CHAPTER 3 METHODOLOGY

The methodologies consist of the heat conduction equation, the governing equations for the two-dimensional heat conduction equation subject to non-local boundary conditions, LWF of heat conduction equation subject to non-local boundary conditions, and discretization. The details of aforementioned topics are described as follows.

3.1 The heat conduction equation

Conduction occurs in a stationary medium (Chris, 2009), which is most likely to be a solid, but conduction can also occur in fluids. Heat is transferred by conduction due to motion of free electrons in metals or atoms in non-metals. Conduction is quantified by Fourier's Law: the heat flux, *q*, is proportional to the temperature gradient in the direction of the outward normal, e.g. in the x-direction:

$$q_x \propto \frac{du}{dx} \tag{3.1}$$

$$q_x = -k\frac{du}{dx} \quad (W/m^2) \tag{3.2}$$

The constant of proportionality, k, is the thermal conductivity, and over an area A, the rate of heat flow in the x-direction, Q_x, is

$$Q_x = -kA\frac{du}{dx} \quad (W) \tag{3.3}$$

Conduction may be treated as either steady-state, where the temperature at a point is constant with time, or as time-dependent (or transient) where temperature varies with time.

The general, time-dependent and multi-dimensional, governing equation for conduction can be derived from an energy balance on an element of dimensions $\delta x, \delta y, \delta z$. Consider the element shown in Figure 3.1.

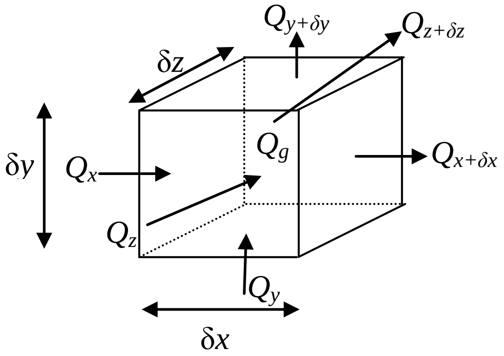

Figure 3.1 Heat Balance for conduction in an infinitesimal element.

The statement of energy conservation applied to this element in a time period δt is that: heat flow in + internal heat generation = heat flow out + rate of increase in internal energy

$$Q_x + Q_y + Q_z + Q_g = Q_{x+\delta x} + Q_{y+\delta y} + Q_{z+\delta z} + mc\frac{du}{dt} \tag{3.4}$$

or

$$Q_x - Q_{x+\delta x} + Q_y - Q_{y+\delta y} + Q_z - Q_{z+\delta z} + Q_g + mc\frac{du}{dt} = 0 \tag{3.5}$$

As noted above, the heat flow is related to temperature gradient through Fourier's Law, so:

$$\begin{aligned} Q_x &= -kA\frac{du}{dx} = -k\delta y\delta z\frac{du}{dx} \\ Q_y &= -kA\frac{du}{dy} = -k\delta x\delta z\frac{du}{dy} \\ Q_z &= -kA\frac{du}{dz} = -k\delta x\delta y\frac{du}{dz} \end{aligned} \tag{3.6}$$

Using a Taylor series expansion:

$$Q_{x+\delta x} = Q_x + \frac{\partial Q_x}{\partial x}\delta x + \frac{1}{2!}\frac{\partial^2 Q_x}{\partial x^2}\delta x^2 + \frac{1}{3!}\frac{\partial^3 Q_x}{\partial x^3}\delta x^3 + ... \tag{3.7}$$

For small values of δx, it is a good approximation to ignore terms containing δx^2 and higher order terms, so:

$$Q_x - Q_{x+\delta x} \cong \frac{\partial Q_x}{\partial x}\delta x = \delta x \delta y \delta z \frac{\partial}{\partial x}(-k\frac{du}{dx}) \tag{3.8}$$

A similar treatment can be applied to the other terms. For time-dependent conduction in three dimensions (x, y, z), with internal heat generation $q_g(W/m^3) = Q_g / \delta x \delta y \delta z$:

$$\frac{\partial}{\partial x}\left(-k\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(-k\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(-k\frac{\partial u}{\partial z}\right) + q_g = \rho c \frac{\partial u}{\partial t} \tag{3.9}$$

For constant thermal conductivity and no internal heat generation (Fourier's Equation):

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\rho c}{k}\frac{\partial u}{\partial t} \tag{3.10}$$

where ($k/\rho c$) is known as α, the thermal diffusivity (m^2/s). For steady state conduction with constant thermal conductivity and no internal heat generation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{3.11}$$

