EMPLOYING CONCRETE MODELS IN A SCAFFOLDED LEARNING UNIT TO FACILITATE SECONDARY AND TERTIARY STUDENTS' DERIVATION OF BASIC SERIES FORMULAS

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ABSTRACT

There are two aspects of this thesis: the mathematical aspect and the educational aspect. In the purely mathematical part, the researcher has presented a geometric derivation of Pascal's formula for sums of powers of integers and has extended the derivation to the formula for sums of powers of arithmetic progressions. In addition, another geometric derivation of the formula for sums of cubes is presented, together with a procedure to generate the coefficients of a polynomial formula for sums of powers of integers when the polynomial is in $\left(n+\frac{1}{2}\right)$.

In the educational part, the researcher has presented two instructional units that employ the newly developed geometric derivations of the formulas for sums of integers, sums of squares, and sums of cubes in the instructional activities.

The first instructional unit encourages students to participate in the derivation of the formulas. The effectiveness of this instructional unit on high-school students was compared to that of traditional instruction. Due to the limitation of the sample, those receiving traditional instruction were higher achievers judging from the previous semester's average mathematics scores. The results indicated that while both groups could recall the formulas almost perfectly in the post-test, the treatment group significantly outperformed the control group in the retention test. However, none could recall the derivations at the time the retention test was administered.

The second instructional unit utilizes the geometric derivations to facilitate students' derivation of the formulas. The instructional activities require groups of students to derive the formulas using only the provided illustrations, and, in the case of sums of squares, concrete models. The instructor monitors and provides suitable scaffolds. This instructional unit was implemented on high-school students and pre-service mathematics teachers. The former significantly outperformed the latter in the pre-test. While both groups could derive the formulas in the activities, most of the latter could not do so in the post-test, indicating that only some of them contributed during the group activities and those who did could not transfer their understanding of the derivations to those who did not.

KEY WORDS: GEOMETRIC DERIVATION / PASCAL'S FORMULA / SUMS OF POWERS OF INTEGERS / SCAFFOLDING / CONCRETE MODEL / INSTRUCTIONAL UNIT / HIGH-SCHOOL STUDENT / PRE-SERVICE TEACHER / COOPERATIVE LEARNING

116 pages

การใช้แบบจำลองที่จับต้องได้ในบทเรียนทางคณิตศาสตร์แบบเสริมต่อการเรียนรู้เพื่ออำนวยความสะดวกสำหรับนักเรียนระดับมัธยมศึกษาตอนปลายและอุดมศึกษาในการหาที่มาของสูตรของอนุกรมพื้นฐาน

EMPLOYING CONCRETE MODELS IN A SCAFFOLDED LEARNING UNIT TO FACILITATE SECONDARY AND TERTIARY STUDENTS' DERIVATION OF BASIC SERIES FORMULAS

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บทคัดย่อ

งานในวิทยานิพนธ์นี้แบ่งเป็นสองส่วน ส่วนแรกเกี่ยวกับคณิตศาสตร์ เสนอการใช้รูปทรงเรขาคณิตในการหาสูตรของปาสคาลสำหรับผลรวมของกำลังของจำนวนเต็ม

ส่วนที่สองเป็นการนำงานทางคณิตศาสตร์ที่เกี่ยวกับการหาสูตรของผลรวมของกำลังต้น ๆ ของจำนวนเต็ม ไปจัดกิจกรรมการเรียนการสอนในหน่วยการเรียนรู้เรื่องอนุกรมสองหน่วย

หน่วยแรกให้ผู้เรียนมีส่วนร่วมในการหาสูตร หน่วยการเรียนรู้นี้ถูกเปรียบเทียบกับการเรียนการสอนแบบดั้งเดิมในแง่ของประสิทธิผล ซึ่งปรากฏว่าผู้เรียนทั้งสองกลุ่มจำสูตรได้อย่างแม่นยำ แต่ผู้เรียนในกลุ่มทดลองคงความทรงจำได้นานกว่าถึงแม้จะเป็นกลุ่มที่ความสามารถทางคณิตศาสตร์ด้อยกว่า ขณะที่ไม่มีผู้เรียนคนใดคงความทรงจำเกี่ยวกับวิธีการหาสูตรได้

หน่วยที่สองจัดเป็นกิจกรรมกลุ่มให้ผู้เรียนแต่ละกลุ่มหาสูตรด้วยตนเองโดยใช้เพียงรูปประกอบที่มาจากงานทางคณิตศาสตร์และแบบจำลองที่จับต้องได้ โดยผู้สอนคอยดูและเสริมต่อการเรียนรู้ตามความเหมาะสม ผู้เรียนประกอบด้วยนักเรียนชั้นมัธยมศึกษาตอนปลายและนักศึกษาฝึกหัดครู ทั้งสองกลุ่มสามารถหาสูตรได้ในการทำกิจกรรม แต่นักศึกษาฝึกหัดครูเกือบทั้งหมดซึ่งเป็นกลุ่มที่ความสามารถทางคณิตศาสตร์ด้อยกว่า ไม่สามารถบอกวิธีหาสูตรได้ในแบบทดสอบหลังเรียน

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LIST OF ABBREVIATIONS

AIM	achievement in mathematics
ATM	attitude toward mathematics
SD	standard deviation
SE	standard error of the means
TIMSS	Trends in International Mathematics and Science Study
ZPD	zone of proximal development

CHAPTER I
INTRODUCTION

Overview

This chapter describes the background and the rationale for the research study. The objectives of the study, the research questions, and the definitions of symbols and terms are also presented.

1.1 Background for the Research Study

This research study focuses on instruction in the formulas for sums of integers, sums of squares, and sums of cubes, which will be collectively referred to as basic summation formulas throughout this document. These formulas are generally taught at the high-school level, with the exception of the first formula, to which students are usually exposed during the middle-school years. Typical instruction at the high-school level begins with the presentation of the formula, coupled with the explanations of the symbols that the instructor deemed unfamiliar to the students, followed by a mathematical proof of the formula. One of the most popular proving techniques employed in this topic is proving by induction. On the one hand, these formulas lend themselves to proofs by induction due to the simplicity of the recursive definitions of the summations. Formally, let p denote the power of integers in a sum. The summation can be recursively defined as

$$S_p(0) = 0, \qquad \text{and}$$

$$S_p(n) = S_p(n-1) + n^p, \qquad n = 1, 2, 3, \ldots$$

From the definition, one can see that $S_p(n)$ can be obtained from $S_p(n-1)$ quite straightforwardly by adding a monomial term to it. Thus, the second step of the proof by induction, wherein one has to proof that the formula is true for $(n+1)$ given that it is true for n, is equally straightforward. On the other hand, using an inductive proof as the first and only proof of the formula can put students under the impression that the formula already exists and all one needs to do is proving that it is correct. So, proper instruction should begin with how to obtain the formula, not how to prove it.

