CHAPTER 2 PRELIMINARIES

The aim of this chapter is to give some definitions, notations and theorems that will be used in the later chapters. Throughout this study, we let $\mathbb{N}$ and $\mathbb{R}$ stand for the set of natural numbers and set of real numbers, respectively, and $CB(X)$ stand for the class of all nonempty closed bounded subsets of X. The space of all real sequences is denoted by w. For $x \in w$, $i \in \mathbb{N}$, we denote

$$
\begin{aligned}
e_i &= (\overbrace{0, 0, \cdots, 0}^{i-1}, 1, 0, 0, 0, \cdots), \\
x \mid_i &= (x(1), x(2), x(3), \cdots, x(i), 0, 0, 0, \cdots), \\
x \mid_{\mathbb{N}-i} &= (0, 0, 0, \cdots, x(i+1), x(i+2), \cdots).
\end{aligned}
$$

2.1 Metric spaces, normed spaces, sequence spaces and ordered sets

Definition 2.1.1. A *metric space* is an order pair (X, d), where X is a nonempty set and d a *metric* on X, that is, $d : X \times X \to \mathbb{R}$ is a mapping satisfying the following conditions:

(i) $d(x, y) \geq 0$ for all $x, y \in X$;

(ii) $d(x, y) = 0$ if and only if $x = y$;

(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Example 2.1.2. Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a mapping define by

$$d(x, y) = |x - y|$$

for all $x, y \in \mathbb{R}$. Then d is a metric on $\mathbb{R}$ and d is called a *usual metric.*

Example 2.1.3. Let $X = \mathbb{R}^2$ and define a mapping $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. Then d is a metric on $\mathbb{R}^2$.

Example 2.1.4. Let X be an arbitrary set and define a mapping $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0, & x = y; \\ 1, & x \neq y. \end{cases}$$

Then d is a metric on X and we called that *discrete metric*.

Example 2.1.5. Let $X = \{f : [a,b] \to \mathbb{R} : f \text{ is continuous on } [a,b]\}$ and define a mapping $d : X \times X \to \mathbb{R}$ by

$$d(f,g) = \max_{x \in [a,b]} |f(x) - g(x)|$$

for all $f, g \in X$. Then d is a metric on X.

Definition 2.1.6. An *ordered set* is a relational structure $(X, \preceq)$ such that the relation "$\preceq$" is an ordering.

Definition 2.1.7. A *partial order* is a binary relation "$\preceq$" over a set X which satisfies the following conditions : for all a, b, and c in X,

(i) $a \preceq a$ (reflexivity);

(ii) if $a \preceq b$ and $b \preceq a$ then $a = b$ (antisymmetry);

(iii) if $a \preceq b$ and $b \preceq c$ then $a \preceq c$ (transitivity).

Definition 2.1.8. A function of positive integer variable, designated by $f(n)$ or x_n, for all $n \geq 1$, is called a *sequence*. The sequence $x_1, x_2, \cdots$ is also designated briefly by $\{x_n\}$.

Definition 2.1.9. A sequence $\{x_n\}$ in a metric space (X,d) is said to *convergent to a point* $x \in X$ if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon$$

for all $n \geq N$. In such case, we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$ and x is called the *limit* of the sequence $\{x_n\}$.

Definition 2.1.10. A sequence $\{x_n\}$ in a metric space (X,d) is called a *Cauchy sequence* if, for any $\varepsilon > 0$, there exits $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon$$

for all $n, m \geq N$.

Definition 2.1.11. A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

Definition 2.1.12. A subset M of metric space (X, d) is said to be *closed* if, any sequence $\{x_n\}$ in M such that $\lim\limits_{n\to\infty} x_n = x$, we have $x \in M$.

Definition 2.1.13. Let (X, d) be a metric space, $a \in X$ and $B \subseteq X$. The *distance* from a point a to $B \subseteq X$ is given by

$$d(a, B) = \inf\{d(a, b) : b \in B\}.$$

Definition 2.1.14. Let X be a vecter space (or linear space). A *norm* on X is a nonnegative real-valued function on X, written as $\|\cdot\|$, satisfying the following conditions: for all $x, y \in X$ and scalar α;

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\alpha x\| = |\alpha|\|x\|$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$ (the triangle inequality).