3.2 The two-dimensional heat conduction equation subject to non-local boundary conditions

The governing equation for the two-dimensional time-dependent heat conduction equation in the domain Ω boundary by Γ with non-local boundary conditions is described by:

$$\frac{\partial u}{\partial t} = \lambda(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + Q(x, y, t); \quad (x, y) \in \Omega \tag{3.12}$$

with initial condition

$$u(x, y, 0) = f(x, y), \quad \Omega = \{(x, y) \mid 0 \le x, y \le 1\}, \tag{3.13}$$

and boundary conditions

$$u = \bar{u}, \qquad (x, y) \in \Gamma_u \tag{3.14}$$

$$u_{,i} n_i \equiv q = \bar{q}, \quad (x, y) \in \Gamma_q \tag{3.15}$$

$$u = \bar{u}_l h(t), \qquad (x, y) \in \Gamma_l. \tag{3.16}$$

The non-local boundary condition is:

$$\int_\Omega u(x, y, t) d\Omega = m(t), \quad 0 \le x \le 1, \quad 0 \le y \le 1. \tag{3.17}$$

The non-local boundary condition is variable-separable, with spatial dependence and time-dependence on the essential boundary conditions can be written as: The boundary conditions are assumed to be:

$$\frac{\partial u(0, y, t)}{\partial x} = g_0(y, t), \quad 0 \le t \le T, \quad 0 \le y \le 1, \tag{3.18}$$

$$\frac{\partial u(1, y, t)}{\partial x} = g_1(y, t), \quad 0 \le t \le T, \quad 0 \le y \le 1, \tag{3.19}$$

$$u(x, 1, t) = h_1(x, t), \quad 0 \le t \le T, \quad 0 \le x \le 1, \tag{3.20}$$

$$u(x, 0, t) = h_0(x)\mu(t), \quad 0 \le t \le T, \quad 0 \le x \le 1, \tag{3.21}$$

where Ω is enclosed by $\Gamma = \Gamma_u + \Gamma_q + \Gamma_l$;

$\bar{u}, \bar{q}$ are the prescribed Dirichlet and Neumann boundary conditions respectively;

n is the outward normal direction to the boundary Γ;

λ is the thermal diffusivity; for the mathematical treatment it is sufficient to consider the case $\lambda = 1$;

t is time-dependent;

T is ending time;

Q is a source/sink term;

g_0, g_1, h_0, h_1, f, *and* m are given functions;

u and μ are unknowns.

The non-local boundary condition is variable-separable, with spatial dependence given by $h_0(x)$ and time dependence given by $\mu(t)$.

3.3 Local weak formulation

The MLPG method constructs the weak form over local sub-domains such as Ω_s, which is a small region taken for each node in the global domain Ω bounded by Γ, as seen in Figure 3.2, and may be of any geometric shape and size. In LWF, it is taken to be of circular shape. Because the weak form is constructed over local sub-domains, the formulation is called the "local weak formulation".

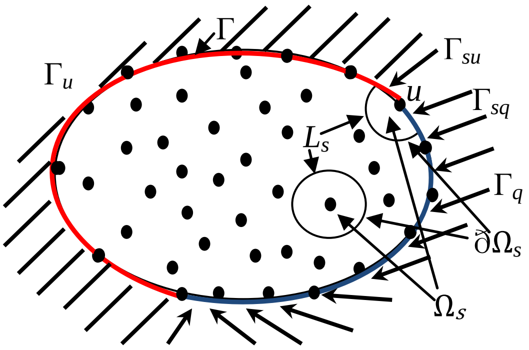

Figure 3.2 Schematic illustration of domain and boundaries in MLPG method: problem domain Ω bounded by Γ including essential boundary Γ_u (red line) and natural boundary Γ_q (blue line), $\partial\Omega_s$ boundary of local sub-domain that Ω_s local sub-domain bounded by Γ_s including the interior boundary L_s, the essential boundary Γ_{su} that intersects with Γ_u and natural boundary Γ_{sq} that intersects with Γ_q.

3.3.1 Local weak formulation I

In this scheme, we used a classical MLS method and penalty parameter imposed at boundary condition in a local weak formulation. The local weak form of equation (3.12) for $\mathbf{x}_i = (x^i, y^i) \in \Omega_s^i$ can be written as follows:

$$\int_{\Omega_s^i} \left(\frac{\partial u}{\partial t} - \lambda \nabla^2 u - Q \right) v_i \, d\Omega = 0, \tag{3.22}$$

where v_i is a test function.

