CHAPTER 3 SOME GEOMETRIC PROPERTIES OF LACUNARY SEQUENCE SPACES AND GENERALIZED CASÀRO SEQUENCE SPACES

The aim of this chapter is to prove the property (β) and the uniform opial property of Lacunary sequence spaces and prove that generalized Cesàro sequence spaces $ces_{(p)}(q)$ have the property (H) and the uniform opial property.

3.1 On the property (β) and the uniform opial property of Lacunary sequence spaces

In this section, we assume that $\lim_{r\to\infty} \inf p_r > 1$ and $\lim_{r\to\infty} \sup p_r < \infty$. First, we give some results in Lacunary sequence spaces which are very important for our consideration.

Lemma 3.1.1. *For any $x \in l(p,\theta)$, there exist $k_0 \in \mathbb{N}$ and $\lambda \in (0,1)$ such that $\varrho(\frac{x^k}{2}) \le \frac{1-\lambda}{2}\varrho(x^k)$ for all $k \in \mathbb{N}$ with $k \ge k_0$, where*

$$x^k = (\overbrace{0, 0, \cdots, 0}^{k-1}, x(k), x(k+1), x(k+2), \cdots).$$

Proof. Let $k \in \mathbb{N}$ be fixed. So there exists $r_k \in \mathbb{N}$ such that k is a minimal element in I_{r_k}. Let α be a real number such that $1 < \alpha \le \lim_{r\to\infty} \inf p_r$. Then there exists $k_0 \in \mathbb{N}$ such that $\alpha < p_{r_k}$ for all $k \ge k_0$. Choose $\lambda \in (0,1)$ be a real number such that $(\frac{1}{2})^\alpha \le \frac{1-\lambda}{2}$. Then, for each $x \in l(p,\theta)$ and $k \ge k_0$, we have

$$\begin{aligned}
\varrho(\frac{x^k}{2}) &= \sum_{r=r_k}^{\infty} \left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x(i)}{2}\right|\right)^{p_r} \\
&= \sum_{r=r_k}^{\infty} \left(\frac{1}{2}\right)^{p_r} \left(\frac{1}{h_r}\sum_{i\in I_r}|x(i)|\right)^{p_r} \\
&\le \left(\frac{1}{2}\right)^{\alpha} \sum_{r=r_k}^{\infty} \left(\frac{1}{h_r}\sum_{i\in I_r}|x(i)|\right)^{p_r} \\
&\le \frac{1-\lambda}{2}\varrho(x^k).
\end{aligned}$$

□

Lemma 3.1.2. *For any $x \in l(p,\theta)$ and $\varepsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that $\varrho(x) \leq 1-\varepsilon$ implies $\|x\| \leq 1-\delta$.*

Proof. Suppose that Lemma does not hold. Then there exist $\varepsilon > 0$ and $x_n \in l(p,\theta)$ such that $\varrho(x_n) \leq 1-\varepsilon$ and $\frac{1}{2} \leq \|x_n\| \nearrow 1$. Let $a_n = \frac{1}{\|x_n\|} - 1$. Then $a_n \to 0$ as $n \to \infty$. Let $L = \sup\{\varrho(2x_n) : n \geq 1\}$. Since $\varrho \in \Delta_2^s$, there exists $K \geq 2$ such that

$$\varrho(2u) \leq K\varrho(u) + 1 \tag{3.1.1}$$

for all $u \in l(p,\theta)$ with $\varrho(u) < 1$. By (3.1.1), we have $\varrho(2x_n) \leq K\varrho(x_n) + 1 \leq K+1$ for all $n \geq 1$. Hence $0 < L < \infty$. By Lemma 2.3.1 and Lemma 2.3.2(ii), we have

$$\begin{aligned} 1 = \varrho(\frac{x_n}{\|x_n\|}) &= \varrho(2a_n x_n + (1-a_n)x_n) \\ &\leq a_n\varrho(2x_n) + (1-a_n)\varrho(x_n) \\ &\leq a_n L + (1-\varepsilon) \to 1-\varepsilon, \end{aligned}$$

which is a contradiction. This completes the proof. □

Theorem 3.1.3. *The space $l(p,\theta)$ is a Banach space with respect to the Luxemburg norm.*