A vecter space X equipped with a norm $\|\cdot\|$ is called a *normed space.*

Every normed space gives rise to the metric $d(x, y) = \|x - y\|$. It is called the *metric* induced by the norm $\|\cdot\|$.

Definition 2.1.15. A complete normed space is called a *Banach space.*

Definition 2.1.16. A *sequence space* is a linear space whose members are sequences. If X is sequence space and $x \in X$, the j^{th} term of x is denote by $x(j)$, that is, $x = \{x(j)\}_{j=1}^{\infty}$.

Definition 2.1.17. Let X be a normed space. The *closed unit ball* of X is the set $\{x \in X : \|x\| \leq 1\}$, which is denoted by $B(X)$. The *unit sphere* of X is the set $\{x \in X : \|x\| = 1\}$, which is denoted by $S(X)$.

Definition 2.1.18. A real-valued continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be *convex* if

$$f\Big(\frac{u+v}{2}\Big) \leq \frac{f(u) + f(v)}{2} \tag{2.1.1}$$

for all $u, v \in \mathbb{R}$. If, in addition, the two sides of (2.1.1) are not equal for all $u \neq v$, then we call f *strictly convex.*

2.2 Modular spaces and Modular metric spaces

Modular spaces

Definition 2.2.1. For a real linear space X, a function $\rho : X \to [0, \infty]$ is called a *modular* if it satisfies the following conditions:

(i) $\rho(x) = 0$ if and only if $x = 0$;

(ii) $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$;

(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

The modular ρ is said to be *convex* if

(iv) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

Example 2.2.2. Let $(X, \|\cdot\|)$ be a normed space. Then $\rho(x) = \|x\|$ is a convex modular.

Example 2.2.3. Let $X = L^p(a, b)$, where $0 < p < 1$. Then

$$\rho(x) = \int_a^b |x(t)|^p dt$$

is a $p-$convex modular.

Example 2.2.4. Let $X = \mathbb{R}$ and $\rho(x) = |x|/(1 + |x|)$. Then ρ is modular (non-convex modular).

Definition 2.2.5. For a modular ρ on X, the space $X_\rho = \{x \in X : \rho(\lambda x) \to 0$ as $\lambda \to 0^+\}$ is called the *modular space*.

Definition 2.2.6. A modular ρ is said to satisfy the $\Delta_2 -$ *condition* (shortly, $\rho \in \Delta_2$) if, for any $\varepsilon > 0$, there exist constants $K \geq 2$ and $a > 0$ such that

$$\rho(2u) \leq K\rho(u) + \varepsilon$$

for all $u \in X_\rho$ with $\rho(u) \leq a$. If ρ satisfies the $\Delta_2 -$ *condition* for any $a > 0$ with $K \geq 2$ dependent on a, then we say that ρ is the *stong* $\Delta_2 -$ *condition* (shortly, $\rho \in \Delta_2^s$).

Lemma 2.2.7. [28, Lemma 2.1] *If $\rho \in \Delta_2^s$, then, for any $L > 0$ and $\varepsilon > 0$, there exists $\delta = \delta(L, \varepsilon) > 0$ such that*

$$|\rho(u + v) - \rho(u)| < \varepsilon$$

whenever $u, v \in X_\rho$ with $\rho(u) \leq L$ and $\rho(v) \leq \delta$.

Lemma 2.2.8. [28, Lemma 2.3] *The convergences in Luxemburg norm and in modular are equivalent in X_ρ if $\rho \in \Delta_2$.*

Lemma 2.2.9. [28, Lemma 2.4] *If $\rho \in \Delta_2^s$, then, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\| x \| \geq 1 + \delta$ whenever $\rho(x) \geq 1 + \varepsilon$.*

Definition 2.2.10. Let X_ρ be a modular space.

(1) The sequence $\{x_n\}$ in X_ρ is said to be ρ-*convergent* to a point $x \in X_\rho$ (shortly, $x_n \xrightarrow{\rho} x$) if $\rho(x_n - x) \to 0$ as $n \to \infty$.