Using the divergence theorem and

$$(\nabla^2 u) v_i = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} v_i\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} v_i\right) - \left(\frac{\partial u}{\partial x}\frac{\partial v_i}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v_i}{\partial y}\right)$$

yields the following expression:

$$\int_{\Omega_s^i} \frac{\partial u}{\partial t} v_i \, d\Omega - \int_{\partial\Omega_s^i} \lambda \frac{\partial u}{\partial n} v_i \, d\Gamma + \int_{\Omega_s^i} \lambda \frac{\partial u}{\partial x}\frac{\partial v_i}{\partial x} + \lambda \frac{\partial u}{\partial y}\frac{\partial v_i}{\partial y} \, d\Omega - \int_{\Omega_s^i} Q v_i \, d\Omega = 0, \tag{3.23}$$

where Ω_s^i is a circle of radius r_0 centered at $\mathbf{x}_i$, $\partial\Omega_s^i$ is the boundary of Ω_s^i, $n = (n_x, n_y)$ is the outward unit normal to the boundary $\partial\Omega_s^i = \Gamma_{su}^i \cup \Gamma_{sq0}^i \cup \Gamma_{sq1}^i$, where Γ_{su}, Γ_{sq0} and Γ_{sq1} are parts of the boundary Γ at each support domain,

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y,$$

and yields the following expression:

$$\begin{aligned} &\int_{\Omega_s^i} \frac{\partial u}{\partial t} v_i \, d\Omega - \int_{\Gamma_{su}^i} \lambda \frac{\partial u}{\partial n} v_i \, d\Gamma - \int_{\Gamma_{sq0}^i} \lambda \frac{\partial u}{\partial n} v_i \, d\Gamma - \int_{\Gamma_{sq1}^i} \lambda \frac{\partial u}{\partial n} v_i \, d\Gamma + \int_{\Omega_s^i} \lambda \frac{\partial u}{\partial x}\frac{\partial v_i}{\partial x} \\ &+ \lambda \frac{\partial u}{\partial y}\frac{\partial v_i}{\partial y} \, d\Omega - \int_{\Omega_s^i} Q v_i \, d\Omega = 0. \end{aligned} \tag{3.24}$$

Substituting trial function $u^h(\mathbf{x},t) = \sum_{j=1}^{n} \phi_j(\mathbf{x})\hat{u}_j(t)$ and $q \equiv \frac{\partial u}{\partial n}$ into equation (3.24) yields:

$$\begin{aligned}&\int_{\Omega_s^i} v_i \sum_{j=1}^{n} \phi_j \frac{\partial \hat{u}_j}{\partial t} d\Omega - \int_{\Gamma_{su}^i} \lambda v_i \sum_{j=1}^{n} \frac{\partial \phi_j}{\partial n} \hat{u}_j d\Gamma - \int_{\Gamma_{sq0}^i} g_0 \lambda v_i d\Gamma - \int_{\Gamma_{sq1}^i} g_1 \lambda v_i d\Gamma \\ &+ \int_{\Omega_s^i} \lambda \sum_{j=1}^{n} \frac{\partial \phi_j}{\partial x} \frac{\partial v_i}{\partial x} \hat{u}_j + \lambda \sum_{j=1}^{n} \frac{\partial \phi_j}{\partial y} \frac{\partial v_i}{\partial y} \hat{u}_j d\Omega - \int_{\Omega_s^i} Q v_i d\Omega = 0,\end{aligned} \tag{3.25}$$

$$\begin{aligned}&\sum_{j=1}^{n} \int_{\Omega_s^i} v_i \phi_j d\Omega \ \frac{\partial \hat{u}_j}{\partial t} - \sum_{j=1}^{n} \int_{\Gamma_{su}^i} \lambda v_i \frac{\partial \phi_j}{\partial n} d\Gamma \ \hat{u}_j + \sum_{j=1}^{n} \int_{\Omega_s^i} \lambda \frac{\partial \phi_j}{\partial x} \frac{\partial v_i}{\partial x} + \lambda \frac{\partial \phi_j}{\partial y} \frac{\partial v_i}{\partial y} d\Omega \ \hat{u}_j \\ &= \int_{\Gamma_{sq0}^i} g_0 \lambda v_i d\Gamma + \int_{\Gamma_{sq1}^i} g_1 \lambda v_i d\Gamma + \int_{\Omega_s^i} Q v_i d\Omega,\end{aligned} \tag{3.26}$$

where $i = 1, 2, 3, \ldots, n.$

By substituting the trial function from equation (2.37) into equation (3.24), the governing equations are transformed into the discretized system, which can be written in matrix form as

$$\mathbf{K} \frac{\partial \mathbf{U}}{\partial t} + \mathbf{F}\mathbf{U} = \mathbf{C}(t), \tag{3.27}$$

where, **K**, **F** and **C** are matrices, described as follows:

$$\mathbf{K} = [K_{ij}], \quad K_{ij} = \int_{\Omega_s^i} \phi_j v_i d\Omega, \tag{3.28}$$