Proof. Let $\{x_n\} = \{x_n(i)\}$ be a Cauchy sequence in $l(p,\theta)$ and $\varepsilon \in (0,1)$. Thus there exists a positive integer N such that $\| x_n - x_m \| < \varepsilon^M$ for all $n,m \geq N$. By Lemma 2.3.2(i), we have

$$\varrho(x_n - x_m) \leq \| x_n - x_m \| < \varepsilon^M \tag{3.1.2}$$

for all $n,m \geq N$. That is,

$$\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} |x_n(i) - x_m(i)| \right)^{p_r} < \varepsilon^M \tag{3.1.3}$$

for all $n,m \geq N$. For fixed r in (3.1.3), we get

$$|x_n(i) - x_m(i)| < \varepsilon$$

for all $n,m \geq N$. Thus $\{x_n(i)\}$ be a Cauchy sequence in $\mathbb{R}$ for all $i \geq 1$. Since $\mathbb{R}$ is complete, there exists $x(i) \in \mathbb{R}$ such that $x_m(i) \to x(i)$ as $m \to \infty$ for all $i \geq 1$. Thus for fixed r in (3.1.3), we have

$$|x_n(i) - x(i)| < \varepsilon$$

$m \to \infty$ for all $n, m \geq N$. This implies that, for all $n \geq N$,

$$\varrho(x_n - x_m) \to \varrho(x_n - x) \tag{3.1.4}$$

$m \to \infty$. This means that, for all $n \geq N$,

$$\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|x_n(i) - x_m(i)|\right)^{p_r} \to \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|x_n(i) - x(i)|\right)^{p_r} \tag{3.1.5}$$

as $m \to \infty$. Hence we have

$$\varrho(x_n - x) \leq \| x_n - x \| < \varepsilon^M \leq \varepsilon$$

for all $n \geq N$, which implies that $x_n \to x$ as $n \to \infty$. By the linearity of the sequence space $l(p, \theta)$, we can write $x = (x - x_N) + x_N \in l(p, \theta)$. Therefore, the sequence space $l(p, \theta)$ is a Banach space with respect to the Luxemburg norm. This completes the proof. $\square$

Theorem 3.1.4. *The space $l(p, \theta)$ has the property (β).*

Proof. Let $\varepsilon > 0$ and $\{x_n\} \subset B(l(p, \theta))$ with $sep(x_n) \geq \varepsilon$. For each $k \geq 1$, there exist $r_k \in \mathbb{N}$ such that k is a minimal element in I_{r_k}. Let

$$x_n^k = (\overbrace{0, 0, \cdots, 0}^{k-1}, x_n(k), x_n(k+1), x_n(k+2), \cdots).$$

Since, for each $i \geq 1$, $\{x_n(i)\}$ is bounded, by using the diagonal method, for each $k \geq 1$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}(i)\}$ converges for each$i \geq 1$. Therefore, for any $k \geq 1$, there exists an increasing sequence $\{t_k\}$ such that $sep((x_{n_j}^k)_{j>t_k}) \geq \varepsilon$. Hence, for each $k \geq 1$, there exists a sequence of positive integers $\{s_k\}_{k=1}^{\infty}$ with $s_1 < s_2 < s_3 < ...$ such that $\|x_{s_k}^k\| \geq \frac{\varepsilon}{2}$ and, since $\varrho \in \Delta_2^s$, by Lemma 2.2.8, we may assume that there exists $\eta > 0$ such that $\varrho(x_{s_k}^k) \geq \eta$ for all $k \geq 1$, that is,

$$\sum_{r=r_k}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|x_{s_k}^k(i)|\right)^{p_r} \geq \eta \tag{3.1.6}$$

for all $k \geq 1$. On the other hand, by Lemma 3.1.1, there exist a positive integer k_0 and $\lambda \in (0, 1)$ such that

$$\varrho\left(\frac{u^k}{2}\right) \leq \frac{1-\lambda}{2}\varrho(u^k) \tag{3.1.7}$$

for all $u \in l(p,\theta)$ and $k \geq k_0$. From Lemma 3.1.2, there exist $\delta > 0$ such that, for any $y \in l(p,\theta)$,

$$\varrho(y) \leq 1 - \frac{\lambda\eta}{4} \Longrightarrow \|y\| \leq 1 - \delta. \tag{3.1.8}$$

Again, since $\varrho \in \Delta_2^s$, by Lemma 2.2.7, there exists δ_0 such that

$$|\varrho(u+v) - \varrho(u)| < \frac{\lambda\eta}{4} \tag{3.1.9}$$

whenever $\varrho(u) \leq 1$ and $\varrho(v) \leq \delta_0$. Since $x \in B(l(p,\theta))$, we have that $\varrho(x) \leq 1$. Then there exits $k \geq k_0$ such that $\varrho(x^k) \leq \delta_0$. Putting $u = x_{s_k}^k$ and $v = x^k$, we have

$$\varrho\left(\frac{u}{2}\right) = \sum_{r=r_k}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x_{s_k}(i)}{2}\right|\right)^{p_r} < 1, \quad \varrho\left(\frac{v}{2}\right) = \sum_{r=r_k}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x(i)}{2}\right|\right)^{p_r} < \delta_0.$$

From (3.1.7) and (3.1.9), we have

$$\begin{aligned} \sum_{r=r_k}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x(i)+x_{s_k}(i)}{2}\right|\right)^{p_r} &= \varrho\left(\frac{u+v}{2}\right) \\ &\leq \varrho\left(\frac{u}{2}\right) + \frac{\lambda\eta}{4} \\ &\leq \frac{1-\lambda}{2}(\varrho(u)) + \frac{\lambda\eta}{4}. \end{aligned} \tag{3.1.10}$$

