CHAPTER 1 INTRODUCTION

1.1 Rationale

Problems described by partial differential equations (PDEs) are useful tools for modeling physical phenomena and other systems, e.g., financial markets, structural mechanics, fluid flow, and heat transfer (Pollard, 1964). Many papers require the researcher to solve PDEs using numerical methods. There are three methods that are well-known: finite elements, finite differential methods and finite volume. The three methods are similar processes in the first step required, creating a mesh or element within the domain. If the domain has a simple shape such as a circle, rectangle or cube, the mesh is not difficult to create. General problems are in complex domains for which a mesh is not easy for the three methods to create, and require a long time in the process for creating the mesh, restrictions on mesh, and the efficiency in solving problems are lower.

There are many works of research that have used the three methods to develop numerical methods to perform the process of creating a mesh. Also, there has also been much research into getting rid of the elements and meshes in the process of numerical methods. This method is called the meshless method.

The Meshless Method (Liu, 2002) is used to establish a system of algebraic equations for the whole problem domain without the use of a predefined mesh for discretization of the domain. The Meshless methods use a set of nodes scattered within the problem domain, as well as sets of nodes scattered on the boundaries of the domain, to represent (not discretize) the problem domain and its boundaries. These sets of scattered nodes are called field nodes, and they do not form a mesh, meaning it does not require any *a priori* information on the relationship between the nodes for the interpolation or approximation of the unknown functions of field variables. The main objective of the meshless methods is to get rid of, or at least alleviate the difficulty of, meshing and remeshing the entire structure; by only adding or deleting nodes in the entire structure instead. Meshless methods may also alleviate some other problems associated with the finite element method, such as locking, element distortion, and others.

The initial idea of meshless methods dates back to the smooth particle hydrodynamics (SPH) method for modeling astrophysical phenomena (Gingold, 1977). Research into meshless methods only became very active after the publication of the Diffuse Element Method (Nayroles, 1992). Several so-called meshless methods, such as the Element Free Galerkin (EFG) method (Belytschko, 1994); Reproducing Kernel Particle Method (RKPM) (Liu, 1996); the Partition of Unity Finite Element Method (PUFEM) (Babuska, 1997); hpcloud method (Duarte, 1996); Natural Element Method (NEM) (Sukumar, 1998); and Meshless Galerkin methods using Radial Basis Functions (RBF) (Wendland, 1998), have also been reported in literature since then.

The major differences in these meshless methods, all of which may be classified as Galerkin methods, come only from the techniques used for interpolating the trial function. Even though no mesh is required in these methods for the interpolation of the trial and test functions for the solution variables, the use of shadow elements is inevitable in these methods, for the integration of the symmetric weak-form, or of the energy. Therefore, these methods are not truly meshless.

The meshless local boundary integral equation (LBIE) method, and the meshless local Petrov-Galerkin (MLPG) method are two truly meshless methods, which have been developed by Atluri and Zhu (1998a; 1998b) and Atluri, et al. (1999a; 1999b) for solving linear and non-linear boundary problems.

During the past 20 years the following meshless methods have been introduced:

1. Smooth Particle Hydrodynamics (SPH).
2. Meshless Local Petrov-Galerkin (MLPG).
3. Meshless Methods Base on Radial Basis Function (RBF).
4. Finite Point Methods (FPM).
5. Meshless Boundary Schemes (MBS).

For a review of meshless methods and their computer implementations, refer to Katz (2009).

In this research, a Meshless Local Petrov-Galerkin method is presented to treat the heat conduction equation with the Dirichlet, Neumann and non-local boundary conditions. The MLPG method was first discovered by Atluri and Zhu (1998a; 1998b). This method has been applied widely and very successfully in recent years. The method is based on local weak forms and moving least squares (MLS) approximation. This method has been described in textbooks by Atluri (2004), Liu and Gu (2005) as allowing for freedom to choose the test function. The main advantage of the MLPG method is that it only requires nodes and a description of the external and internal boundary conditions, therefore, no element connectivity, neither total nor part, is needed. Effective implementations of MLPG method are key to success (Atluri, 1998a; Lin, 2000; Atluri, 2000; Atluri, 2002c; Abbasbandy, 2010).