$$\begin{aligned}&\mathbf{F} = [F_{ij}], \quad F_{ij} = \int_{\Omega_s^i} \lambda \frac{\partial \phi_j}{\partial x} (v_i)_x + \lambda \frac{\partial \phi_j}{\partial y} (v_i)_y d\Omega + \int_{\Gamma_{su0}^i} \lambda v_i \frac{\partial \phi_j}{\partial y} d\Gamma \\ &- \int_{\Gamma_{su1}^i} \lambda v_i \frac{\partial \phi_j}{\partial y} d\Gamma,\end{aligned} \tag{3.29}$$

$$\mathbf{C}(t) = [C_i(t)], \quad C_i(t) = \int_{\Gamma_{sq0}^i} g_0 \lambda v_i d\Gamma + \int_{\Gamma_{sq1}^i} g_1 \lambda v_i d\Gamma + \int_{\Omega_s^i} Q v_i d\Omega. \tag{3.30}$$

There is a problem for MLS at boundary conditions, because the trial function does not pass through the nodal values, so we impose a penalty parameter, α, at the Dirichlet boundary conditions and the Neumann boundary conditions with $\alpha \gg 1$:

$$\begin{aligned}&\alpha\int_{\Gamma^i_{su_0}} (u-h_0)v_i d\Gamma + \alpha\int_{\Gamma^i_{su_1}} (u-h_1\mu)v_i d\Gamma + \alpha\int_{\Gamma^i_{sq_0}} (\frac{\partial u}{\partial x}-g_0)v_i d\Gamma \\ &+\alpha\int_{\Gamma^i_{sq_1}} (\frac{\partial u}{\partial x}-g_1)v_i d\Gamma = 0.\end{aligned} \tag{3.31}$$

Substituting the trial function $u^h(\mathbf{x}) = \sum_{j=1}^{n}\phi_j(\mathbf{x})\hat{u}_j$ in equation (3.31) yields:

$$\begin{aligned}&\sum_{j=1}^{n}\left[\alpha\int_{\Gamma^i_{su_0}} \phi_j(\mathbf{x})v_i d\Gamma + \alpha\int_{\Gamma^i_{sq_0}} \frac{\partial\phi_j}{\partial x}v_i d\Gamma + \alpha\int_{\Gamma^i_{su_1}} \phi_j(\mathbf{x})v_i d\Gamma + \alpha\int_{\Gamma^i_{sq_1}} \frac{\partial\phi_j}{\partial x}v_i d\Gamma\right]\hat{u}_j \\ &-\alpha\int_{\Gamma^i_{su_0}} h_0 v_i d\Gamma \quad \mu = \alpha\int_{\Gamma^i_{su_1}} h_1 v_i d\Gamma + \alpha\int_{\Gamma^i_{sq_0}} g_0 v_i d\Gamma + \alpha\int_{\Gamma^i_{sq_1}} g_1 v_i d\Gamma.\end{aligned} \tag{3.32}$$

By substituting the trial function from equation (2.37) into equation (3.31), the governing equations are transformed into the discretized system, which can be written in matrix form as:

$$\mathbf{GU} - \mathbf{E}\hat{\boldsymbol{\mu}} = \mathbf{H}(t), \tag{3.33}$$

where **G**, **E** and **H** are matrices, described as follows:

$$G_{ij} = \alpha\int_{\Gamma^i_{su_0}} \phi_j(\mathbf{x})v_i d\Gamma + \alpha\int_{\Gamma^i_{sq_0}} \frac{\partial\phi_j}{\partial x}v_i d\Gamma + \alpha\int_{\Gamma^i_{su_1}} \phi_j(\mathbf{x})v_i d\Gamma + \alpha\int_{\Gamma^i_{sq_1}} \frac{\partial\phi_j}{\partial x}v_i d\Gamma, \tag{3.34}$$

$$\mathbf{E} = [E_{ij}], \quad E_{ij} = \alpha\int_{\Gamma^i_{su_0}} h_0 v_i d\Gamma, \tag{3.35}$$

$$\mathbf{H}(t) = [H_i(t)], \quad H_i(t) = \alpha\int_{\Gamma^i_{su_1}} h_1 v_i d\Gamma + \alpha\int_{\Gamma^i_{sq_0}} g_0 v_i d\Gamma + \alpha\int_{\Gamma^i_{sq_1}} g_1 v_i d\Gamma. \tag{3.36}$$

Setting a time-stepping scheme to overcome the time derivative and applying the Crank-Nicolson technique of approximation to equation (3.27) yields:

$$\mathbf{K}\frac{\mathbf{U}^{k+1}-\mathbf{U}^{k}}{\Delta t}+\frac{\mathbf{F}}{2}(\mathbf{U}^{k+1}+\mathbf{U}^{k})=\mathbf{C}^{k+\frac{1}{2}}. \tag{3.37}$$