There are several strategies that can be used to obtain basic summation formulas, some of which employ more advanced mathematics. Since this research study focuses on instruction that can be implemented at the high-school level, only the strategies involving simple mathematics will be discussed. These strategies can be classified into four categories as follows:

1.1.1 Algebraic derivation

An algebraic derivation employs algebraic manipulation to arrive at the required formula. For example, if one knows that $S_p(n)$ is a polynomial in n of degrees $(p+1)$, a well-known fact that is far from obvious, one can find the coefficients by solving a system of linear equations for the first $(p+2)$ values of the formula. When $p = 1$, the system of linear equations for the first three values is

$$a0^2 + b0 + c = 0 \Rightarrow c = 0 \tag{1.1}$$

$$a1^2 + b1 + c = 0 + 1 \underset{(1.1)}{\Longrightarrow} a + b = 1 \tag{1.2}$$

$$a2^2 + b2 + c = 0 + 1 + 2 \Rightarrow 4a + 2b = 3.$$

Solving the equations yields $a = b = \frac{1}{2}$, resulting in the formula

$$S_1(n) = \frac{n^2+n}{2} = \frac{n(n+1)}{2}. \tag{1.3}$$

Equation (1.1) implies that there is no constant term in these formulas while equation (1.2) means that the sum of their coefficients is always one.

This derivation typifies derivations in this category, i.e., they are rather straightforward. In addition, the set-up (in this case, the system of linear equations) for the sums of a higher power of integers can be discerned without much difficulty. However, the terms in the formula seem to have no meaning other than being just the results of some algebraic manipulation.

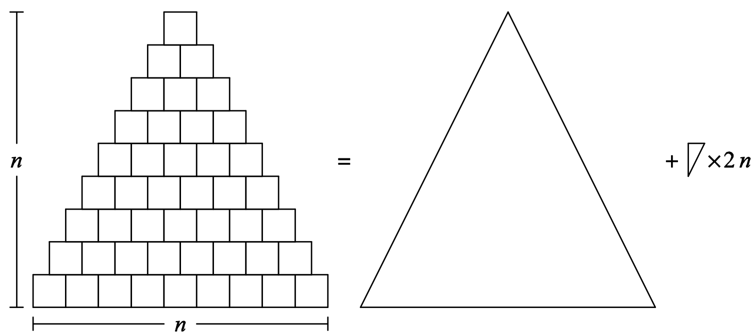


Figure 1.1 Adding right triangles to inscribed triangle yields sum of integers.

1.1.2 Geometric derivation

In a geometric derivation, one represents the required sum geometrically and tries to find its measure from other geometric objects with known measures. For example, if one represents 1 with a unit square, a sum of integers can be represented by the stack of rectangles in Figure 1.1, whose area can be found by adding the area of all the small right triangles to that of the inscribed triangle. Since each tier contains two small right triangles, each with an area of a quarter, and the area of the inscribed triangle is $\frac{n^2}{2}$, the sum of these areas yields equation (1.3).

One can see that this derivation is also straightforward. In addition, the derivation gives a geometric interpretation of each term in the resulting formula, which is the most important advantage of a geometric derivation in general. In this case, the term $\frac{n^2}{2}$ is the area of the inscribed triangle and the term $\frac{n}{2}$ is the total area of all the small right triangles needed to transform the inscribed triangle into the stack of rectangles that represents the required sum. It is also quite straightforward to extend the derivation to find the formula for sums of squares, which will be covered in section 2.1.3.6, but it is much harder to do the same for sums of cubes or of higher powers of integers because it would take us into the fourth dimension and beyond.

1.1.3 Combinatorial derivation

A combinatorial derivation utilizes two ways of counting something where one of the two ways yields the required sum directly. For example, suppose there are $(n + 1)$ people attending a party, how many handshakes will there be if each person has to shake hand with everyone else exactly once? Of course, this is a simple combinatorial question whose answer is $\binom{n+1}{2}$ where the notation $(\cdot)$ denotes the binomial coefficient. Another way to find the answer is by considering one person at a time. The first person has to shake hand with n others. The second person has to shake hand with $(n-1)$ others, excluding the first person because that handshake has just been counted. Continuing in this fashion, there is only one handshake unaccounted for when the n^{th} person is considered. Thus, the total number of handshakes is $(1+2+3+\cdots+n)$. Equating the two answers yields equation (1.3).

A question for this kind of derivation is very specific and usually has nothing to do with sums of other powers of integers. Besides, it can be quite difficult to find an answer and, unless one has an excellent feel for permutations and combinations, the answer tends to have very little meaning. In this case, one may wonder why the sum from 1 to n is equivalent to the number of ways of selecting 2 things out of $(n+1)$ things.

Another approach to combinatorial derivation utilizes two sets whose numbers of members can be easily found. The number of members of a set is usually the required sum while that of the other is the required formula. One then needs to find a bijection between the two sets to establish the equality.

1.1.4 Heuristic derivation

A heuristic derivation can be algebraic, geometric, or even combinatorial. Its typical feature is the ingenious arrangement of numbers or figures that helps simplify the derivation. The following familiar derivation of the formula for sums of integers serves as a good example:

$$S_1(n) = 1 + 2 + \cdots + (n-1) + n$$
$$S_1(n) = n + (n-1) + \cdots + 2 + 1$$
$$2S_1(n) = (n+1) + (n+1) + \cdots + (n+1) + (n+1) = n(n+1).$$

This last equation yields equation (1.3). Figure 1.2 illustrates the geometric version of this derivation.

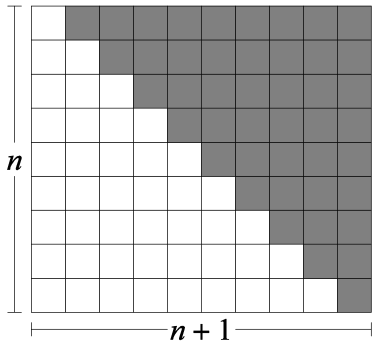


Figure 1.2 Geometric version of the familiar derivation of the formula for $S_1(n)$

Due to the ingenious arrangement, this kind of derivation usually gives an interpretation of the resulting formula. In this case, n is the number of integers to be added and $(n + 1)$ is twice the average of the integers. However, like a combinatorial derivation, the arrangement is very specific and cannot be extended to derive the formulas for sums of other powers of integers.

1.2 Rationale for the Investigation

From the researcher's experience, simply demonstrating a derivation or a proof of a basic summation formula to students has very little effect on their understanding: they usually end up committing the formula to memory for the purpose of the exam, only to lose it soon after the exam is over, without a way to find it again except by looking it up. After a while, some of them cannot make sense of the formula presented to them because it involves quite a few symbols (the summation sign, the index, and the two parameters) that alienate them. The main reason might be that even though students can follow the derivation or the proof, they do not thoroughly understand the idea behind it and thus resort to rote learning of the formula. So, the key is to find a way to help them see the idea behind the derivation. To accomplish this, students need to be deeply involved in the derivation from the start, which means that the derivation has to be manageable to at least some of the students and it has to be meaningful to almost all of them. This narrows the kinds of derivations down to geometric or heuristic ones.

According to the findings by the committee on *How People Learn*, the second principle states that *understanding requires factual knowledge and conceptual frameworks* (Fuson, Kalchman, & Bransford, 2005, p. 231). A significant part of factual knowledge in mathematics can be thought of as procedural fluency, which can be achieved by repeatedly applying the procedure to a variety of problems. However, without conceptual understanding, students tend to forget the knowledge they have no opportunity utilizing. Conceptual frameworks help support an effective organization of knowledge so as to facilitate recall, strategy development, and adaptive reasoning. Since a heuristic derivation is usually very specific, it is difficult to organize a collection of heuristic derivations into a common conceptual framework. That leads the researcher to the first objective of this research study, i.e., to find a family of geometric derivations for sums of powers of integers. It is hopeful that the first few derivations in this family will be so manageable that students can be let to rediscover them after being provided with only the set-ups. As an extra benefit, understanding the first few derivations could help some of them see the general pattern in the family and might provoke a few to rediscover the whole family and experience the joy of mathematical discovery.

