{i=0}^{m} k^{m-i}\varepsilon_i &= \sum_{i=0}^{N} k^{m-i}\varepsilon_i + \sum_{i=N+1}^{m} k^{m-i}\varepsilon_i \\
&\leq k^{m-N}\sum_{i=0}^{N} k^{N-i}\varepsilon_i + \varepsilon \sum_{i=N+1}^{m} k^{m-i}.
\end{aligned} \qquad (4.2.4)$$

Taking limit as $m \to \infty$ in (4.2.4), we have

$$\lim_{m\to\infty} \sum_{i=0}^{m} k^{m-i}\varepsilon_i = 0. \qquad (4.2.5)$$

Since x^* is a fixed point of T, using Theorem 4.2.2, it follows that the sequence $\{T^n x\}$ converge to x^*. This implies that

$$\lim_{m\to\infty} \omega_{\frac{\lambda}{2}}(T^{m+1}x, x^*) = 0 \qquad (4.2.6)$$

for all $\lambda > 0$. Therefore, from (4.2.3), (4.2.5) and (4.2.6), we have

$$\lim_{m\to\infty} \omega_\lambda(y_{m+1}, x^*) = 0 \tag{4.2.7}$$

for all $\lambda > 0$, which implies that $\lim_{n\to\infty} y_n = x^*$. This completes the proof. $\square$

Theorem 4.2.4. *Let X_ω be a complete modular metric space and, for any $x^* \in X_\omega$, define*

$$B_\omega(x^*, \gamma) := \{x \in X_\omega \, : \, \omega_\lambda(x, x^*) \leq \gamma, \ \forall \lambda > 0\}.$$

If $T : B_\omega(x^*, \gamma) \to X_\omega$ is a contraction mapping with

$$\omega_{\frac{\lambda}{2}}(Tx^*, x^*) \leq (1-k)\gamma \tag{4.2.8}$$

for all $\lambda > 0$, where $0 \leq k < 1$, then T has a fixed point in $B_\omega(x^*, \gamma)$.

Proof. By Theorem 4.2.2, we only prove that $B_\omega(x^*, \gamma)$ is complete and $Tx \in B_\omega(x^*, \gamma)$ for all $x \in B_\omega(x^*, \gamma)$. Suppose that $\{x_n\}$ is a Cauchy sequence in $B_\omega(x^*, \gamma)$, and then also $\{x_n\}$ is a Cauchy sequence in X_ω. Since X_ω is complete, there exists $x \in X_\omega$ such that

$$\lim_{n\to\infty} \omega_{\frac{\lambda}{2}}(x_n, x) = 0 \tag{4.2.9}$$

for all $\lambda > 0$. Since, for each $n \geq 1$, $x_n \in B_\omega(x^*, \gamma)$, using the property of a metric modular, we get

$$\begin{aligned} \omega_\lambda(x^*, x) &\leq \omega_{\frac{\lambda}{2}}(x^*, x_n) + \omega_{\frac{\lambda}{2}}(x_n, x) \\ &\leq \gamma + \omega_{\frac{\lambda}{2}}(x_n, x^*) \end{aligned} \tag{4.2.10}$$

for all $\lambda > 0$. It follows the inequalities (4.2.9) and (4.2.10) that $w_\lambda(x^*, x) \leq \gamma$, which implies that $x \in B_\omega(x^*, \gamma)$. Therefore, $\{x_n\}$ is a convergent sequence in $B_\omega(x^*, \gamma)$ and also $B_\omega(x^*, \gamma)$ is complete.

Next, we prove that $Tx \in B_\omega(x^*, \gamma)$ for all $x \in B_\omega(x^*, \gamma)$. Let $x \in B_\omega(x^*, \gamma)$. From the inequalities (4.2.8), the contraction of T and the notion of a metric modular, we have

$$\begin{aligned} \omega_\lambda(x^*, Tx) &\leq \omega_{\frac{\lambda}{2}}(x^*, Tx^*) + \omega_{\frac{\lambda}{2}}(Tx^*, Tx) \\ &\leq (1-k)\gamma + k\omega_{\frac{\lambda}{2}}(x^*, x) \\ &\leq (1-k)\gamma + k\gamma \\ &= \gamma. \end{aligned}$$

Therefore, $Tx \in B_\omega(x^*, \gamma)$. This completes the proof. $\square$

Theorem 4.2.5. *Let X_ω be a complete modular metric space and T be a self-mapping on X_ω satisfying*

$$\omega_\lambda(Tx,Ty) \ \leq \ k(\omega_{2\lambda}(Tx,x)+\omega_{2\lambda}(Ty,y)) \tag{4.2.11}$$

for all $x,y\in X_\omega$, where $k\in\left[0,\frac{1}{2}\right)$. Assume that there exists $x_0\in X$ such that $\omega_\lambda(x_0,Tx_0)<\infty$ for all $\lambda>0$. Then T has a fixed point in $x\in X_\omega$ and the sequence $\{T^n x_0\}$ converges to x. Moreover, if, $z\in F(X_\omega)$, where $F(X_\omega)$ is a set of fixed point of T such that $\omega_\lambda(x_*,z)<\infty$ for all $\lambda>0$, then $x_*=z$.

Proof. Let x_0 be an element in X_ω such that $\omega_\lambda(x_0,Tx_0)<\infty$ for all $\lambda>0$. We write $x_1=Tx_0, x_2=Tx_1=T^2x_0$ and, in general, $x_n=Tx_{n-1}=T^nx_0$ for all $n\geq 1$. If $Tx_{n_0-1}=Tx_{n_0}$ for some $n_0\geq 1$, then $Tx_{n_0}=x_{n_0}$. Thus x_{n_0} is a fixed point of T.