By (3.1.6), (3.1.9), (3.1.10) and the convexity of function $f(t) = |t|^{p_r}$ for all $r \geq 1$, we have

$$\begin{aligned} \varrho\left(\frac{x+x_{s_k}}{2}\right) &= \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x(i)+x_{s_k}(i)}{2}\right|\right)^{p_r} \\ &= \sum_{r=1}^{r_k-1}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x(i)+x_{s_k}(i)}{2}\right|\right)^{p_r} + \sum_{r=r_k}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x(i)+x_{s_k}(i)}{2}\right|\right)^{p_r} \\ &\leq \frac{1}{2}\left(\sum_{r=1}^{r_k-1}\left(\frac{1}{h_r}\sum_{i\in I_r}|x(i)|\right)^{p_r} + \sum_{r=1}^{r_k-1}\left(\frac{1}{h_r}\sum_{i\in I_r}|x_{s_k}(i)|\right)^{p_r}\right) \\ &\quad + \sum_{r=r_k}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x_{s_k}(i)}{2}\right|\right)^{p_r} + \frac{\lambda\eta}{4} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\left(\sum_{r=1}^{r_k-1}\left(\frac{1}{h_r}\sum_{i\in I_r}|x(i)|\right)^{p_r}+\sum_{r=1}^{r_k-1}\left(\frac{1}{h_r}\sum_{i\in I_r}|x_{s_k}(i)|\right)^{p_r}\right)\\
&\quad+\frac{1-\lambda}{2}\sum_{r=r_k}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|x_{s_k}(i)|\right)^{p_r}+\frac{\lambda\eta}{4}\\
&= \frac{1}{2}\sum_{r=1}^{r_k-1}\left(\frac{1}{h_r}\sum_{i\in I_r}|x(i)|\right)^{p_r}+\frac{1}{2}\sum_{r=1}^{r_k-1}\left(\frac{1}{h_r}\sum_{i\in I_r}|x_{s_k}(i)|\right)^{p_r}\\
&\quad+\frac{1-\lambda}{2}\sum_{r=r_k}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|x_{s_k}(i)|\right)^{p_r}+\frac{\lambda\eta}{4}\\
&= \frac{1}{2}\sum_{r=1}^{r_k-1}\left(\frac{1}{h_r}\sum_{i\in I_r}|x(i)|\right)^{p_r}+\frac{1}{2}\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|x_{s_k}(i)|\right)^{p_r}\\
&\quad-\frac{\lambda}{2}\sum_{r=r_k}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|x_{s_k}(i)|\right)^{p_r}+\frac{\lambda\eta}{4}\\
&\leq \frac{1}{2}+\frac{1}{2}-\frac{\lambda\eta}{2}+\frac{\lambda\eta}{4}\\
&= 1-\frac{\lambda\eta}{4}.
\end{aligned}
$$

So, it follow from (3.1.8) that

$$\left\|\frac{x+x_{s_k}}{2}\right\|\leq 1-\delta.$$

Therefore, the space $l(p,\theta)$ has the property (β). This completes the proof. □

By the facts presented in the section 2.5, the following results are obtained directly from Theorem 3.1.4.

Corollary 3.1.5. *The space $l_p(\theta)$ has the property (β).*

Corollary 3.1.6. *The space $l(p,\theta)$ is the nearly uniform convexity and has the drop property. Also, the spaces $l(p,\theta)$ is reflexive.*

Corollary 3.1.7. *The space $l(p,\theta)$ has the property (UKK).*

Corollary 3.1.8. [37, Theorem 2.9] *The space $l(p,\theta)$ has the property (H).*

Corollary 3.1.9. *The space $l_p(\theta)$ is the nearly uniform convexity and has the drop property. Also, the spaces $l_p(\theta)$ is reflexive.*

Corollary 3.1.10. *The space $l_p(\theta)$ has the property (UKK) and the property (H).*

Theorem 3.1.11. *The space $l(p,\theta)$ has the uniform Opial property.*

Proof. Take any $\varepsilon > 0$ and $x \in l(p,\theta)$ with $\| x \| \geq \varepsilon$. Let (x_n) be a weakly null sequence in $S(l(p,\theta))$. By $\limsup_{r\to\infty} p_r < \infty$, i.e., $\varrho \in \Delta_2^s$, by Lemma 2.2.8, there exists $\delta \in (0,1)$ independent of x such that $\varrho(x) > \delta$. Also, by $\varrho \in \Delta_2^s$ and Lemma 2.2.7, there exists $\delta_1 \in (0,\delta)$ such that

$$|\varrho(y+z) - \varrho(y)| < \frac{\delta}{4} \tag{3.1.11}$$

whenever $\varrho(y) \leq 1$ and $\varrho(z) \leq \delta_1$. Choose a positive integer r_0 such that

$$\sum_{r=r_0+1}^{\infty} \left(\frac{1}{h_r} \sum_{i\in I_r} |x(i)| \right)^{p_r} < \frac{\delta_1}{4}. \tag{3.1.12}$$