Figure 1.1 and 1.2 illustrate that a geometric derivation of the formula for sums of integers resides in a two-dimensional space. The idea behind these derivations suggests that a derivation of the formula for sums of squares would reside in a three-dimensional space and that for sums of cubes in a four-dimensional space, which is beyond most students' imaginations. Fortunately, the equality between sums of cubes and squares of sums of integers implies that a geometric derivation of the formula for sums of cubes can be constructed two-dimensionally. There are several two-dimensional geometric derivations that exclusively employ squares to elicit the equality. Surprisingly, from the researcher's experience, demonstrating this kind of derivation to students leads to confusion about the equality between 1^3 and 1^2, perhaps due to their forming tight connections between 1^3 and a cube and between 1^2 and a square. Thus, the second goal of this research study is to find a geometric derivation of the formula for sums of cubes that does not involve any square.

Traditional instruction in mathematics places more emphasis on procedural fluency than on conceptual understanding. When introducing a new topic that relies heavily on some formulas, the formulas are presented to students right away, with or without their derivations. If the derivations are also demonstrated, they are done without students' participation, with the effect that almost all of the students lose interest in or cannot follow the derivations right from the start. So, they resort to rote learning of the formulas. This is a loss of opportunity to foster reasoning and sense-making ability in students in general and to help them develop conceptual understanding of the topic in particular. In fact, most, if not all, mathematical formulas from the primary-school level to the high-school level are sufficiently manageable for students not only to understand but also to rediscover them. One could argue that the time saved on later reviews and remedies more than compensates for that spent on the activities that help students rediscover the formulas, perhaps several times over.

Even though the derivations of most formulas are manageable for students, one cannot just state the problems and let them find the solutions on their own resources. One needs to provide them with at least some helpful initial set-ups to make sure that they are actively engaged in the rediscovering activities, together with relevant and useful suggestions along the way to keep their attention on the tasks when faced with difficulties. In other words, one needs to *scaffold* students' derivation of the formulas. The term *scaffolding* was introduced by Wood, Bruner, and Ross (1976) to describe the role of an expert in facilitating the problem solving of a novice. They defined scaffolding as the *process that enables a child or novice to solve a problem, carry out a task or achieve a goal which would be beyond his unassisted efforts* (p. 90). Wood et al. (1976) did not provide a theory behind scaffolding but Rogoff and Wertsch (1984) indicated the connection between scaffolding and Vygotsky's (1978) *zone of proximal development* (ZPD), which was defined as *the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers* (p. 86). From the two definitions, one can see that scaffolding is one of the possible processes that enable the learner to benefit from being in the ZPD.

ZPD is one of the three tenets of Vygotsky's theory of teaching, learning, and development, which emphasizes the role of learning in fostering the learner's developmental level. Wood et al. (1976) might have similar idea while investigating the scaffolding process, as evidenced by their supposition that the scaffolding process *may result, eventually, in development of task competence by the learner at a pace that would far outstrip his unassisted efforts* (p. 90). They also drew analogy between the acquisition of skill in completing a task and the advancement of problem-solving skill in general. So, in order to fulfill students' potential, one needs to engage them in a challenging activity that is properly scaffolded whenever possible. From this perspective, it is imperative to design and develop mathematics instruction that not only focuses on conceptual understanding but also provides impetus for the development of higher psychological processes. So, the third aim of this research study is to develop a scaffolded learning unit on the derivations of basic summation formulas wherein students have opportunity to employ mathematical thinking in order to rediscover the formulas. In order for such a learning unit to be effective, students would have to be given ample time to think things through and to discuss with their peers and with the teacher. As a result, the instruction would be considerably longer than a traditional one in basic summation formulas, which makes them incomparable. Thus, another, time-comparable, learning unit that utilizes worksheets and class participation is developed so as to compare the effectiveness of geometric derivations with that of algebraic derivations in helping students recall and retain basic summation formulas. This learning unit will be called the *funneled* learning unit throughout this document.

Most mathematical functions are defined analytically but their computations are implemented numerically. This is true even for some discrete functions whose arguments and outcomes are both integers, due to the limitation on the number of significant digits of the calculating devices. Sums of powers of integers are an example of such functions. To implement the calculations of these functions on a calculating device, one may use Jacques Bernoulli's (1654–1705) formulas, which express these functions as polynomials in n (Bernoulli, 1959). The implementation of Bernoulli's formulas is simple enough for an average programmer. However, their rates of convergence, judging from the numerical values of the most significant terms in the formulas, are inferior to those of Johann Faulhaber's (1580–1653) formulas, which express these functions as polynomials in $S_1(n)$ (Edwards, 1986; Knuth, 1993), with the caveat that the latter is not that easy to implement. So, the last objective of this research study is to find a family of formulas for sums of powers of integers that is almost as simple as Bernoulli's but with rates of convergence comparable to that of Faulhaber's.

1.3 Objectives and Research Questions

This research study has four main objectives.

1. To develop a family of geometric derivations of the formulas for sums of powers of integers, whose derivations of the formulas for sums of integers and for sums of squares are manageable for high-school students and pre-service mathematics teachers.
2. To develop a geometric derivation of the formula for sums of cubes that does not involve a square.
3. To develop a learning unit on the derivations of basic summation formulas for high-school students and pre-service mathematics teachers that utilizes the newly developed geometric derivations as parts of the scaffolding processes.
4. To develop a family of formulas for sums of powers of integers that is easy to implement and converges quickly to the true sums.

These four objectives lead to the following four questions.

1. To what extent can the funneled learning unit enhance high-school students' performance on and retention of basic summation formulas, compared to traditional instruction?
2. What are high-school students' perceptions of the funneled learning unit?
3. To what extent can the scaffolded learning unit facilitate high-school students' and pre-service mathematics teachers' derivation of basic summation formulas?
4. What are high-school students' and pre-service teachers' perceptions of the scaffolded learning unit?

1.4 Definitions of Symbols and Terms

Due to the lengthiness of mathematical symbols and their names, the following symbols and terms will be used throughout this dissertation.

$S_p(n) = \sum_{i=0}^{n} i^p = 0^p + 1^p + 2^p + 3^p + \cdots + n^p$ where $0^0 \stackrel{\text{def}}{=} 0$

Summation	A sum that can be represented by $S_p(n)$
Basic summation	A sum of integers, a sum of squares, or a sum of cubes
General summation	A family of sums of powers of integers, i.e., $S_p(n), p = 0, 1, 2, 3, ...$
Progression summation	A family of sums of powers of arithmetic progressions, i.e., $\sum_{i=0}^{n-1}(a + id)^p, p = 1, 2, 3, ...$

The four terms above can be used as adjectives, especially with the noun *formula*.

The following terms will be used to distinguish between the learning units.

Funneled learning unit	The learning unit used to assess the effectiveness of newly developed derivations compared to a traditional learning unit
Scaffolded learning unit	The learning unit aimed to facilitate students' derivation of basic summation formulas

The following terms will be used to refer to different theories on learning and development.