(2) The sequence $\{x_n\}$ in X_ρ is called a ρ-*Cauchy sequence* if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

(3) A subset C of X_ρ is said to be $\rho -$ *closed* if the $\rho -$ *limit* of a ρ-convergent sequence of C always belongs to C.

(4) A subset C of X_ρ is said to be $\rho -$ *complete* if every ρ-Cauchy sequence in C is $\rho -$ *convergent* to a point in C.

(5) A subset C of X_ρ is said to be $\rho -$ *bounded* if

$$\delta_\rho(C) = \sup\{\rho(x - y) \, : \, x, y \in C\} < \infty.$$

Definition 2.2.11. Let X_ρ be a modular space. A mappings $T : X_\rho \to X_\rho$ is said to be *continuous* at a point $x_0 \in X_\rho$ if, for any sequence $\{x_n\}$ in X_ρ with $x_n \xrightarrow{\rho} x$ we have $T(x_n) \xrightarrow{\rho} T(x)$ as $n \to \infty$.

Definition 2.2.12. [29] Let X_ρ be a modular space, where ρ satisfies the $\Delta_2 -$ *condition*. Two self-mappings T and f of X_ρ are said to be $\rho -$ *compatible* if $\rho(Tfx_n - fTx_n) \to 0$ as $n \to \infty$, whenever $\{x_n\}$ is a sequence in X_ρ such that $fx_n \to z$ and $Tx_n \to z$ for some point $z \in X_\rho$.

Modular metric spaces Let X be a nonempty set and $\lambda \in (0,\infty)$. Due to the disparity of the arguments, a function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ will be written as $\omega_\lambda(x,y) = \omega(\lambda, x, y)$ for all $\lambda > 0$ and $x, y \in X$.

Definition 2.2.13. [30] Let X be a nonempty set. A function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is called a *metric modular* on X is satisfies the following condition: for all $x, y, z \in X$,

(i) $\omega_\lambda(x,y) = 0$ for all $\lambda > 0$ if and only if $x = y$;

(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$ for all $\lambda > 0$;

(iii) $\omega_{\lambda+\mu}(x,y) \leq \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda, \mu > 0$.

If, instead of (i), we have only the following condition:

(i′) $\omega_\lambda(x,x) = 0$ for all $\lambda > 0$, then ω called a *(metric) pseudomodular* on X.

If $\omega_\lambda(x,y) = \omega(x,y)$ does not depend on $\lambda > 0$ and has only finite values, then the axioms (i)-(iii) mean that ω is a metric on X if (i) is replaced by (i′).

Example 2.2.14. Let $\lambda > 0$ and $x, y \in X$. Define a mapping $\omega : (0,\infty) \times X \times X \to [0,\infty]$ by

$$\omega_\lambda(x,y) = \begin{cases} \infty, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Then ω is a metric modular on X.

Example 2.2.15. Let (X,d) be a metric space , $\lambda > 0$ and $x, y \in X$. Define a mapping $\omega : (0,\infty) \times X \times X \to [0,\infty]$ by

$$\omega_\lambda(x,y) = \frac{d(x,y)}{\varphi(\lambda)},$$

where $\varphi : (0,\infty) \to (0,\infty)$ is a nondecreasing function. Then ω is a metric modular on X.

The main property of a (pseudo) modular ω on a set X is as follows: for all $x, y \in X$, the function $0 < \lambda \mapsto \omega_\lambda(x,y) \in [0,\infty]$ is a nonincreasing on $(0,\infty)$. In fact, if $0 < \mu < \lambda$, then (iii), (i′) and (ii) imply

$$\omega_\lambda(x,y) \leq \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y). \tag{2.2.1}$$

It follows that, at each point $\lambda > 0$, the right limit $\omega_{\lambda+0}(x,y) := \lim_{\epsilon\to+0} \omega_{\lambda+\epsilon}(x,y)$, the left limit $\omega_{\lambda-0}(x,y) := \lim_{\varepsilon\to+0} \omega_{\lambda-\varepsilon}(x,y)$ exist in $[0,\infty]$ and the following two inequalities hold:

$$\omega_{\lambda+0}(x,y) \ \le \omega_{\lambda}(x,y) \ \le \omega_{\lambda-0}(x,y). \tag{2.2.2}$$