So, we have

$$\begin{aligned} \delta &< \sum_{r=1}^{r_0} \left(\frac{1}{h_r} \sum_{i\in I_r} |x(i)| \right)^{p_r} + \sum_{r=r_0+1}^{\infty} \left(\frac{1}{h_r} \sum_{i\in I_r} |x(i)| \right)^{p_r} \\ &\leq \sum_{r=1}^{r_0} \left(\frac{1}{h_r} \sum_{i\in I_r} |x(i)| \right)^{p_r} + \frac{\delta_1}{4}, \end{aligned} \tag{3.1.13}$$

which implies that

$$\sum_{r=1}^{r_0} \left(\frac{1}{h_r} \sum_{i\in I_r} |x(i)| \right)^{p_r} > \delta - \frac{\delta_1}{4} > \delta - \frac{\delta}{4} = \frac{3\delta}{4}. \tag{3.1.14}$$

Since $x_n \xrightarrow{w} 0$, there exists a positive integer n_0 such that

$$\frac{3\delta}{4} \leq \sum_{r=1}^{r_0} \left(\frac{1}{h_r} \sum_{i\in I_r} |x_n(i) + x(i)| \right)^{p_r} \tag{3.1.15}$$

for all $n > n_0$ since the weak convergence implies the coordinatewise convergence. Again, by $x_n \xrightarrow{w} 0$, there exists a positive integer n_1 such that

$$\| x_{n|_{k_o}} \| < 1 - \left(1 - \frac{\delta}{4} \right)^{\frac{1}{M}} \tag{3.1.16}$$

for all $n > n_1$, where k_0 is a minimal element in I_{r_0+1} and a positive integer M with $p_r \leq M$ for all $r \geq 1$. Hence, by the triangle inequality of the norm, we get

$$\| x_{n|_{\mathbb{N}-k_o}} \| > \left(1 - \frac{\delta}{4} \right)^{\frac{1}{M}}. \tag{3.1.17}$$

It follows from Lemma 2.3.2(iii) that

$$\begin{aligned} 1 &< \varrho\left(\frac{x_{n|_{\mathbb{N}-k_o}}}{\left(1-\frac{\delta}{4}\right)^{1/M}}\right) \\ &= \sum_{r=r_0+1}^{\infty}\left(\frac{\frac{1}{h_r}\sum_{i\in I_r}|x_n(i)|}{(1-\frac{\delta}{4})^{1/M}}\right)^{p_r} \\ &\leq \left(\frac{1}{(1-\frac{\delta}{4})^{1/M}}\right)^{M}\sum_{r=r_0+1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|x_n(i)|\right)^{p_r} \end{aligned} \tag{3.1.18}$$

implies

$$\sum_{r=r_0+1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|x_n(i)|\right)^{p_r} > 1-\frac{\delta}{4} \tag{3.1.19}$$

for all $n > n_1$. By the inequality (3.1.11), (3.1.15) and (3.1.19), it follows that, for any $n > n_1$

$$\begin{aligned} \varrho(x_n + x) &= \sum_{r=1}^{r_0}\left(\frac{1}{h_r}\sum_{i\in I_r}|x_n(i)+x(i)|\right)^{p_r} + \sum_{r=r_0+1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|x_n(i)+x(i)|\right)^{p_r} \\ &\geq \frac{3\delta}{4} + \sum_{r=r_0+1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|x_n(i)|\right)^{p_r} - \frac{\delta}{4} \\ &\geq \frac{3\delta}{4} + (1-\frac{\delta}{4}) - \frac{\delta}{4} \\ &\geq 1+\frac{\delta}{4}. \end{aligned}$$

Since $\varrho \in \Delta_2^s$, by Lemma 2.2.9, there exists τ depending on δ only such that

$$\| x_n + x \| \geq 1 + \tau,$$

which implies that $\liminf_{n\to\infty} \| x_n + x \| \geq 1+\tau$. This completes the proof. $\square$

By the facts presented in section 2.5, we get the following results:

Corollary 3.1.12. *The space $l_p(\theta)$ has the uniform Opial property.*

3.2 On the property (H) and the uniform opial property of generalized Cesàro sequence spaces

In this section, we prove the property (H) and the uniform opial property of generalized Cesàro sequence space $ces_{(p)}(q)$. Let $M = \sup p_k < \infty$ for all $k \geq 1$. The following results are very important for our consideration.