Constructivism	Piaget's theory of learning and development
Social constructionism	Vygotsky's theory of learning and development

CHAPTER II
LITERATURE REVIEW

Overview

This chapter summarizes the literature related to the content of the thesis. It comprises two parts, namely, mathematical review and educational review. The mathematical review focuses on derivations of summation formulas. The educational review focuses on Vygotsky's social constructionism, scaffolding, and the use of models.

2.1 Mathematical Review

The derivations of summation formulas can be classified into two categories according to the number of powers of integers in the derivations: those for a single integral power and those for all positive-integer powers. Due to its historical significance, the latter will be reviewed first, followed by the former, which will be further separated into those for sums of integers, for sums of squares, and for sums of cubes.

2.1.1 Derivations of general summation formulas

The derivations will be presented more or less chronologically. Those that require mathematics beyond the high-school level will be omitted. Most of the information in this section came from an article by Janet Beery (2009).

2.1.1.1 *Abu Ali al-Hasan ibn al-Hasan ibn al-Haytham's (965–1039) possible derivation*

According to Beery (2009), Al-Haytham was probably the first mathematician to come up with a derivation of a general summation formula. Al-Haytham might have used the diagram similar to Figure 2.1 to arrive at the recursive formula

$$(n+1)S_p(n) = S_{p+1}(n) + \sum_{i=1}^{n} S_p(i). \qquad (2.1)$$

There are two ways to find the area of the biggest rectangle in the figure. The more fundamental way is to multiply its height by its width, resulting in the left side of equation (2.1). The other way is to add the areas of all the smaller rectangles together, yielding the right side of the equation. To see how one can get a summation formula from the equation, consider the equation when $p = 0$.

$$(n+1)S_0(n) = S_1(n) + \sum_{i=1}^{n} S_0(i)$$

$$(n+1)n = S_1(n) + S_1(n)$$

$$S_1(n) = \frac{n(n+1)}{2}.$$

Once S_1 is known, S_2 can be found from the equation using $p = 1$.

$$(n+1)S_1(n) = S_2(n) + \sum_{i=1}^{n} S_1(i)$$

$$\frac{n(n+1)^2}{2} = S_2(n) + \frac{S_2(n)}{2} + \frac{n(n+1)}{4}$$

$$\frac{3}{2}S_2(n) = \frac{n(n+1)}{2}\left(n+1-\frac{1}{2}\right)$$

$$S_2(n) = \frac{n(n+1)(2n+1)}{6}.$$

Continuing this way, one can find all the summation formulas. Although this derivation is rather elegant, the last term in equation (2.1) actually contains two summation signs, making the derivation too complicated for average high-school students, except when $p = 0$ where the inner summation is just the sum of 1s. Figure 2.2 shows that when $p = 0$, Al-Haytham's derivation is equivalent to the familiar derivation described in section 1.1.4.

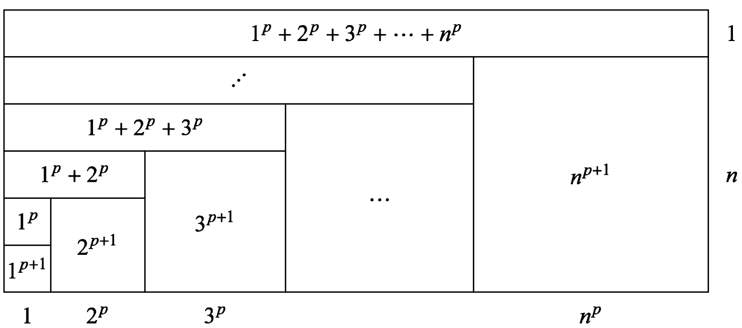


Figure 2.1 Al-Haytham's possible derivation of general summation formula

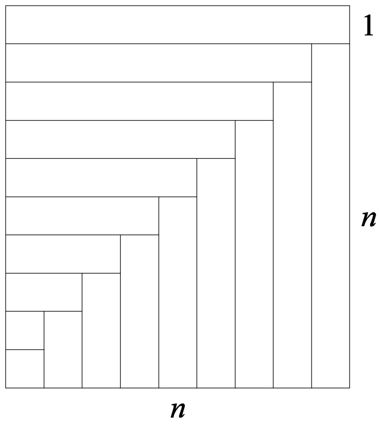


Figure 2.2 Al-Haytham's derivation of the formula for sums of integers

Table 2.1 Harriot's difference table for sums of integers

S_1 (p^2)	First difference (p)	Second difference (e)
1	1	1
3	2	1
6	3	1
10	4	1
15	5	1
21	6	1
28	7	1
36	8	1

Table 2.2 Harriot's difference table for sums of squares

S_2 (p^3)	First difference (p^2)	Second difference (p)	Third difference (e)
1	1	1	2
5	4	3	2
14	9	5	2
30	16	7	2
55	25	9	2
91	36	11	2
140	49	13	2
204	64	15	2

Table 2.3 Harriot's difference table for sums of cubes

S_3 (p^4)	First diff. (p^3)	Second diff. (p^2)	Third diff. (p)	Fourth diff. (e)
1	1	1	0	6
9	8	7	6	6
36	27	19	12	6
100	64	37	18	6
225	125	61	24	6
441	216	91	30	6
784	343	127	36	6
1296	512	169	42	6

Table 2.4 Harriot's general difference table

p^4	p^3	p^2	p	e
$p^4+p^3+p^2+p+e$	p^3+p^2+p+e	p^2+p+e	$p+e$	e
$p^4+2p^3+3p^2+4p+5e$	$p^3+2p^2+3p+4e$	$p^2+2p+3e$	$p+2e$	e
$p^4+3p^3+6p^2+10p+15e$	$p^3+3p^2+6p+10e$	$p^2+3p+6e$	$p+3e$	e
$p^4+4p^3+10p^2+20p+35e$	$p^3+4p^2+10p+20e$	$p^2+4p+10e$	$p+4e$	e
$p^4+5p^3+15p^2+35p+70e$	$p^3+5p^2+15p+35e$	$p^2+5p+15e$	$p+5e$	e
$p^4+6p^3+21p^2+56p+126e$	$p^3+6p^2+21p+56e$	$p^2+6p+21e$	$p+6e$	e
$p^4+7p^3+28p^2+84p+210e$	$p^3+7p^2+28p+84e$	$p^2+7p+28e$	$p+7e$	e

2.1.1.2 *Thomas Harriot's (1560–1621) difference table*

Harriot's motivation for his derivation of a general summation formula probably came from observing that the differences of consecutive integers, the second-level differences of consecutive squares, the third-level differences of consecutive cubes, and so forth are constant. Thus, the second-level differences of consecutive sums of integers, the third-level differences of consecutive sums of squares, and the fourth-level differences of consecutive sums of cubes are also constant. Tables 2.1–2.3 demonstrate the difference tables similar to those constructed by Harriot. The first columns in these tables are the values of $S_p(n)$. If (r,c) denotes the entry at row r, column c, then $(r,c)=(r,c-1)-(r-1,c-1)$ provided that the two entries between which the difference is taken exist. After all the differences have been taken, the last column is filled with the constant and the remaining entries can be obtained by extrapolation.