Definition 2.2.16. [30] A function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a *convex (metric) modular* on X if it is satisfies the conditions (i) and (ii) in Definition 2.2.13 and the following condition holds;

(iv) $\omega_{\lambda+\mu}(x,y) \le \frac{\lambda}{\lambda+\mu}\omega_{\lambda}(x,z) + \frac{\mu}{\lambda+\mu}\omega_{\mu}(z,y)$ for all $\lambda,\mu > 0$ and $x,y,z \in X$.

If, instead of (i), we have only the condition (i′) in Definition 2.2.13, then ω is called a *convex (metric) pseudomodular* on X.

From [30, 31], if $x_0 \in X$, the set $X_\omega = \{x \in X : \lim_{\lambda\to\infty} \omega_{\lambda}(x,x_0) = 0\}$ is called *a modular set*. We note that condition of X_ω is an anlogue of the condition of X_ρ and the set X_ω is a metric space with a metric is given by

$$d^{\circ}_{\omega}(x,y) = \inf\{\lambda > 0 : \omega_{\lambda}(x,y) \le \lambda\}$$

for all $x,y \in X_\omega$ (see Theorem 2.6 in [30]). Furthermore, if ω is convex, then the modular set X_ω is equal to

$$X^*_\omega = \{x \in X : \exists\lambda = \lambda(x) > 0 \text{ such that } \omega_{\lambda}(x,x_0) < \infty\}$$

and metrizable by

$$d^*_\omega(x,y) = \inf\{\lambda > 0 : \omega_{\lambda}(x,y) \le 1\}$$

for all $x,y \in X^*_\omega$. We know that (see [30, Theorem 3.11]) if X is a real linear space, $\rho : X \to [0,\infty]$ and we set

$$\omega_{\lambda}(x,y) = \rho\left(\frac{x-y}{\lambda}\right) \tag{2.2.3}$$

for all $\lambda > 0$ and $x,y \in X$, then ρ is modular (convex modular) on X in the sense of Definition 2.2.1 if and only if ω is metric modular(convex metric modular, respectively) on X. On the other hand, if ω satisfy the following two conditions

(a) $\omega_{\lambda}(\mu x, 0) = \omega_{\lambda/\mu}(x,0)$ for all $\lambda,\mu > 0$ and $x \in X$;

(b) $\omega_{\lambda}(x+z, y+z) = \omega_{\lambda}(x,y)$ for all $\lambda > 0$ and $x,y,z \in X$.

For any $x \in X$, if we set $\rho(x) = \omega_1(x, 0)$ with (2.2.3), then

(i) $X_\rho = X_\omega(0)$ is a linear subspace of X and the functional $\|x\|_\rho = d^\circ_\omega(x, 0)$, $x \in X_\rho$ is an *F-norm* on X_ρ;

(ii) if ω is convex, $X^*_\rho \equiv X^*_\omega(0) = X_\rho$ is a linear subspace of X and the functional $\| x \|_\rho = d^*_\omega(x, 0)$ for any $x \in X^*_\rho$ is a norm on X^*_ρ.

Similar assertions hold if we replace the word *modular* by *pseudomodular*. If ω is metric modular in X, then the set X_ω is called a *modular metric space*.

By the some properties of metric spaces and modular spaces, we have the following:

Definition 2.2.17. Let X_ω be a modular metric space.

(1) The sequence $\{x_n\}$ in X_ω is said to be convergent to a point $x \in X_\omega$ if $\omega_\lambda(x_n, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$.

(2) The sequence $\{x_n\}$ in X_ω is called a *Cauchy sequence* if $\omega_\lambda(x_m, x_n) \to 0$ as $m, n \to \infty$ for all $\lambda > 0$.

(3) A subset C of X_ω is said to be *closed* if the limit of a convergent sequence of C always belong to C.

(4) A subset C of X_ω is said to be *complete* if every Cauchy sequence in C is a convergent sequence and its limit is in C.