Finding the numerical solution with non-local boundary conditions is important for research in many fields of science and engineering, such as chemical diffusion, diffusion equation, thermoelasticity, heat conduction process, heat transfer, control theory, medical science and so on (Capasso, 1988; Day, 1982; Cannon, 1990; Wang, 1989; Martin-Vaquero, 2009; Abbasbandy, 2010; Kazem, 2012; Mauricio, 2012; Syed, 2011; Pisano, 2009). The problem with all these phenomena characteristic problems as well as a nonclassical parabolic equation, diffusion equation with integral condition found application. It is most widely used, and very important, in thermoelasticity. In 1963, Cannon (1963) first introduced non-local boundary condition problems, and most investigations developed various problems with one dimension, two dimensions, the Dirichlet boundary condition, and the Neumann boundary condition. In 2010, Abbasbandy and Shirzadi (2010) researched the MLPG method for two-dimensional diffusion equations with the Neumann boundary condition and the non-classical boundary condition, and a meshless method for two-dimensional diffusion equations with an integral condition. The proposed method worked very well for the two-dimensional diffusion equations with a non-classical boundary condition, because of its simplicity and high accuracy.

1.2 Literature Review

Most research has focused on the heat conduction equation, where many research studies have used different methods to obtain answers with high accuracy and low computation time. Boundary conditions of the problem vary according to the researcher's interests relevant to the research, presented as follows:

Wu and Tao (2007) applied the meshless local Petrov-Galerkin (MLPG) method to compute two steady-state heat conduction problems of irregular complex domain in two-dimensional space. The moving least-squares approximation (MLS) is applied to construct the trial functions, and the transformation method is employed to deal with the essential boundary condition. The MLPG results have been compared with the results of FVM, and the results of MLPG method are in good agreement with those obtained by FVM. The present study demonstrates that the MLPG method is a highly-accurate numerical method for problems within irregular domains.

Mehdi (2002) presented three fully explicit finite-difference schemes: the standard FTCS method, the 9-point finite-difference scheme, and the (1,13) fully explicit technique, which were applied to the two-dimensional diffusion equation with an integral condition replacing a one boundary condition. The last technique worked very well for two-dimensional diffusion with an integral condition, because of its fourth-order accuracy. These methods seem particularly suited for parabolic partial differential equations with continuous boundary conditions.

Mehdi (2003) presented two new methods for one-dimensional parabolic equations, with non-local boundary specifications replacing boundary conditions. One of the new techniques developed can be implemented on a parallel architecture at every time step. This technique works very well for one-dimensional diffusion with integral conditions. For the model problem considered, the parallel algorithm in equation (36) was found to be about 10 times faster than both the implicit finite difference scheme presented in this article and the Crank-Nicolson technique of Wang (1990). A comparison with the one existing sequential technique in the literature clearly demonstrates that the new parallel technique is computationally superior. The numerical tests obtained by using this method give acceptable results and suggests convergence to the exact solution when h goes to zero. Further work remains to be done in several areas, such as efficient numerical techniques for the higher-dimensional case, computational schemes for Neumann boundary conditions and also developing numerical methods for the case where the non-local boundary specifications have more complicated structures. In these cases, the solutions will be more difficult to compute.

Marjan (2008) used derivation and implementation of a numerical solution of a time-dependent diffusion equation given in detail, based on the meshless local Petrov Galerkin method (MLPG). A simple method is proposed that ensures a constant number of support nodes for each point. Numerical integrations are carried out over local square domains. The implicit Crank-Nicolson scheme is used for time discretization. A detailed convergence study was performed experimentally to optimize the number of support nodes, quadrature domain size and other parameters. The accuracy of the MLPG solution is compared with that of standard methods on a unit square and on an irregularly shaped test domain. As expected, the finite difference method on a regular mesh is uncompetitive on irregularly shaped domains. MLPG is significantly more accurate when using moving least square shape functions of degree two than with degree one. It is comparable to the finite element method of degree two in the H_1 error norm and about two times less accurate in the L_2 error norm.