Harriot named the columns e, p, p^2, p^3, and so forth and used the column names to also denote their first rows (Beery, 2009). Using these notations, he investigated the entries in the difference table if only the first row and the constant column were known. The entries can be constructed from the relation $(r, c) = (r-1, c) + (r, c+1)$, resulting in a table similar to Table 2.4 (Beery & Stedall, 2009, p. 9). One can see that the coefficients in each row are the same across columns provided that one looks at their orders, not at the column names to which the coefficients associate. To help the investigation, let us put them into Table 2.5.

Table 2.5 Coefficients from Table 2.4

S_p	First difference	Second difference	Third difference	Fourth difference
1	0	0	0	0
1	1	1	1	1
1	2	3	4	5
1	3	6	10	15
1	4	10	20	35
1	5	15	35	70
1	6	21	56	126
1	7	28	84	210

One may notice the familiar pattern in Table 2.5. In fact, it is Pascal's triangle rotated forty five degrees counterclockwise. So, these coefficients can be represented by binomial coefficients. According to Beery and Stedall (2009), Harriot learned about the formulas for these coefficients through Girolamo Cardano's (1501–1576) work on combinations (pp. 6–7). Cardano described the formula for the row 1, 11, 55, 165, 330, 462, 330, 165, 55, 11, 1 starting from 11 as follows:

> Or if there are eleven [objects], I want to know quickly the numbers that arise from three choices. First, for the second place, I take 1 from 11 which makes 10, I divide by 2, the number of the place, there comes out 5, I multiply with 11 to

> make 55, the number of the second place. Next I subtract 2, which is the number of the difference of all the places from the first, from 11, there remains 9, I divide 9 by 3, the number of the place, there comes out 3, I multiply 3 by 55, the second number, to make 165, the number in the third place. Similarly, if I want the number of four choices, I take 3, the difference of 4 from the first place, from 11, there is left 8, I divide 8 by 4, the number of the place, there comes out 2, I multiply 2 by 165 to make 330, the number of the fourth place. Similarly for the fifth I subtract 4, the difference from the first place, there remains 7, I divide by 5, the number of the place, there comes out $1\frac{2}{5}$, I multiply by 330, the number in the previous place, to make 462, the number of the fifth place (p. 7).

Notice that the entries in Table 2.5 satisfy the relation $(r, c) = \binom{r+c-3}{c-1}$ where $\binom{-1}{0} = \binom{0}{0} = 1$ and $\binom{i}{j} = 0$ for other pairs of i and j such that $i < j$. The term $(c-1)$ can be interpreted as the level of difference. Thus, if p_r^d denotes the entry at row r of column p^d, then Table 2.4 implies that

$$p_r^d = \sum_{i=0}^{d} \binom{r+d-i-2}{d-i} p^i \qquad (2.2)$$

where $p^0 = e$ and $p^1 = p$. But if p^d is the first column in a difference table, then p_r^d would be the r^{th} element in the summation being investigated. So, the formula for that summation can be found by plugging the values from the first row of the corresponding difference table into equation (2.2). For sums of integers, the first row of Table 2.1 yields the formula

$$\begin{aligned} S_1(n) &= \binom{n-2}{0}1 + \binom{n-1}{1}1 + \binom{n}{2}1 \\ &= 1 + (n-1) + \frac{n(n-1)}{2} \\ &= \frac{n(n+1)}{2} \\ &= \binom{n+1}{2}. \end{aligned}$$

Similarly, the first row of Table 2.2 yields the following formula for sums of squares:

$$\begin{aligned} S_2(n) &= 1 + (n-1) + \binom{n}{2} + \binom{n+1}{3} 2 \\ &= \binom{n+1}{2} + 2\binom{n+1}{3} \\ &= \binom{n+2}{3} + \binom{n+1}{3} \\ &= \tfrac{n(n+1)}{6}(n+2+n-1) \\ &= \tfrac{n(n+1)(2n+1)}{6}. \end{aligned}$$

Here, the basic combinatorial identity $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ is employed. For sums of cubes, the top row of Table 2.3 yields the formula

$$\begin{aligned} S_3(n) &= 1 + (n-1) + \binom{n}{2} + \binom{n+1}{3} 0 + \binom{n+2}{4} 6 \\ &= \binom{n+1}{2} + \binom{n+2}{4} 6 \\ &= \tfrac{n(n+1)}{4}[2 + (n+2)(n-1)] \\ &= \left[\tfrac{n(n+1)}{2}\right]^2 \\ &= S_1(n)^2. \end{aligned} \tag{2.3}$$

Since his associates preferred to see the formulas in terms of n, Harriot transformed all the binomial coefficients into polynomials in n and used them for all the calculations involving constant difference tables. The significance of Harriot's derivation is that the technique can be applied to any sequence whose differences finally come to a constant, which includes polynomials.

It is interesting to note that while investigating other matters, Harriot employed the relation $(r, c) = (r+1, c-1) - (r, c-1)$ instead of the one described above to construct a general difference table whose resulting coefficients are shown in Table 2.6 (Beery & Stedall, 2009, p. 9) whose entries satisfy the relation

$$(r, c) = \binom{r-1}{c-1}. \tag{2.4}$$

As before, the term $(c-1)$ can be interpreted as the level of difference and the term $(r-1)$ suggests that one should construct the difference table starting with the 0th element of the sequence so as to yield the equation

$$S(n) = \sum_{i=0}^{d} \binom{n}{d-i} p^i \tag{2.5}$$

where $S(n)$ is the n^{th} element in the sequence and d is the level at which the difference becomes constant. This particular technique was rediscovered by Sir Isaac Newton (1642–1727) in 1665 (Beery & Stedall, 2009, p. 4), prompting Conway and Guy (2006) to call it *Newton's useful little formula* (pp. 81–83). To see how this works, let us transform Table 2.3 into Table 2.7 using the new relation.

Table 2.6 Coefficients from Harriot's other works

Sequence	First diff.	Second diff.	Third diff.	Fourth diff.	Fifth diff.
1	0	0	0	0	0
1	1	0	0	0	0
1	2	1	0	0	0
1	3	3	1	0	0
1	4	6	4	1	0
1	5	10	10	5	1
1	6	15	20	15	6
1	7	21	35	35	21

Table 2.7 Newton's difference table for sums of cubes

S_3 (p^4)	First diff. (p^3)	Second diff. (p^2)	Third diff. (p)	Fourth diff. (e)
0	1	7	12	6
1	8	19	18	6
9	27	37	24	6
36	64	61	30	6
100	125	91	36	
225	216	127		
441	343			
784				

Notice that the first entry in the first column of Table 2.7 is now 0 instead of 1. Applying equation (2.5) yields the formula

$$S_3(n) = \binom{n}{0}0 + \binom{n}{1}1 + \binom{n}{2}7 + \binom{n}{3}12 + \binom{n}{4}6$$
$$= n + \binom{n}{2} + 6\left[\binom{n+1}{3} + \binom{n+1}{4}\right]$$
$$= \binom{n+1}{2} + 6\binom{n+2}{4}$$

which is the same as equation (2.3).

2.1.1.3 *Johann Faulhaber's (1580–1635) ingenious insight*

While other mathematicians tried to express summation formulas as polynomials in n, Faulhaber had the incredible insight that $S_{2p-1}(n)$ and $\frac{S_{2p}(n)}{2n+1}$ are polynomials in $S_1(n)$ of degrees p (Edwards, 1986; Knuth, 1993). However, it is not clear how he came up with the insight nor how he obtained the coefficients in these polynomials. So, let us apply the algebraic derivation described in section 1.1.1 in order to find the coefficients for $S_4(n)$ and $S_5(n)$, taking his insight as a given fact.