(5) A subset C of X_ω is said to be *bounded* if, for all $\lambda > 0$,

$$\delta_\omega(C) = \sup\{\omega_\lambda(x, y); x, y \in C\} < \infty.$$

2.3 Lacunary sequence spaces

By a Lacunary sequence $\theta = (k_r)$, where $k_0 = 0$, we will mean an increasing sequence of nonnegative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. We write $h_r = k_r - k_{r-1}$ and the ratio k_r/k_{r-1} will denoted by q_r. The space of Lacunary strongly convergent sequence N_θ was defined by Freedman et al. [32] as follow:

$$N_\theta = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - l| = 0 \ \text{ for some } l \right\}.$$

It is well know that very closed connection between the space of Lacunary strongly convergent sequence and the space of strongly Cesàro summability sequences. Some more details can be found in [32, 33, 34, 35, 36].

Let $p = (p_r)$ be a bounded sequence of the positive real numbers. In 2007, Karakaya [37] introduced the new sequence spaces $l(p,\theta)$ involving Lacunary sequence as follows:

$$l(p,\theta) = \left\{ x = (x(i)) : \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{i\in I_r} |x(i)| \right)^{p_r} < \infty \right\} \tag{2.3.1}$$

and the paranorm on $l(p,\theta)$ is given by

$$\| x \|_{l(p,\theta)} = \left(\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{i\in I_r} |x(i)| \right)^{p_r} \right)^{\frac{1}{M}}, \tag{2.3.2}$$

where $M = \sup_r p_r$. If $p_r = p$ for all $r \geq 1$, then we use the notation $l_p(\theta)$ in place of $l(p,\theta)$. The norm on $l_p(\theta)$ is given by

$$\| x \|_{l_p(\theta)} = \left(\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{i\in I_r} |x(i)| \right)^{p} \right)^{\frac{1}{p}}. \tag{2.3.3}$$

By using the properties of the Lacunary sequence in the space $l(p,\theta)$, we get the following sequences. If $\theta = (2^r)$, then $l(p,\theta) = ces(p)$. If $\theta = (2^r)$ and $p_r = p$ for all $r \in \mathbb{N}$, then $l(p,\theta) = ces_p$. For all $x \in l(p,\theta)$ defined the modular on $l(p,\theta)$ by

$$\varrho(x) = \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{i\in I_r} |x(i)| \right)^{p_r}. \tag{2.3.4}$$

The Luxemberg norm on $l(p,\theta)$ is defined by

$$\| x \| = \inf\{\varepsilon > 0 : \varrho(\tfrac{x}{\varepsilon}) \leq 1\}.$$

The Luxemberg norm on $l_p(\theta)$ can be reduced to a usual norm on $l_p(\theta)$ (see [37]), that is,

$$\| x \| = \| x \|_{l_p(\theta)} .$$

Lemma 2.3.1. [37, Lemma 2.3] *The functional ϱ is a convex modular on $l(p,\theta)$.*

Lemma 2.3.2. [37, Lemma 2.5]

(i) *For any* $x \in l(p,\theta)$, *if* $\| x \| < 1$, *then* $\varrho(x) \leq \| x \|$.

(ii) *For any* $x \in l(p,\theta)$, $\| x \| = 1$ *if and only if* $\varrho(x) = 1$.

(iii) *For any* $x \in l(p,\theta)$, $\| x \| > 1$ *if and only if* $\varrho(x) > 1$.

2.4 Cesàro sequence spaces

Let w be the space of all real sequences. For $1 \leq p < \infty$, the Cesàro sequence space (ces_p, for short) is defined by

$$ces_p = \{x \in w : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p < \infty\}$$

equipped with the norm

$$\|x\| = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p \right)^{\frac{1}{p}}. \tag{2.4.1}$$