Time Discretization I

In this scheme, we discretize equation (3.33) at time step $k+\dfrac{1}{2}$, obtaining:

$$\mathbf{G}\mathbf{U}^{k+\frac{1}{2}}-\mathbf{E}\hat{\mu}^{k+\frac{1}{2}}=\mathbf{H}^{k+\frac{1}{2}}, \tag{3.38}$$

$$\frac{\mathbf{G}}{2}(\mathbf{U}^{k+1}+\mathbf{U}^{k})-\frac{\mathbf{E}}{2}(\hat{\boldsymbol{\mu}}^{k+1}+\hat{\boldsymbol{\mu}}^{k})=\mathbf{H}^{k+\frac{1}{2}}. \tag{3.39}$$

Subtracting equation (3.39) from equation (3.37), we obtain:

$$\mathbf{K}\frac{\mathbf{U}^{k+1}-\mathbf{U}^{k}}{\Delta t}+\frac{\mathbf{F}}{2}(\mathbf{U}^{k+1}+\mathbf{U}^{k})-\frac{\mathbf{G}}{2}(\mathbf{U}^{k+1}+\mathbf{U}^{k})+\frac{\mathbf{E}}{2}(\hat{\mu}^{k+1}+\hat{\mu}^{k})=\mathbf{C}^{k+\frac{1}{2}}-\mathbf{H}^{k+\frac{1}{2}}, \tag{3.40}$$

$$\begin{aligned}(\mathbf{K}+\frac{\Delta t}{2}\mathbf{F}-\frac{\Delta t}{2}\mathbf{G})\mathbf{U}^{k+1}+\frac{\Delta t}{2}\mathbf{E}\hat{\mu}^{k+1}&=\Delta t(\mathbf{C}^{k+\frac{1}{2}}-\mathbf{H}^{k+\frac{1}{2}})+(\mathbf{K}-\frac{\Delta t}{2}\mathbf{F}+\frac{\Delta t}{2}\mathbf{G})\mathbf{U}^{k}\\&-\frac{\Delta t}{2}\mathbf{E}\hat{\mu}^{k}.\end{aligned} \tag{3.41}$$

From equation (3.41), assuming that $\hat{u}_i^k$ for $i=1,2,\ldots,\mathrm{N}$ and $\hat{\mu}^k$ are known, our aim is to compute $\hat{u}_i^{k+1}$ for $i=1,2,\ldots,\mathrm{N}$ and $\hat{\mu}^{k+1}$. Now we have N + 1 unknowns, so we need one equation to compute these unknowns, which can be obtained from the non-local boundary condition from equation (3.17), and substituting the trial function $u^h(\mathbf{x})=\sum_{j=1}^{n}\phi_j(\mathbf{x})\hat{u}_j$, yields:

$$\int_{\Omega}(u^h)^{k+1}(\mathbf{x})d\Omega=\int_{\Omega}\sum_{j=1}^{n}\phi_j(\mathbf{x})\widehat{u}_j^{\,k+1}d\Omega=\sum_{j=1}^{n}\left[\int_{\Omega}\phi_j(\mathbf{x})d\Omega\right]\widehat{u}_j^{\,k+1}=m^{k+1}, \tag{3.42}$$

which can be written in matrix form as

$$\mathbf{S}\mathbf{U}^{k+1} = m^{k+1}, \tag{3.43}$$

where **S** is a matrix, described as follows:

$$\mathbf{S} = [S_j], \quad S_j = \int_{\Omega} \phi_j(\mathbf{x}) d\Omega. \tag{3.44}$$

The matrix form equation (3.41) and equation (3.43) can be written in the matrix form as

$$\mathbf{A}\mathbf{U}^{k+1} = \mathbf{B}\mathbf{U}^{k} + \mathbf{C}, \tag{3.45}$$