Proposition 3.2.1. *The functional ϱ is a convex modular on $ces_{(p)}(q)$.*

Proof. Let $x, y \in ces_{(p)}(q)$. It is obvious that $\varrho(x) = 0$ if and only if $x = 0$ and $\varrho(\alpha x) = \varrho(x)$ for scalar α with $|\alpha| = 1$. Let $\alpha \geq 0$ and $\beta \geq 0$ with $\alpha + \beta = 1$. By the convexity of the function $t \mapsto |t|^{p_k}$ for all $k \geq 1$, we have

$$\begin{aligned}
\varrho(\alpha x + \beta y) &= \sum_{k=1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |\alpha q_i x(i) + \beta q_i y(i)| \right)^{p_k} \\
&\leq \sum_{k=1}^{\infty} \left(\alpha \frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| + \beta \frac{1}{Q_k} \sum_{i=1}^{k} |q_i y(i)| \right)^{p_k} \\
&\leq \alpha \sum_{k=1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k} + \beta \sum_{k=1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i y(i)| \right)^{p_k} \\
&= \alpha \varrho(x) + \beta \varrho(y).
\end{aligned}$$

This completes the proof. $\square$

Proposition 3.2.2. *For all $x \in ces_{(p)}(q)$, the modular ϱ on $ces_{(p)}(q)$ satisfies the following properties:*

(i) *If $0 < a < 1$, then $a^M \varrho(\frac{x}{a}) \leq \varrho(x)$ and $\varrho(ax) \leq a\varrho(x)$;*

(ii) *If $a > 1$, then $\varrho(x) \leq a^M \varrho(\frac{x}{a})$;*

(iii) *If $a \geq 1$, then $\varrho(x) \leq a\varrho(x) \leq \varrho(ax)$.*

Proof. (i) Let $0 < a < 1$. Then, we have

$$\begin{aligned}
\varrho(x) &= \sum_{k=1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k} \\
&= \sum_{k=1}^{\infty} \left(\frac{a}{Q_k} \sum_{i=1}^{k} |\frac{q_i x(i)}{a}| \right)^{p_k} \\
&= \sum_{k=1}^{\infty} a^{p_k} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |\frac{q_i x(i)}{a}| \right)^{p_k} \\
&\geq \sum_{k=1}^{\infty} a^{M} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |\frac{q_i x(i)}{a}| \right)^{p_k} \\
&= a^M \sum_{k=1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |\frac{q_i x(i)}{a}| \right)^{p_k} \\
&= a^M \varrho(\frac{x}{a}).
\end{aligned}$$

Thus, by convexity of modular ϱ, we have $\varrho(ax) \leq a\varrho(x)$ and so (i) is obtained.
(ii) Let $a > 1$. Then, we have

$$\begin{aligned}
\varrho(x) &= \sum_{k=1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x(i)|\right)^{p_k} \\
&= \sum_{k=1}^{\infty} a^{p_k}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|\frac{q_i x(i)}{a}|\right)^{p_k} \\
&\leq a^M \sum_{k=1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|\frac{q_i x(i)}{a}|\right)^{p_k} \\
&= a^M \varrho\left(\frac{x}{a}\right).
\end{aligned}$$

Hence (ii) is satisfies. (iii) follows from the convexity of ϱ. This completes the proof. $\square$

Proposition 3.2.3. *For any $x \in ces_{(p)}(q)$, we have*

(i) *If $\|x\| < 1$, then $\varrho(x) \leq \|x\|$;*

(ii) *If $\|x\| > 1$, then $\varrho(x) \geq \|x\|$;*

(iii) *$\|x\| = 1$ if and only if $\varrho(x) = 1$;*

(iv) *$\|x\| < 1$ if and only if $\varrho(x) < 1$;*

(v) *$\|x\| > 1$ if and only if $\varrho(x) > 1$.*

Proof. (i) Let $\varepsilon > 0$ be such that $0 < \varepsilon < 1 - \|x\|$ and so $\|x\| + \varepsilon < 1$. By the definition of $\|\cdot\|$, there exits $\lambda > 0$ such that $\|x\| + \varepsilon > \lambda$ and $\varrho(\frac{x}{\lambda}) \leq 1$. By (i) and (iii) of Proposition 3.2.2, we have

$$\begin{aligned}
\varrho(x) &\leq \varrho\left(\frac{(\|x\| + \varepsilon)}{\lambda}x\right) \\
&= \varrho\left((\|x\| + \varepsilon)\frac{x}{\lambda}\right) \\
&\leq (\|x\| + \varepsilon)\varrho\left(\frac{x}{\lambda}\right) \\
&\leq \|x\| + \varepsilon,
\end{aligned}$$

which implies that $\varrho(x) \leq \|x\|$. Hence (i) is satisfies.