This space was first introduced by Shiue [38]. It is useful in the theory of matrix operators and others (see [39]). In 2003, Suantai [40, 41] defined the generalized Cesàro sequence space $ces_{(p)}$ when $p = \{p_k\}$ is a bounded sequence of positive real numbers with $p_k \geq 1$ for all $k \geq 1$ by

$$ces_{(p)} = \left\{ x \in w : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0 \right\},$$

where

$$\varrho(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{k} |x(i)| \right)^{p_n}$$

equipped with the Luxemburg norm

$$\|x\| = \inf\{\varepsilon > 0 : \varrho(\tfrac{x}{\varepsilon}) \leq 1\}.$$

In the case of $p_k = p$, $1 \leq p < \infty$, for all $k \geq 1$, the generalized Cesàro sequence space, $ces_{(p)}$, is the Cesàro sequence space, ces_p, and the Luxemburg norm is expressed by the formula (2.4.1). In 2010, Khan [42] defined the generalized Cesàro sequence space for $1 \leq p < \infty$ with is a bounded sequence $q = (q_k)$ of positive real numbers by

$$ces_p(q) = \left\{ x \in w : \left(\sum_{k=1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^p \right)^{1/p} < \infty \right\},$$

where $Q_k = \sum_{k=1}^{n} q_k$, for all $n \geq 1$. If $q_k = 1$ for all $k \geq 1$, then $ces_p(q)$ reduces to ces_p.

Now, we define the generalized Cesàro sequence space for bounded sequences $p = \{p_k\}$ and $q = \{q_k\}$ of positive real numbers with $p_k \geq 1$ for all $k \geq 1$ by

$$ces_{(p)}(q) = \Big\{ x \in w : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0 \Big\},$$

where

$$\varrho(x) = \sum_{k=1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k}$$

with $Q_k = \sum_{k=1}^{n} q_k$ and consider $ces_{(p)}(q)$ equipped with the Luxemburg norm

$$\|x\| = \inf\{\varepsilon > 0 : \varrho(\tfrac{x}{\varepsilon}) \leq 1\}.$$

Thus we see that $p_k = p$, $1 \leq p < \infty$, for all $k \geq 1$, then $ces_{(p)}(q)$ reduces to $ces_p(q)$ and, if $q_k = 1$ for all $k \geq 1$, then $ces_{(p)}(q)$ reduces to $ces_{(p)}$.

2.5 Geometric properties of Banach spaces

Let the $(X, \|\cdot\|)$ be a real Banach space. Let $B(X)$ and $S(X)$ be the closed unit ball and the unit sphere of X, respectively. For any subset A of X, we denote by $conv(A)$ (resp., ($\overline{conv}(A)$) the convex hull (resp., the closed convex hull) of A.

Definition 2.5.1. A Banach space X is said to have the *property* (H) if, for any sequence $\{x_n\}$ in X and $x \in X$ such that $x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\|$, we have $x_n \to x$ as $n \to \infty$.

Definition 2.5.2. A point $x \in S(X)$ is called an *extreme point* of $B(X)$ if, for any sequence $y, z \in S(X)$, the inequality $2x = y + z$ implies $y = z$.

Definition 2.5.3. A point $x \in S(X)$ is called a *locally uniformly rotund point* of $B(X)$ (LUR-point, for short) if, for any sequence $\{x_n\}$ in $B(X)$ such that $\|x_n + x\| \to 2$ as $n \to \infty$, we have $\|x_n - x\| \to 0$ as $n \to \infty$.

Definition 2.5.4. A Banach space X is said to be *rotund* (R) (or *strictly convex* (SC)) if every point of $S(X)$ is an extreme point of $B(X)$. If every point of $S(X)$ is a LUR-point of $B(X)$, then X is said to be *locally uniformly rotund* (LUR).

Definition 2.5.5. A Banach space X is said to be *uniformly rotund* (UR) (or *uniformly convex* (UC)) if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in S(X)$, the inequality $\|x - y\| > \varepsilon$ implies

$$\left\|\frac{1}{2}(x+y)\right\| < 1 - \delta.$$

Definition 2.5.6. A sequence $\{x_n\}$ of element X is called an *ε-separated sequence* for some $\varepsilon > 0$ if

$$sep(x_n) = \inf\{\|x_n - x_m\| : n \neq m\} \geq \varepsilon.$$

Definition 2.5.7. A Banach space X is said to have the *uniform Kadec-Klee property* (for short, (UKK)) if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any sequence $\{x_n\}$ in $S(X)$ with $sep(x_n) > \varepsilon$ and $x_n \xrightarrow{w} x$, we have $\|x\| < 1 - \delta$.