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{K} + \frac{\Delta t}{2}\mathbf{F} - \frac{\Delta t}{2}\mathbf{G} & \frac{\Delta t}{2}\mathbf{E} \\ \mathbf{S} & 0 \end{pmatrix}, \tag{3.46}$$

$$\mathbf{B} = \begin{pmatrix} \mathbf{K} - \frac{\Delta t}{2}\mathbf{F} + \frac{\Delta t}{2}\mathbf{G} & -\frac{\Delta t}{2}\mathbf{E} \\ \mathbf{O}^{T} & 0 \end{pmatrix}, \tag{3.47}$$

$$\mathbf{C} = \begin{pmatrix} \Delta t(\mathbf{C}^{k+\frac{1}{2}} - \mathbf{H}^{k+\frac{1}{2}}) \\ m^{k+1} \end{pmatrix}, \tag{3.48}$$

$$\mathbf{U}^{k+1} = \begin{pmatrix} \mathbf{U}^{k+1} \\ \hat{\mu}^{k+1} \end{pmatrix}, \tag{3.49}$$

for $i = 1, 2, 3, ..., n;\ j = 1, 2, 3, ..., n.$

Time Discretization II

In this second scheme, we discretize equation (3.33) at time step k+1, obtaining:

$$\mathbf{G}\mathbf{U}^{k+1} - \mathbf{E}\hat{\mu}^{k+1} = \mathbf{H}^{k+1}. \tag{3.50}$$

Subtracting equation (3.50) from equation (3.37), we obtain:

$$\mathbf{K}\frac{\mathbf{U}^{k+1} - \mathbf{U}^{k}}{\Delta t} + \frac{\mathbf{F}}{2}(\mathbf{U}^{k+1} + \mathbf{U}^{k}) - \mathbf{G}\mathbf{U}^{k+1} + \mathbf{E}\hat{\mu}^{k+1} = \mathbf{C}^{k+\frac{1}{2}} - \mathbf{H}^{k+1}, \tag{3.51}$$

$$(\mathbf{K} + \frac{\Delta t}{2}\mathbf{F} - \Delta t\mathbf{G})\mathbf{U}^{k+1} + \Delta t\mathbf{E}\hat{\mu}^{k+1} = \Delta t(\mathbf{C}^{k+\frac{1}{2}} - \mathbf{H}^{k+1}) + (\mathbf{K} - \frac{\Delta t}{2}\mathbf{F})\mathbf{U}^{k}. \tag{3.52}$$

So we need one equation, same as in the first scheme, that can be obtained from the non-local boundary condition from equation (3.17):

$$\int_{\Omega} (u^h)^{k+1}(\mathbf{x})d\Omega = \int_{\Omega} \sum_{j=1}^{n} \phi_j(\mathbf{x})\widehat{u}_j^{\,k+1} d\Omega = \sum_{j=1}^{n} \left[\int_{\Omega} \phi_j(\mathbf{x})d\Omega \right] \widehat{u}_j^{\,k+1} = m^{k+1}, \tag{3.53}$$

which can be written in the matrix form as

$$\mathbf{S}\mathbf{U}^{k+1} = m^{k+1}, \tag{3.54}$$

where **S** is a matrix described as follows:

$$\mathbf{S} = [S_j], \quad S_j = \int_{\Omega} \phi_j(\mathbf{x})d\Omega. \tag{3.55}$$

The matrix form equation (3.52) and equation (3.54) can be written in matrix form as

$$\mathbf{A}\mathbf{U}^{k+1} = \mathbf{B}\mathbf{U}^{k} + \mathbf{C}, \tag{3.56}$$

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{K} + \frac{\Delta t}{2}\mathbf{F} - \Delta t\mathbf{G} & \Delta t\mathbf{E} \\ \mathbf{S} & 0 \end{pmatrix}, \tag{3.57}$$

$$\mathbf{B} = \begin{pmatrix} \mathbf{K} - \frac{\Delta t}{2}\mathbf{F} & \mathbf{O} \\ \mathbf{O}^T & 0 \end{pmatrix}, \tag{3.58}$$

$$\mathbf{C} = \begin{pmatrix} \Delta t(\mathbf{C}^{k+\frac{1}{2}} - \mathbf{H}^{k+1}) \\ m^{k+1} \end{pmatrix}, \tag{3.59}$$

for $i = 1, 2, 3, ..., n;\ j = 1, 2, 3, ..., n.$

3.3.2 Local weak formulation II

In this scheme, we implement the problems under a non-local boundary condition as in equation (3.17). Also, we used the weight function, designed by Most and Bucher (2005) in our implementation. The MLS approximation lacks the Kronecker delta function property, so it should be noted that the weight function leads to the MLS approximation shape functions fulfilling the interpolation condition exactly, enabling a direct application of the Dirichlet boundary conditions, where solving the two-dimensional time-dependent heat conduction equation is shown to be accurate and easier.