Suppose that $Tx_{n-1}\neq Tx_n$ for all $n\geq 1$. For any $k\in[0,\frac{1}{2})$, we have

$$\begin{aligned}
\omega_\lambda(x_{n+1},x_n) &= \omega_\lambda(Tx_n,Tx_{n-1}) \\
&\leq k(\omega_{2\lambda}(Tx_n,x_n)+\omega_{2\lambda}(Tx_{n-1},x_{n-1})) \\
&\leq k(\omega_\lambda(x_{n+1},x_n)+\omega_\lambda(x_n,x_{n-1}))
\end{aligned} \tag{4.2.12}$$

for all $\lambda>0$ and $n\geq 1$. Hence we have

$$\omega_\lambda(x_{n+1},x_n) \ \leq \ \tfrac{k}{1-k}\omega_\lambda(x_n,x_{n-1}) \tag{4.2.13}$$

for all $\lambda>0$ and $n\geq 1$. Put $\beta:=\frac{k}{1-k}$. Since $k\in(0,\frac{1}{2})$, we get $\beta\in(0,1)$ and hence

$$\begin{aligned}
\omega_\lambda(x_{n+1},x_n) &\leq \beta\omega_\lambda(x_n,x_{n-1}) \\
&\leq \beta^2\omega_\lambda(x_{n-1},x_{n-2}) \\
&\ \ \vdots \\
&\leq \beta^n\omega_\lambda(x_1,x_0)
\end{aligned} \tag{4.2.14}$$

for all $\lambda>0$ and $n\geq 1$. Similar to the proof of Theorem 4.2.2, we can conclude that $\{x_n\}$ is a Cauchy sequence and, by the completeness of X_ω there exists a point $x\in X_\omega$ such that $x_n\to x$ as $n\to\infty$. By the property of a metric modular and the inequality (4.2.11), we have

$$\begin{aligned}
\omega_\lambda(Tx,x) &\leq \omega_{\frac{\lambda}{2}}(Tx,Tx_n)+\omega_{\frac{\lambda}{2}}(Tx_n,x) \\
&\leq k(\omega_\lambda(Tx,x)+\omega_\lambda(Tx_n,x_n))+\omega_{\frac{\lambda}{2}}(Tx_n,x) \\
&\leq k(\omega_\lambda(Tx,x)+\omega_{\frac{\lambda}{2}}(Tx_n,x)+\omega_{\frac{\lambda}{2}}(x,x_n))+\omega_{\frac{\lambda}{2}}(Tx_n,x) \\
&= k(\omega_\lambda(Tx,x)+\omega_{\frac{\lambda}{2}}(x_{n+1},x)+\omega_{\frac{\lambda}{2}}(x,x_n))+\omega_{\frac{\lambda}{2}}(x_{n+1},x)
\end{aligned} \tag{4.2.15}$$

for all $\lambda > 0$ and $n \geq 1$. Taking $n \to \infty$ in the inequality (4.2.15), we obtain

$$\omega_\lambda(Tx, x) \quad \leq \quad k\omega_\lambda(Tx, x). \tag{4.2.16}$$

Since $k \in [0, \frac{1}{2})$, we have $Tx = x$. Thus x is a fixed point of T.

Let z be another fixed points of T such that $\omega_\lambda(x_*, z) < \infty$ for all $\lambda > 0$, then we get

$$\begin{aligned}
\omega_\lambda(x, z) &= \omega_\lambda(Tx, Tz) \\
&\leq k(\omega_{2\lambda}(Tx, x) + \omega_{2\lambda}(Tz, z)) \\
&= 0
\end{aligned}$$

for all $\lambda > 0$, which implies that $x = z$. This completes the proof. □

Now, we give an example to illustrate Theorem 4.2.2.

Example 4.2.6. Let $X = \{(a, 0) \in \mathbb{R}^2 : 0 \leq a \leq 1\} \cup \{(0, b) \in \mathbb{R}^2 : 0 \leq b \leq 1\}$. Defined a mapping $\omega : (0, \infty) \times X \times X \to [0, \infty]$ by

$$\omega_\lambda((a_1, 0), (a_2, 0)) = \frac{4|a_1 - a_2|}{3\lambda},$$

$$\omega_\lambda((0, b_1), (0, b_2)) = \frac{|b_1 - b_2|}{\lambda}$$

and

$$\omega_\lambda((a, 0), (0, b)) = \frac{4a}{3\lambda} + \frac{b}{\lambda} = \omega_\lambda((0, b), (a, 0)).$$

We note that , if we take $\lambda \to \infty$, then we see that $X = X_\omega$ and also X_ω is a complete modular metric space. Define a mapping $T : X_\omega \to X_\omega$ by

$$T((a, 0)) = (0, a), \quad T((0, b)) = \left(\frac{b}{2}, 0\right).$$

Simple computations show that

$$\omega_\lambda(T((a_1, b_1)), T((a_2, b_2))) \quad \leq \quad \frac{3}{4}\omega_\lambda((a_1, b_1), (a_2, b_2))$$

for all $(a_1, b_1), (a_2, b_2) \in X_\omega$. Thus T is a contraction mapping with constant $k = \frac{3}{4}$. Therefore, T has a unique fixed point $(0, 0) \in X_\omega$. In the Euclidean metric d on X_ω, we see that

$$d(T((0, 0)), T((1, 0))) = d((0, 0), (0, 1)) = 1 > k = kd((0, 0), (1, 0))$$

for all $k \in [0, 1)$. Thus T is not a contraction mapping and then Banach's contraction mapping cannot be applied to this example.