Definition 2.5.8. A Banach space X is said to *nearly uniformly convex* (for short, (NUC)) if, for any $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that, for any $\{x_n\} \subset B(X)$ with $sep(x_n) \geq \varepsilon$, we have

$$conv(x_n \cap ((1-\delta)B(X)) \neq \emptyset.$$

Definition 2.5.9. A Banach space X is said to have the *drop property* (for short, (D)) if, for every closed set C disjoint with $B(X)$, there exists $x \in C$ such that

$$D(x, B(x)) \cap C = \{x\},$$

where $D(x, B(X)) = \mathrm{conv}(B(X) \cup \{x\})$ such that $x \notin B(X)$.

Definition 2.5.10. A Banach space X is said to have the *property* (β) if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $x \in B(X)$ and each sequence $\{x_n\}$ in $B(X)$ with $sep(x_n) \geq \varepsilon$ there exists an index k such that

$$\left\|\frac{x + x_k}{2}\right\| \leq 1 - \delta.$$

Rolewicz [43] showed that the property (β) follows from the uniform convexity and the property (β) implies (NUC) and (NUC) implies property (D). He also proved that a Banach space X has property (D), then X is reflexive [44]. Moreover, the property (β) is different from both of them (see [45]). Montesions [46] extended this result by showing that X has property (D) if and only if X is reflexive and property (H). It is also known that (UKK) implies the property (H). Summarizing the above discussion, we have the foloowing:

$$\begin{array}{c} \qquad\qquad\qquad\qquad (D) \Longrightarrow (Rfx) \\ \qquad\qquad\qquad \Uparrow \\ (UC) \Longrightarrow \text{the property}(\beta) \Longrightarrow (NUC) \Longrightarrow (UKK) \Longrightarrow \text{the property } (H), \end{array}$$

where (Rfx) denotes the property of reflexivity (see [43, 46, 47, 48]). The converse of these implications are not true, in general, such as Kutzarova [45] provided an example of (NUC) space which does not have the property (β).

Definition 2.5.11. [49] A Banach space X is said to have *Opial's property* if, for any weakly null sequence $\{x_n\}$ and $x \neq 0$ in X,

$$\liminf_{n\to\infty} \| x_n \| \leq \liminf_{n\to\infty} \| x_n + x \|.$$

Opial proved in [49] that the sequence space $l_p(1 < p < \infty)$ have this property, but $L_p[0, \pi](p \neq 2, 1 < p < \infty)$ do not have it. Opial's property is important because Banach spaces with this property have the weak fixed point property (see [50]).

Definition 2.5.12. [51] A Banach space X is said to have the *uniform Opial property* if, for any $\varepsilon > 0$, there exists $\tau > 0$ such that, for any weakly null sequence $\{x_n\}$ in $S(X)$ and $x \in X$ with $\| x \| \geq \varepsilon$,

$$1 + \tau \leq \liminf_{n\to\infty} \| x_n + x \|.$$

2.6 Fixed points and best proximity points

Definition 2.6.1. Let X be an nonempty set and $f, g : X \to X$ be single-valued mappings.

(1) A point $x \in X$ is a *fixed point* of f if $fx = x$. The set of all fixed points of f is denoted by $F(f)$.

(2) A point $x \in X$ is a *common fixed point* of f and g if $x = fx = gx$. The set of all common fixed points of f and g is denoted by $F(f, g)$.

Definition 2.6.2. [52] Let (X, d) be a metric space and $f, g : X \to X$. The pair (f, g) is said to be *commuting* if $fgx = gfx$ for all $x \in X$.

Let A and B be nonempty subsets of a metric space (X, d). We recall the following notations and notions that will be used in what follows:

$$d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\},$$

$$A_0 := \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$

$$B_0 := \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

In the setting of normed spaces, if the sets A and B are closed such that $d(A, B) > 0$, then it follows that A_0 and B_0 are contained in the boundaries of A and B respectively (see [53]).