Martin-Vaquero and Vigo-Aguiar (2008) used the modeled by non-classical parabolic boundary value problems with non-local boundary conditions, and several methods were compared to approach the numerical solution of the one-dimensional heat equation subject to the specifications of mass. One of them was the Crandall formula which the scheme showed in paper is of order $O(h^2)$, not of order $O(h^4)$. However, it is possible with several changes to derive a Crandall algorithm of order $O(h^4)$. The new method compared with the previous is efficient and results on the same tests produce errors 10^3 to 10^5 times smaller with the new scheme.

Abbasbandy and Shirzadi (2010) presented a new approach based on the meshless local Petrov-Galerkin (MLPG) and collocation methods to treat the parabolic partial differential equations with non-classical boundary conditions. The MLPG method is applied to the interior nodes, while the meshless collocation method is applied to the nodes on the boundaries, and so the Dirichlet boundary condition is imposed directly. The proposed method worked very well, because of its simplicity and high accuracy.

Abbasbandy and Shirzadi (2010) used a meshless local Petrov-Galerkin (MLPG) method to treat parabolic partial differential equations with Neumann's and non-classical boundary conditions. A difficulty in implementing the MLPG method is imposing boundary conditions. To overcome this difficulty, two new techniques are presented for use on square domains. These techniques are based on the finite differences and the Moving Least Squares (MLS) approximations. Non-classical integral boundary conditions are approximated using Simpson's composite numerical integration rule and the MLS approximation. Two test problems are presented to verify the efficiency and accuracy of the method. Also, from experience, we understood that for problems with Neumann's boundary conditions, our proposed algorithms have higher precision than the classical MLPG.

Syed and Ahmet (2011) employed the HAM for the solutions of two-dimensional diffusion equations subject to non-standard boundary specifications. Unlike the traditional techniques used by other numerical algorithms, the solutions here are given in series forms, which can lead to exact closed form solutions. The approximate solutions to the equations were computed without any need for transformation techniques, linearization and discretization, and then compared with exact solutions. It was shown that the method is reliable, efficient and requires less computation.

Most and Bucher (2005) presented the application of the MLS interpolation, solved with a new weighting function, which makes the MLS interpolation more attractive especially within a Galerkin method. A new weighting function was designed for meshless shape functions to fulfill these essential conditions with very high accuracy without any additional effort. Due to the approximative character of this interpolation, the obtained shape functions do not fulfill the interpolation conditions, which causes additional numerical effort for the application of the boundary conditions. This will be made clear by pointing out that the choice of the base polynomial is arbitrary; thus, the accuracy can be increased by choosing higher order polynomials.

Yang, et al. (2011) used an improved hybrid boundary node method (hybrid BNM) for solving steady fluid flow problems, Miao, et al. (2009) used a meshless hybrid boundary-node method for Helmholtz problems, and Wang, et al. (2011) used the multi-domain hybrid boundary node method for three-dimensional elasticity. These works of research used the Hybrid BNM, proposed by Zhang, et al. (2002), for potential and elasticity problems, which is a new boundary type meshless method developed by Miao, et al. (2005). Belytschko, et al. (1996) used this method, combining MLS approximation and a modified variational principle. It only requires nodes constructed on the boundary of the domain, and does not require any mesh for the interpolation of variables, nor for the integration. The accuracy of the hybrid BNM is rather high. However, shape functions for the classical MLS approximation lack the delta function property, for which the paper by Yang, et al. (2011) used the improved MLS approach in the hybrid boundary node method. They proposed adopting a regularized weight function and used a new weight function designed by Most and Bucher. This method leads to the MLS shape functions fulfilling the interpolation condition exactly, which enables a direct application of essential boundary conditions without additional numerical effort.

From the literature review, meshless methods with new techniques are used to solve many problems. In this research, the researchers used the MLPG method to treat the two-dimensional time-dependent heat conduction equation subject to non-local boundary conditions. Two techniques were presented in implementation: the first technique uses a penalty parameter to impose at Dirichlet’s boundary conditions and Neumann’s boundary conditions and the second technique improves the weight function in the classical MLS method by using the new weight function designed by Most and Bucher.

1.3 Objective of the Thesis

The purpose of this study is to develop a meshless method to treat the two-dimensional time-dependent heat conduction equation subject to non-local boundary conditions, reducing calculation time, reducing memory cost, and increasing accuracy.

1.4 Scope of the Thesis

In this study, we consider the two-dimensional problem, time-dependent heat conduction equation and non-local boundary conditions.