(ii) Let $\varepsilon > 0$ such that $0 < \varepsilon < \frac{\|x\|-1}{\|x\|}$. Then $0 < (1-\varepsilon)\|x\| \leq \|x\|$. By the definition of $\|\cdot\|$ and Proposition 3.2.2(i), we have

$$1 < \varrho(\frac{x}{(1-\varepsilon)\|x\|}) < \frac{1}{(1-\varepsilon)\|x\|}\varrho(x),$$

and so $(1-\varepsilon)\|x\| < \varrho(x)$ for all $\varepsilon \in (0, \frac{\|x\|-1}{\|x\|})$, which implies that $\|x\| \le \varrho(x)$.

(iii) Assume that $\|x\| = 1$. Let $\varepsilon > 0$. Then there exits $\lambda > 0$ such that $1+\varepsilon > \lambda > \|x\|$ and $\varrho(\frac{x}{\lambda}) \le 1$. By Proposition 3.2.2(ii), we have

$$\varrho(x) \le \lambda^M \varrho(\frac{x}{\lambda}) \le \lambda^M < (1+\varepsilon)^M,$$

and so $(\varrho(x))^{\frac{1}{M}} < 1+\varepsilon$ for all $\varepsilon > 0$, which implies that $\varrho(x) \le 1$. If $\varrho(x) < 1$, let $a \in (0,1)$ such that $\varrho(x) < a^M < 1$. From Proposition 3.2.2(i), we have $\varrho(\frac{x}{a}) \le \frac{1}{a^M}\varrho(x) < 1$. Hence $\|x\| < a < 1$, which is a contradiction. Thus we have $\varrho(x) = 1$.

Conversely, assume that $\varrho(x) = 1$. If $\|x\| > 1$, by (ii), we get $\varrho(x) \ge \|x\| > 1$, which is a contradiction. Thus $\|x\| \le 1$. Suppose that $\|x\| < 1$. Then it follows from (i) that $\varrho(x) \le \|x\| < 1$, which is a contradiction. Thus we obtain $\|x\| = 1$. (iv) follows from (i) and (iii), (v) follows from (iii) and (iv). This completes the proof. □

Proposition 3.2.4. *For any $x \in ces_{(p)}(q)$, we have*

(i) *If $0 < a < 1$ and $\|x\| > a$, then $\varrho(x) > a^M$;*

(ii) *If $a \ge 1$ and $\|x\| < a$, then $\varrho(x) < a^M$.*

Proof. (i) Let $0 < a < 1$ and $\|x\| > a$. Then $\|\frac{x}{a}\| > 1$. Thus by Proposition 3.2.3(v), we have $\varrho(\frac{x}{a}) > 1$. Hence, by Proposition 3.2.2(i), we have $\varrho(x) \ge a^M\varrho(\frac{x}{a}) > a^M$ and so we obtain (i).

(ii) Suppose $a \ge 1$ and $\|x\| < a$. Then $\|\frac{x}{a}\| < 1$. Thus by Proposition 3.2.3(iv), we have $\varrho(\frac{x}{a}) < 1$. If $a = 1$, it is obvious that $\varrho(x) < 1 = a^M$. If $a > 1$, then, by Proposition 3.2.2(ii), we obtain $\varrho(x) \le a^M\varrho(\frac{x}{a}) < a^M$. This completes the proof. □

Proposition 3.2.5. *Let $\{x_n\}$ be a sequence in $ces_{(p)}(q)$.*

(i) *If $\|x_n\| \to 1$ as $n \to \infty$, then $\varrho(x_n) \to 1$ as $n \to \infty$;*

(ii) *If $\varrho(x_n) \to 0$ as $n \to \infty$, then $\|x_n\| \to 0$ as $n \to \infty$.*

Proof. (i) Assume that $\|x_n\| \to 1$ as $n \to \infty$. Let $\varepsilon \in (0,1)$. Then there exists a positive integer N such that $1-\varepsilon < \|x_n\| < 1+\varepsilon$ for all $n \ge N$. By Proposition 3.2.4, we have $(1-\varepsilon)^M < \varrho(x_n) < (1+\varepsilon)^M$ for all $n \ge N$, which implies that $\varrho(x_n) \to 1$ as $n \to \infty$.

(ii) Suppose that $\|x_n\| \nrightarrow 0$ as $n \to \infty$. Then there exists $\varepsilon \in (0,1)$ and a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \varepsilon$ for all $k \geq 1$. By Proposition 3.2.4(i) we obtain $\varrho(x_{n_k}) > (\varepsilon)^M$ for all $k \geq 1$. This implies that $\varrho(x_n) \nrightarrow 0$ as $n \to \infty$. This completes the proof. $\square$

Lemma 3.2.6. *Let $x \in ces_{(p)}(q)$ and $\{x_n\} \subseteq ces_{(p)}(q)$. If $\varrho(x_n) \to \varrho(x)$ as $n \to \infty$ and $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \geq 1$, then $x_n \to x$ as $n \to \infty$.*