Definition 2.6.3. A point $x \in A$ is said to be a *best proximity point* of the mapping $S : A \to B$ if it satisfies the following condition:

$$d(x, Sx) = d(A, B).$$

It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

Definition 2.6.4. Let $S : A \to B$ and $T : A \to B$. An element $x^* \in A$ is said to be a *common best proximity point* if it satisfies the following condition:

$$d(x^*, Sx^*) = d(x^*, Tx^*) = d(A, B).$$

Observe that a common best proximity point is an element at which the multi-objective functions $x \mapsto d(x, Sx)$ and $x \mapsto d(x, Tx)$ attain a common global minimum since $d(x, Sx) \geq d(A, B)$ and $d(x, Tx) \geq d(A, B)$ for all x.

Definition 2.6.5. A mapping $T : A \cup B \to A \cup B$ is called a *cyclic mapping* if $T(A) \subset B$ and $T(B) \subset A$.

Definition 2.6.6. [54] A mapping $S : A \to B$ and $T : A \to B$ is said to be *commute proximally* if they satisfy the following condition:

$$[d(u, Sx) = d(v, Tx) = d(A, B)] \Longrightarrow Sv = Tu$$

for all $u, v, x \in A$.

It is easy to see that proximal commutativity of self-mappings become commutativity of the mappings.

Definition 2.6.7. [54] A mapping $S : A \to B$ and $T : A \to B$ is said to be *swapped proximally* if they satisfy the following condition:

$$[d(y, u) = d(y, v) = d(A, B),\ Su = Tv] \Longrightarrow Sv = Tu$$

for all $u, v, \in A$ and $y \in B$.

Definition 2.6.8. [55] A mapping $T : A \to B$ is called a *proximal contraction of the first kind* if there exists $k \in [0, 1)$ such that

$$\left.\begin{aligned} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{aligned}\right\} \quad \Longrightarrow \quad d(u, v) \le kd(x, y)$$

for all $u, v, x, y \in A$.

Definition 2.6.9. [55] A mapping $T : A \to B$ is called a *proximal contraction of the second kind* if there exists $k \in [0, 1)$ such that

$$\left.\begin{aligned} d(u, Tx) = d(A, B)) \\ d(v, Ty) = d(A, B) \end{aligned}\right\} \quad \Longrightarrow \quad d(Tu, Tv) \le kd(Tx, Ty)$$

for all $u, v, x, y \in A$.

It is easy to see that the self-mappings that is a proximal contraction of the first kind and second kind are precisely a contraction.

Definition 2.6.10. Let $S : A \to B$ and $T : B \to A$ be two mappings. The pair (S, T) is called a *proximal cyclic contraction pair* if there exists $k \in [0, 1)$ such that

$$\left.\begin{aligned} d(a, Sx) = d(A, B) \\ d(b, Ty) = d(A, B) \end{aligned}\right\} \quad \Longrightarrow \quad d(a, b) \le kd(x, y) + (1 - k)d(A, B)$$

for all $a, x \in A$ and $b, y \in B$.

Definition 2.6.11. Let $S : A \to B$ and $g : A \to A$ be an isometry. The mapping S is said to preserve the *isometric distance* with respect to g if

$$d(Sgx, Sgy) = d(Sx, Sy)$$

for all $x, y \in A$.

Definition 2.6.12. A mapping $T : A \to B$ is said to be *increasing* if

$$x \preceq y \Longrightarrow Sx \preceq Sy$$

for all $x, y \in A$.

Definition 2.6.13. [56] A mapping $T : A \to B$ is said to be *proximally order-preserving* if it satisfies the following condition:

$$\left.\begin{array}{l} x \preceq y \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \Longrightarrow \; u \preceq v \tag{2.6.1}$$

for all $u, v, x, y \in A$.

It is easy to observe that, for a self-mapping, the notion of proximally order-preserving mapping reduces to that of increasing mapping.

Definition 2.6.14. A set A is said to be *approximatively compact* with respect to a set B if every sequence $\{x_n\}$ in A satisfying the condition that $d(y, x_n) \to d(y, A)$ for some $y \in B$ has a convergent subsequence.

We observe that every set is approximatively compact with respect to itself. Also, every compact set is approximatively compact with respect to any set.