For brevity, the argument of S_p will be omitted. Suppose that

$$S_4 = (2n+1)(aS_1^2 + bS_1 + c),$$

three equations are needed to determine the coefficients a, b, and c. Evaluating the equation at 0, 1, and 2 yields the following system of equations:

$$0 = 1(0a + 0b + c)$$
$$1 = 3(1a + 1b + c)$$
$$17 = 5(9a + 3b + c).$$

Solving the equations yields $a = \frac{2}{5}, b = -\frac{1}{15}$, and $c = 0$. Thus,

$$S_4 = \frac{2n+1}{15}S_1(6S_1 - 1) = \frac{1}{5}S_2(6S_1 - 1).$$

Now, suppose that

$$S_5 = aS_1^3 + bS_1^2 + cS_1 + d,$$

Evaluating the equation at 0, 1, 2, and 3 yields

$$0 = 0a + 0b + 0c + d$$
$$1 = 1a + 1b + 1c + d$$
$$33 = 27a + 9b + 3c + d$$
$$276 = 216a + 36b + 6c + d.$$

Solving the equations yields $a = \frac{4}{3}, b = -\frac{1}{3}$, and $c = d = 0$. Thus,

$$S_5 = \frac{1}{3} S_1^2 (4S_1 - 1).$$

Notice how simple these formulas turn out to be. It turns out that S_1^2 is a factor of an odd-power summation and S_2 is a factor an even-power summation. Since S_1 is a second-degree polynomial in n, the second terms of the formulas are much smaller than the first, which means that calculating only the first terms would yield close approximations to the true sums.

2.1.1.4 *Blaise Pascal's (1623–1662) telescoping sum*

Although Al-Haytham probably employed the technique that can lead to a general summation formula, he had no interest in such a formula (Beery, 2009). Other derivations yield a formula for a specific power, but the formula cannot be written in a general form. Pascal was the first mathematician who explicitly stated a general summation formula. He utilized the *telescoping-sum* technique (Beery, 2009; Edwards, 1987, pp. 82–83), which will be demonstrated here to find the formula for sums of cubes. Consider the following binomial expansion:

$$(i+1)^4 - i^4 = 4i^3 + 6i^2 + 4i + 1.$$

Writing the equality from $i = 1$ to n yields

$$2^4 - 1^4 = 4 \cdot 1^3 + 6 \cdot 1^2 + 4 \cdot 1 + 1$$
$$3^4 - 2^4 = 4 \cdot 2^3 + 6 \cdot 2^2 + 4 \cdot 2 + 1$$
$$4^4 - 3^4 = 4 \cdot 3^3 + 6 \cdot 3^2 + 4 \cdot 3 + 1$$
$$\vdots$$
$$n^4 - (n-1)^4 = 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 1$$
$$(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1.$$

Summing all the equalities yields

$$(n+1)^4 - 1 = 4S_3(n) + 6S_2(n) + 4S_1(n) + n$$
$$4S_3(n) = (n+1)^4 - 1 - 6S_2(n) - 4S_1(n) - n$$
$$4S_3(n) = (n+1)^4 - (n+1) - n(n+1)(2n+1) - 2n(n+1)$$
$$= (n+1)(n^3 + 3n^2 + 3n + 1 - 1 - 2n^2 - n - 2n)$$
$$= (n+1)(n^3 + n^2)$$
$$= [n(n+1)]^2.$$

Applying this technique to the expansion of $[(i+1)^{p+1} - i^{p+1}]$ yields

$$(n+1)^{p+1} - 1 = \binom{p+1}{1} S_p(n) + \binom{p+1}{2} S_{p-1}(n) + \cdots + \binom{p+1}{1} S_1(n) + n$$

or

$$(n+1)^{p+1} = 1 + \sum_{k=0}^{p} \binom{p+1}{k} S_k(n) \tag{2.6}$$

from which $S_p(n)$ can be found if all the lower-power formulas are known. Rearranging and dividing the equality by $(p+1)$ yields

$$S_p(n) = \frac{(n+1)^{p+1}}{p+1} - \left[\frac{n+1}{p+1} + \sum_{k=1}^{p-1} \frac{\binom{p+1}{p-k}}{p+1} S_{p-k}(n)\right].$$

Since

$$\begin{aligned}\frac{\binom{p+1}{p-k}}{p+1} &= \frac{(p+1)p(p-1)\cdots(p+1-k)}{(k+1)!(p+1)} \\ &= \frac{p(p-1)\cdots(p+1-k)}{k!(k+1)} \\ &= \frac{\binom{p}{k}}{k+1},\end{aligned} \tag{2.7}$$

the general summation formula can be written as

$$S_p(n) = \frac{(n+1)^{p+1}}{p+1} - \left[\frac{n+1}{p+1} + \sum_{k=1}^{p-1} \frac{\binom{p}{k}}{k+1} S_{p-k}(n)\right]. \tag{2.8}$$

In fact, Pascal applied the telescoping-sum technique to derive a progression summation formula, the description of which was translated as follows (Beery, 2009):

> Given any numbers whatever in arithmetic progression, each being raised to the same (integral) power, to find the sum of these powers. Form a binomial having as its first term a literal quantity *A* and for second term the difference of the given progression. Raise this binomial to a power of which the exponent is one more than the power proposed, noting the coefficients of the successive powers of *A* in the resulting development.

Symbolically, this is equivalent to

$$(A+d)^{p+1} = A^{p+1} + (p+1)dA^{p} + \binom{p+1}{2} d^2 A^{p-1} + \cdots + (p+1)d^{p}A + d^{p+1}.$$

> Raise to the same power as the binomial the number which in the progression follows immediately after the last given term. From the result obtained subtract the following:

1st. The first term of the progression—that is, the smallest of the given terms, itself raised to this same power (one greater than the degree proposed).

2nd. The difference of the progression raised to this same power, then multiplied by the number of given terms.

3rd. The sums of similar powers of degree lower than the degree proposed, multiplied, respectively, by the coefficients of the same powers of *A* in the development of the above binomial.

The number which in the progression follows immediately after the last given term is $(a+nd)$ since the last term in the progression of n numbers is $[a+(n-1)d]$. This number is to be raised to the same power as the binomial, i.e., $(a+nd)^{p+1}$. From this number, one has to subtract (1) the first term of the progression, i.e., a, raised to this same power, i.e., a^{p+1}, (2) the difference of the progression, i.e., d, raised to this same power, i.e., d^{p+1}, multiplied by the number of given terms, i.e., nd^{p+1}, and (3) the sum

$$\binom{p+1}{p-k} d^k \sum_{i=0}^{n-1}(a+id)^{p-k},\ k=1,2,3,\ldots,p-1.$$

The remainder found is a multiple of the sum sought, and contains this sum as many times as unity is contained in the coefficient of the power of *A* of which the exponent is equal to the degree of the power proposed.