Proof. Let $\varepsilon > 0$ be given. Since $\varrho(x) = \sum_{k=1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k} < \infty$, there exists a positive integer k_0 such that

$$\sum_{k=k_0+1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k} < \frac{\varepsilon}{3 \cdot 2^{M+1}}. \tag{3.2.1}$$

Since

$$\varrho(x_n) - \sum_{k=1}^{k_0} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x_n(i)| \right)^{p_k} \to \varrho(x) - \sum_{k=1}^{k_0} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k}$$

and $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \geq 1$, there exists a positive integer n_0 such that

$$\varrho(x_n) - \sum_{k=1}^{k_0} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x_n(i)| \right)^{p_k} < \varrho(x) - \sum_{k=1}^{k_0} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k} + \frac{\varepsilon}{3 \cdot 2^M} \tag{3.2.2}$$

for al $n \geq n_0$ and

$$\sum_{k=1}^{k_0} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x_n(i) - q_i x(i)| \right)^{p_k} < \frac{\varepsilon}{3} \tag{3.2.3}$$

for al $n \geq n_0$. It follow from (3.2.1), (3.2.2) and (3.2.3) that, for all $n \geq n_0$,

$$\begin{aligned}
\varrho(x_n - x) &= \sum_{k=1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x_n(i) - q_i x(i)| \right)^{p_k} \\
&= \sum_{k=1}^{k_0} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x_n(i) - q_i x(i)| \right)^{p_k} \\
&\quad + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x_n(i) - q_i x(i)| \right)^{p_k} \\
&< \frac{\varepsilon}{3} + 2^M \left(\sum_{k=k_0+1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x_n(i))| \right)^{p_k} \right) \\
&\quad + 2^M \left(\sum_{k=k_0+1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i))| \right)^{p_k} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\varepsilon}{3} + 2^M \left(\varrho(x_n) - \sum_{k=1}^{k_0} \left(\tfrac{1}{Q_k} \sum_{i=1}^{k} |q_i x_n(i)| \right)^{p_k} \right) \\
&\quad +2^M \left(\sum_{k=k_0+1}^{\infty} \left(\tfrac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i))| \right)^{p_k} \right) \\
&< \frac{\varepsilon}{3} + 2^M \left(\varrho(x) - \sum_{k=1}^{k_0} \left(\tfrac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k} + \frac{\varepsilon}{3 \cdot 2^M} \right) \\
&\quad +2^M \left(\sum_{k=k_0+1}^{\infty} \left(\tfrac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i))| \right)^{p_k} \right) \\
&= \frac{\varepsilon}{3} + 2^M \left(\sum_{k=k_0+1}^{\infty} \left(\tfrac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k} + \frac{\varepsilon}{3 \cdot 2^M} \right) \\
&\quad +2^M \left(\sum_{k=k_0+1}^{\infty} \left(\tfrac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i))| \right)^{p_k} \right) \\
&= \frac{\varepsilon}{3} + 2^M \left(2 \sum_{k=k_0+1}^{\infty} \left(\tfrac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k} + \frac{\varepsilon}{3 \cdot 2^M} \right) \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}
$$

This show that $\varrho(x_n - x) \to 0$ as $as \to \infty$. Hence, by Proposition 3.2.5(ii), we have $\|x_n - x\| \to 0$ as $\to \infty$. This completes the proof. □

Theorem 3.2.7. *The space $ces_{(p)}(q)$ has the property (H).*

Proof. Let $x \in S(ces_{(p)}(q))$ and $\{x_n\} \subseteq ces_{(p)}(q)$ such that $\|x_n\| \to 1$ and $x_n \xrightarrow{w} x$ as $n \to \infty$. By Proposition 3.2.3(iii), we have $\varrho(x) = 1$, it follow form Proposition 3.2.5(i) that $\varrho(x_n) \to \varrho(x)$ as $n \to \infty$. Since the mapping $\pi_i : ces_{(p)}(q) \to \mathbb{R}$ defined by $\pi_i(y) = y(i)$ is a continuous linear functional on $ces_{(p)}(q)$, it follow that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \geq 1$. Thus, by Lemma 3.2.6, we obtain $x_n \to x$ as $n \to \infty$ and hence the space $ces_{(p)}(q)$ has the property (H). This completes the proof. □

Corollary 3.2.8. *For any $1 < p < \infty$, the space $ces_p(q)$ has the property (H).*

Corollary 3.2.9. [40, Theorem 2.6] *The space $ces_{(p)}$ has the property (H).*

Corollary 3.2.10. *For any $1 < p < \infty$, the space ces_p has the property (H).*