The local weak form of equation (3.12) for $\mathbf{x}_i = (x^i, y^i) \in \Omega_s^i$ can be written as follows:

$$\int_{\Omega_s^i} \left(\frac{\partial u}{\partial t} - \lambda \nabla^2 u - Q \right) v_i \, d\Omega = 0, \tag{3.60}$$

where v_i is a test function. Using the divergence theorem and

$(\nabla^2 u) v_i = \frac{\partial}{\partial x}(\frac{\partial u}{\partial x} v_i) + \frac{\partial}{\partial y}(\frac{\partial u}{\partial y} v_i) - (\frac{\partial u}{\partial x}\frac{\partial v_i}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v_i}{\partial y})$ yields the following expression:

$$\int_{\Omega_s^i} \frac{\partial u}{\partial t} v_i \, d\Omega - \int_{\partial\Omega_s^i} \lambda \frac{\partial u}{\partial n} v_i \, d\Gamma + \int_{\Omega_s^i} \lambda \frac{\partial u}{\partial x}\frac{\partial v_i}{\partial x} + \lambda \frac{\partial u}{\partial y}\frac{\partial v_i}{\partial y} \, d\Omega - \int_{\Omega_s^i} Q v_i \, d\Omega = 0, \tag{3.61}$$

where Ω_s^i is a circle of radius r_0 centered at $\mathbf{x}_i$, $\partial\Omega_s^i$ is the boundary of Ω_s^i, $n=(n_x,n_y)$ is the outward unit normal to the boundary $\partial\Omega_s^i=\Gamma_{su}^i\cup\Gamma_{sq0}^i\cup\Gamma_{sq1}^i$, where Γ_{su}, Γ_{sq0} and Γ_{sq1} are parts of the boundary Γ at each support-domain,

$$\frac{\partial u}{\partial n}=\frac{\partial u}{\partial x}n_x+\frac{\partial u}{\partial y}n_y,$$

and yields the following expression:

$$\begin{aligned}&\int_{\Omega_s^i}\frac{\partial u}{\partial t}v_i d\Omega-\int_{\Gamma_{su}^i}\lambda\frac{\partial u}{\partial n}v_i d\Gamma-\int_{\Gamma_{sq0}^i}\lambda\frac{\partial u}{\partial n}v_i d\Gamma-\int_{\Gamma_{sq1}^i}\lambda\frac{\partial u}{\partial n}v_i d\Gamma+\int_{\Omega_s^i}\lambda\frac{\partial u}{\partial x}\frac{\partial v_i}{\partial x}\\&+\lambda\frac{\partial u}{\partial y}\frac{\partial v_i}{\partial y}d\Omega-\int_{\Omega_s^i}Qv_i d\Omega=0.\end{aligned}\tag{3.62}$$

Substituting the trial function $u^h(\mathbf{x},t)=\sum_{j=1}^{n}\phi_j(\mathbf{x})\hat{u}_j(t)$ and $q\equiv\dfrac{\partial u}{\partial n}$ into equation (3.62) yields:

$$\begin{aligned}&\int_{\Omega_s^i}v_i\sum_{j=1}^{n}\phi_j\frac{\partial\hat{u}_j}{\partial t}d\Omega-\int_{\Gamma_{su}^i}\lambda v_i\sum_{j=1}^{n}\frac{\partial\phi_j}{\partial n}\hat{u}_j d\Gamma-\int_{\Gamma_{sq0}^i}g_0\lambda v_i d\Gamma-\int_{\Gamma_{sq1}^i}g_1\lambda v_i d\Gamma\\&+\int_{\Omega_s^i}\lambda\sum_{j=1}^{n}\frac{\partial\phi_j}{\partial x}\frac{\partial v_i}{\partial x}\hat{u}_j+\lambda\sum_{j=1}^{n}\frac{\partial\phi_j}{\partial y}\frac{\partial v_i}{\partial y}\hat{u}_j d\Omega-\int_{\Omega_s^i}Qv_i d\Omega=0,\end{aligned}\tag{3.63}$$

$$\begin{aligned}&\sum_{j=1}^{n}\int_{\Omega_s^i}v_i\phi_j d\Omega\;\frac{\partial\hat{u}_j}{\partial t}-\sum_{j=1}^{n}\int_{\Gamma_{su}^i}\lambda v_i\frac{\partial\phi_j}{\partial n}d\Gamma\;\hat{u}_j+\sum_{j=1}^{n}\int_{\Omega_s^i}\lambda\frac{\partial\phi_j}{\partial x}\frac{\partial v_i}{\partial x}+\lambda\frac{\partial\phi_j}{\partial y}\frac{\partial v_i}{\partial y}d\Omega\;\hat{u}_j\\&=\int_{\Gamma_{sq0}^i}g_0\lambda v_i d\Gamma+\int_{\Gamma_{sq1}^i}g_1\lambda v_i d\Gamma+\int_{\Omega_s^i}Qv_i d\Omega,\end{aligned}\tag{3.64}$$

where $i=1,2,3,\ldots,n.$

By substituting the trial function from equation (2.37) into equation (3.62), the governing equations are transformed into the discretized system, which can be written in the matrix form as