The coefficient of A^p, the degree of the power proposed, is $(p+1)d$, which is the multiple of the sum sought. Symbolically, the instruction is equivalent to

$$(p+1)d\sum_{i=0}^{n-1}(a+id)^p =$$
$$(a+nd)^{p+1} - a^{p+1} - nd^{p+1} - \sum_{k=1}^{p-1}\binom{p+1}{p-k} d^k \sum_{i=0}^{n-1}(a+id)^{p-k},$$

which, after rearranging and using relation (2.7), can be written as

$$\sum_{i=0}^{n-1}(a+id)^p = \frac{(a+nd)^{p+1}}{(p+1)d} - \left[\frac{a^{p+1}}{(p+1)d} + \sum_{k=1}^{p}\frac{\binom{p}{k}d^k}{k+1}\sum_{i=0}^{n-1}(a+id)^{p-k}\right]. \quad (2.9)$$

2.1.1.5 *Jacques Bernoulli's (1654–1705) numbers*

Bernoulli began his derivation of summation formulas by observing the *table of combinations* (Bernoulli, 1959) which is the same as Table 2.6.

The relations (2.4) and $(r+1, c+1) = \sum_{i=1}^{r}(i, c)$ were well known, which together yields

$$\binom{r}{c} = \sum_{i=1}^{r}\binom{i-1}{c-1}. \qquad (2.10)$$

When $c = 2$, this equation becomes

$$\frac{r(r-1)}{2} = \sum_{i=1}^{r}(i-1)$$
$$\frac{r^2-r}{2} = S_1(r) - r$$
$$S_1(r) = \frac{r^2}{2} + \frac{r}{2}.$$

When $c = 3$, equation (2.10) becomes

$$\frac{r(r-1)(r-2)}{6} = \sum_{i=1}^{r}\frac{(i-1)(i-2)}{2}$$
$$\frac{r^3-3r^2+2r}{6} = \frac{S_2(r)}{2} - \frac{3S_1(r)}{2} + \frac{2r}{2}$$
$$S_2(r) = \frac{r^3-3r^2+2r}{3} + \frac{3r^2}{2} + \frac{3r}{2} - 2r$$
$$= \frac{r^3}{3} + \frac{r^2}{2} + \frac{r}{6}.$$

When $c = 4$, equation (2.10) becomes

$$\frac{r(r-1)(r-2)(r-3)}{24} = \sum_{i=1}^{r}\frac{(i-1)(i-2)(i-3)}{6}$$
$$\frac{r^4-6r^3+11r^2-6r}{24} = \frac{S_3(r)}{6} - \frac{6S_2(r)}{6} + \frac{11S_1(r)}{6} - \frac{6r}{6}$$
$$S_3(r) = \frac{r^4-6r^3+11r^2-6r}{4} + 2r^3 + 3r^2 + r - \frac{11r^2}{2} - \frac{11r}{2} + 6r$$
$$= \frac{r^4}{4} + \frac{r^3}{2} + \frac{r^2}{4}.$$

Reaching this point, Bernoulli (1959) stated that *thus we can step by step reach higher and higher powers* (p. 89). Having found the first ten formulas (p. 90), he aligned their coefficients in the fashion similar to that of Table 2.8. Bernoulli then claimed that *whoever will examine the series as to their regularity may be able to continue the table* (p. 90). This may not be so even though the regularity is rather simple. Besides the fact that all the coefficients of a formula sum to one, which is established in section 1.1.1, the regularity is

$$(r, c) = \frac{r}{r+3-c}(r-1, c). \qquad (2.11)$$

Table 2.8 Bernoulli's (1959) coefficients of the first ten summation formulas

S_1	1/2	1/2									
S_2	1/3	1/2	1/6								
S_3	1/4	1/2	1/4	0							
S_4	1/5	1/2	1/3	0	-1/30						
S_5	1/6	1/2	5/12	0	-1/12	0					
S_6	1/7	1/2	1/2	0	-1/6	0	1/42				
S_7	1/8	1/2	7/12	0	-7/24	0	1/12	0			
S_8	1/9	1/2	2/3	0	-7/15	0	2/9	0	-1/30		
S_9	1/10	1/2	3/4	0	-7/10	0	1/2	0	-3/20	0	
S_{10}	1/11	1/2	5/6	0	-1	0	1	0	-1/2	0	5/66

That is, the coefficients of S_p can be obtained from those of S_{p-1} by multiplying the latter's coefficient in the same position by $\frac{p}{(p+2-i)}$ where i is the position of the coefficient starting from the most significant one. Let a_p^k denote the coefficient of n^k in $S_p(n)$. The ratio of the coefficients implies that $a_p^k = \frac{p}{k} a_{p-1}^{k-1}$. Applying this relation recursively yields

$$a_p^k = a_{p+1-k}^1 \prod_{i=0}^{k-2} \frac{p-i}{k-i}.$$

For the first coefficients, $k = p + 1$, resulting in the *telescoping product*

$$\frac{p}{p+1}\frac{p-1}{p}\cdots\frac{2}{3}\frac{1}{2} = \frac{1}{p+1}.$$

Thus,

$$a_p^{p+1} = \frac{1}{p+1} a_0^1.$$

For the second coefficients, $k = p$, implying that the terms in the product are 1. So,

$$a_p^p = a_1^1.$$

The third coefficients involve another telescoping product $\frac{p}{p-1}\frac{p-1}{p-2}\cdots\frac{4}{3}\frac{3}{2} = \frac{p}{2}$. So,

$$a_p^{p-1} = \frac{p}{2} a_2^1.$$

The product for the fourth coefficients is $\frac{p}{p-2}\frac{p-1}{p-3}\cdots\frac{5}{3}\frac{4}{2} = \frac{p(p-1)}{3\cdot 2}$. So,

$$a_p^{p-2} = \frac{p(p-1)}{3\cdot 2} a_3^1.$$

In general, for $i = 0, 1, 2, \dots, p-1$, one can write

$$a_p^{p+1-i} = \frac{p(p-1)\cdots(p+2-i)}{i(i-1)\cdots 3\cdot 2} a_i^1$$

$$= \frac{1}{p+1} \frac{(p+1)p(p-1)\cdots(p+2-i)}{i(i-1)\cdots 3\cdot 2} a_i^1$$

$$= \frac{\binom{p+1}{i}}{p+1} a_i^1. \tag{2.12}$$

Thus, Bernoulli's general summation formula is

$$S_p(n) = \frac{1}{p+1} \sum_{i=0}^{p} \binom{p+1}{i} a_i^1 n^{p+1-i}, \tag{2.13}$$

from which a summation formula can be found without having to know the lower-power formulas. Instead, one needs $a_i^1, i = 0, 1, 2, \dots, p$, which can be found one at time using the coefficients from formula (2.13) and the fact that they sum to one. That is, a_i^1 can be recursively defined by the two equations:

$$a_0^1 = 1 \qquad \text{and}$$

$$a_p^1 = 1 - \frac{1}{p+1} \sum_{i=0}^{p-1} \binom{p+1}{i} a_i^1, \; p = 1, 2, 3, \dots$$

For example,

$$a_1^1 = 1 - \frac{\binom{2}{0}}{2} a_0^1 = 1 - \frac{1}{2} = \frac{1}{2},$$

$$a_2^1 = 1 - \frac{\binom{3}{0}}{3} a_0^1 - \frac{\binom{3}{1}}{3} a_1^1 = 1 - \frac{1}{3} - \frac{1}{2} = \frac{1}{6}, \qquad \text{and}$$

$$a_3^1 = 1 - \frac{\binom{4}{0}}{4} a_0^1 - \frac{\binom{4}{1}}{4} a_1^1 - \frac{\binom{4}{2}}{4} a_2^1 = 1 - \frac{1}{4} - \frac{1}{2} - \frac{1}{4} = 0.$$

Nonetheless, probably the simplest way to find them is to construct Table 2.8 using the relation (2.11), which can be easily accomplished with a spreadsheet.