Theorem 3.2.11. *The space $ces_{(p)}(q)$ has the uniform Opial property.*

Proof. Take any $\varepsilon > 0$ and $x \in ces_{(p)}(q)$ with $\|x\| \geq \varepsilon$. Let $\{x_n\}$ be a weakly null sequence in $S(ces_{(p)}(q))$. By $\sup_{k\geq 1} p_k < \infty$, i.e., $\varrho \in \Delta_2^s$, and Lemma 2.2.8, there exists $\delta \in (0,1)$ independent of x such that $\varrho(x) > \delta$. Also, by $\varrho \in \Delta_2^s$ and Lemma 2.2.7, there exists $\delta_1 \in (0, \delta)$ such that

$$|\varrho(y+z) - \varrho(y)| < \frac{\delta}{4} \tag{3.2.4}$$

whenver $\varrho(y) \leq 1$ and $\varrho(z) \leq \delta_1$. Choose a positive integer k_0 such that

$$\sum_{k=k_0+1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=k_0+1}^{k} |q_i x(i)| \right)^{p_k} < \sum_{k=k_0+1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k} < \frac{\delta_1}{4}. \tag{3.2.5}$$

So, we have

$$\begin{aligned} \delta &< \sum_{k=1}^{k_0} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k} \\ &\leq \sum_{k=1}^{k_0} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k} + \frac{\delta_1}{4}, \end{aligned} \tag{3.2.6}$$

which implies that

$$\sum_{k=1}^{k_0} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k} > \delta - \frac{\delta_1}{4} > \delta - \frac{\delta}{4} = \frac{3\delta}{4}. \tag{3.2.7}$$

Since $x_n \xrightarrow{w} 0$, there exists a positive integer n_0 such that

$$\frac{3\delta}{4} \leq \sum_{k=1}^{k_0} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x_n(i) + q_i x(i)| \right)^{p_k} \tag{3.2.8}$$

for all $n > n_0$ since the weak convergence implies the coordinatewise convergence. Again, by $x_n \xrightarrow{w} 0$, there exists a positive integer n_1 such that

$$\| x_{n|_{k_o}} \| < 1 - \left(1 - \frac{\delta}{4}\right)^{\frac{1}{M}} \tag{3.2.9}$$

for all $n > n_1$, where $p_k \leq M$ for all $k \geq 1$. Hence, by the triangle inequality of the norm, we get

$$\| x_{n|_{\mathbb{N}-k_o}} \| > \left(1 - \frac{\delta}{4}\right)^{\frac{1}{M}}. \tag{3.2.10}$$

It follows from Proposition 3.2.3(v) , that

$$
\begin{aligned}
1 < \varrho\left(\frac{x_n|_{\mathbb{N}-k_o}}{\left(1-\frac{\delta}{4}\right)^{\frac{1}{M}}}\right) &= \sum_{k=k_0+1}^{\infty}\left(\frac{\frac{1}{Q_k}\sum_{i=k_0+1}^{k}|q_i x_n(i)|}{(1-\frac{\delta}{4})^{\frac{1}{M}}}\right)^{p_k} \\
&\leq \left(\frac{1}{(1-\frac{\delta}{4})^{\frac{1}{M}}}\right)^{M}\sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=k_0+1}^{k}|q_i x_n(i)|\right)^{p_k}
\end{aligned}
\tag{3.2.11}
$$

implies

$$
\sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=k_0+1}^{k}|q_i x_n(i)|\right)^{p_k} \quad > \quad 1-\frac{\delta}{4} \tag{3.2.12}
$$

for all $n > n_1$. By the inequalities (3.2.4), (3.2.5), (3.2.8), and (3.2.12), it follows that, for any $n > n_1$,

$$
\begin{aligned}
& \varrho(x_n + x) \\
= \;& \sum_{k=1}^{k_0}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x_n(i) + q_i x(i)|\right)^{p_k} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x_n(i) + q_i x(i)|\right)^{p_k} \\
\geq \;& \sum_{k=1}^{k_0}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x_n(i) + q_i x(i)|\right)^{p_k} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=k_0+1}^{k}|q_i x_n(i) + q_i x(i)|\right)^{p_k} \\
\geq \;& \frac{3\delta}{4} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=k_0+1}^{k}|q_i x_n(i)|\right)^{p_k} - \frac{\delta}{4} \\
\geq \;& \frac{3\delta}{4} + (1-\frac{\delta}{4}) - \frac{\delta}{4} \\
\geq \;& 1 + \frac{\delta}{4}.
\end{aligned}
$$

Since $\varrho \in \Delta_2^s$, by Lemma 2.2.9, there exists τ depending on δ only such that

$$
\| x_n + x \| \geq 1 + \tau,
$$

which implies that $\lim_{n\to\infty}\inf \| x_n + x \| \geq 1 + \tau$. This completes the proof. □

Corollary 3.2.12. *For any $1 < p < \infty$, the space $ces_p(q)$ has the uniform Opial property.*

Corollary 3.2.13. [57, Theorem 2.6] *The space $ces_{(p)}$ has the uniform Opial property.*

Corollary 3.2.14. [6, Theorem 2] *For any $1 < p < \infty$, the space ces_p has the uniform Opial property.*