$$\mathbf{K}\frac{\partial\mathbf{U}}{\partial t}+\mathbf{F}\mathbf{U}=\mathbf{C}(t),\tag{3.65}$$

where, **K**, **F** and **C** are matrices described as follows:

$$\mathbf{K}=[K_{ij}],\quad K_{ij}=\int_{\Omega_s^i}\phi_j v_i d\Omega, \tag{3.66}$$

$$\mathbf{F}=[F_{ij}],\ F_{ij}=\int_{\Omega_s^i}\frac{\partial\phi_j}{\partial x}(v_i)_x+\frac{\partial\phi_j}{\partial y}(v_i)_y d\Omega+\int_{\Gamma_{su0}^i}v_i\frac{\partial\phi_j}{\partial y}d\Gamma\ -\int_{\Gamma_{su1}^i}v_i\frac{\partial\phi_j}{\partial y}d\Gamma, \tag{3.67}$$

$$\mathbf{C}(t)=[C_i(t)],\quad C_i(t)=\int_{\Gamma_{sq0}^i}g_0 v_i d\Gamma+\int_{\Gamma_{sq1}^i}g_1 v_i d\Gamma+\int_{\Omega_s^i}Qv_i d\Omega. \tag{3.68}$$

Setting a time-stepping scheme to overcome the time derivative and applying the Crank-Nicolson technique of approximation to equation (3.65) yields:

$$\mathbf{K}\frac{\mathbf{U}^{k+1}-\mathbf{U}^{k}}{\Delta t}+\frac{\mathbf{F}}{2}(\mathbf{U}^{k+1}+\mathbf{U}^{k})=\mathbf{C}^{k+\frac{1}{2}} \tag{3.69}$$

which can be rearranged into:

$$(\mathbf{2K}+\Delta t\mathbf{F})\mathbf{U}^{k+1}=(\mathbf{2K}-\Delta t\mathbf{F})\mathbf{U}^{k}+2\Delta t\mathbf{C}^{k+\frac{1}{2}} \tag{3.70}$$

where $\mathbf{U}$ is a matrix described as follows:

$$\mathbf{U}=\begin{pmatrix}\hat{\mathbf{U}}\\ \hat{\mu}\end{pmatrix}. \tag{3.71}$$

From equation (3.70), assuming that $\hat{u}_i^k$ for $i=1,2,\ldots,\mathrm{N}$ and $\hat{\mu}^k$ are known, our aim is to compute $\hat{u}_i^{k+1}$ for $i=1,2,\ldots,\mathrm{N}$ and $\hat{\mu}^{k+1}$. Now we have N + 1 unknowns, so we need one equation to compute these unknowns, which can be obtained from the non-local boundary condition from equation (3.17), and substituting the trial function $u^h(\mathbf{x})=\sum_{j=1}^{n}\phi_j(\mathbf{x})\hat{u}_j$, yields:

$$\int_{\Omega}(u^h)^{k+1}(\mathbf{x})d\Omega = \int_{\Omega}\sum_{j=1}^{n}\phi_j(\mathbf{x})\widehat{u}_j^{\,k+1}d\Omega = \sum_{j=1}^{n}\left[\int_{\Omega}\phi_j(\mathbf{x})d\Omega\right]\widehat{u}_j^{\,k+1} = m^{k+1}, \tag{3.72}$$

which can be written in matrix form as

$$\mathbf{S}\mathbf{U}^{k+1} = m^{k+1}, \tag{3.73}$$

where **S** is a matrix described as follows:

$$\mathbf{S} = [S_j], \quad S_j = \int_{\Omega}\phi_j(\mathbf{x})d\Omega. \tag{3.74}$$

We use the matrix forms in equation (3.70) and equation (3.73) to compute $\hat{u}_i^{k+1}$ and $\hat{\mu}^{k+1}$ for nodes located inside the domain and on the Neumann boundary conditions. For nodes on the Dirichlet boundary conditions, we have to impose boundary conditions as follows:

For nodes $\mathbf{x}^l = (x^l, 1)$ on the top horizontal boundary $(0 \le x^l \le 1)$, using equation (3.20), we have

$$\widehat{u}_i^{k+1}(\mathbf{x}^l) = h_1(\mathbf{x}^l, (k+1)\Delta t). \tag{3.75}$$

For nodes $\mathbf{x}^l = (x^l, 0)$ on the bottom horizontal boundary $(0 \le x^l \le 1)$, using equation (3.21), we have

$$\widehat{u}_i^{k+1} - h_0(\mathbf{x}^l)\hat{\mu}^{k+1} = 0. \tag{3.76}$$