2.1.1.6 *Gottfried Wilhelm Leibniz' (1646–1716) derivation*

The chronology of Bernoulli's and Leibniz' discovery was uncertain but since Bernoulli was Leibniz' mentor in mathematics, the former should be mentioned first. Leibniz' derivation is quite simple, comparable to the algebraic derivation presented in section 1.1.1. First, he assumed that $S_p(n)$ is a polynomials in n of degrees $(p+1)$. He then proceeded to take the difference between two consecutive terms of the summation and match the coefficients (Leibniz, 2008, pp. 51–52). Let us consider the derivation of the formula for sums of squares:

$$S_2(n) = an^3 + bn^2 + cn;$$
$$n^2 = S_2(n) - S_2(n-1) = a[n^3 - (n-1)^3] + b[n^2 - (n-1)^2] + c$$
$$= 3an^2 - 3an + a + 2bn - b + c.$$

Comparing the coefficients yields $a = \frac{1}{3}, b = \frac{1}{2}$, and $c = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}$. Although this derivation requires one to solve a system of linear equations, the system is usually very easy to solve because the coefficient matrix is a triangular one.

Table 2.9 Witmer's (1935) coefficients of the first ten summation formulas

S_0	1	-1/2										
S_1	1/2	0	-1/8									
S_2	1/3	0	-1/12	0								
S_3	1/4	0	-1/8	0	1/64							
S_4	1/5	0	-1/6	0	7/240	0						
S_5	1/6	0	-5/24	0	7/96	0	-1/128					
S_6	1/7	0	-1/4	0	7/48	0	-31/1344	0				
S_7	1/8	0	-7/24	0	49/192	0	-31/384	0	17/2048			
S_8	1/9	0	-1/3	0	49/120	0	-31/144	0	127/3840	0		
S_9	1/10	0	-3/8	0	49/80	0	-31/64	0	381/2560	0	-31/2048	
S_{10}	1/11	0	-5/12	0	7/8	0	-31/32	0	127/256	0	-2555/33792	0

2.1.1.7 *Witmer's (1935) polynomials*

Witmer (1935) knew that S_{2p-1} and $\frac{S_{2p}}{2n+1}$ are polynomials in S_1 of degrees p. Perhaps by reasoning that since $\left(n + \frac{1}{2}\right)$ is the average of n and $(n + 1)$, and is also a half of $(2n + 1)$, he investigated general summation formulas when they are polynomials in $\left(n + \frac{1}{2}\right)$. Intuitively, this should yield formulas whose first terms give approximations that are almost as good as those of Faulhaber's. In addition, there should be cohesion between the formulas for even and odd powers. Burrows and Talbot (1984) focused on the approximation issue of the same polynomials. Both groups utilized Pascal's telescoping sum technique to find the formulas for the coefficients, which are rather complicated looking. Witmer (1935) also presented the formulas up to the tenth power, whose coefficients are shown in Table 2.9. For example,

$$S_0(n) = \left(n + \tfrac{1}{2}\right) - \tfrac{1}{2},$$

$$S_1(n) = \tfrac{1}{2}(n + \tfrac{1}{2})^2 - \tfrac{1}{8},$$

$$S_2(n) = \tfrac{1}{3}(n + \tfrac{1}{2})^3 - \tfrac{1}{12}\left(n + \tfrac{1}{2}\right), \text{ and}$$

$$S_3(n) = \tfrac{1}{4}(n + \tfrac{1}{2})^4 - \tfrac{1}{8}(n + \tfrac{1}{2})^2 + \tfrac{1}{64}.$$

2.1.1.8 *Paul's (1971) combinatorial derivation*

It is fascinating that a combinatorial derivation can be used to derive a general summation formula. Paul (1971) did just that for Pascal's formula (2.6) by employing two ways of counting the number of integral points in a $(p+1)$-dimensional *cube* whose sides contain the integers 0, 1, 2, … , n. Obviously, there are $(n+1)^{p+1}$ points. The trick is to partition them into $(p+1)$ disjoint sets whose numbers of points match the terms in the summation sign. To accomplish this, he associated with each k in the formula the set of points for which exactly $(p+1-k)$ coordinates have a common value and the remaining k coordinates are strictly smaller. For example, when $k = 0$, all the coordinates have a common value, which means that only the $(n+1)$ diagonal points are in the set. When $k = 1$, all but one coordinates have a common value, which is larger than that of the singled-out coordinate. Since there are $\binom{p+1}{1}$ ways of selecting the singled-out coordinate, and for each common value $i = 1, 2, 3, \dots, n$, there are i ways of selecting a lower value, the number of points is $\binom{p+1}{1} S_1(n)$. When $k = 2$, there are $\binom{p+1}{2}$ ways of selecting the coordinates having the largest value, and for each largest value i, there are i^2 ways of selecting two lower values. Thus, the number of points is $\binom{p+1}{2} S_2(n)$. Continuing this line of argument, it is not too difficult to see how formula (2.6) emerges.

Figure 2.3 illustrates Paul's (1971) derivation when $p = 1$ after replacing points with unit squares. The gray squares correspond to $k = 0$ and the white squares correspond to $k = 1$. One can see that there are two ways to select the lower coordinate, resulting in two copies of white-square steps, each of which represents $S_1(n)$. Similarly, Figure 2.4 shows that, when $p = 2$, the black unit cubes correspond to $k = 0$, the three steps, one of which is directly behind the black cubes in Figure 2.4(a), correspond to $k = 1$, and the three empty spaces that can be filled with step pyramids correspond to $k = 2$. One such pyramid is upside-down while the other two point *into* the paper from Figure 2.4(b)'s perspective. In this illustration, the point (0, 0, 0) is at the bottom of Figure 2.4(a) whose right steps correspond to $y < x = z$ and left steps correspond to $x < y = z$. The remaining steps, which can be seen most clearly from Figure 2.4(c), correspond to $z < x = y$. Again, one can see that there are three ways to single out the lower coordinate, resulting in three steps that represent $S_1(n)$. Similarly, the step pyramid at the front of Figure 2.4(a) corresponds to $x, y < z$, the one on the right corresponds to $y, z < x$, and the one to the left corresponds to $x, z < y$. Hence, there are three ways to select two lower coordinates, resulting in three copies of the pyramids, each of which represents $S_2(n)$. Since one can visualize the derivation geometrically, it should be classified as a geometric rather than a combinatorial derivation of formula (2.6). However, looking at these figures without referring to the formula, one may classify the corresponding derivations as heuristic.

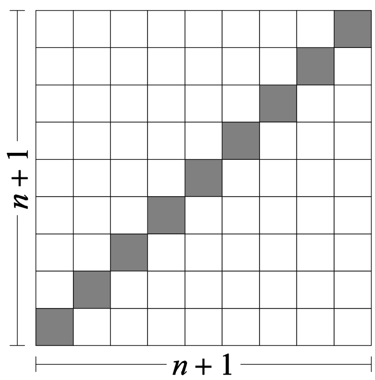


Figure 2.3 A geometric version of Paul's (1971) derivation when $p = 1$

2.1.2 Derivations of formulas for sums of integers

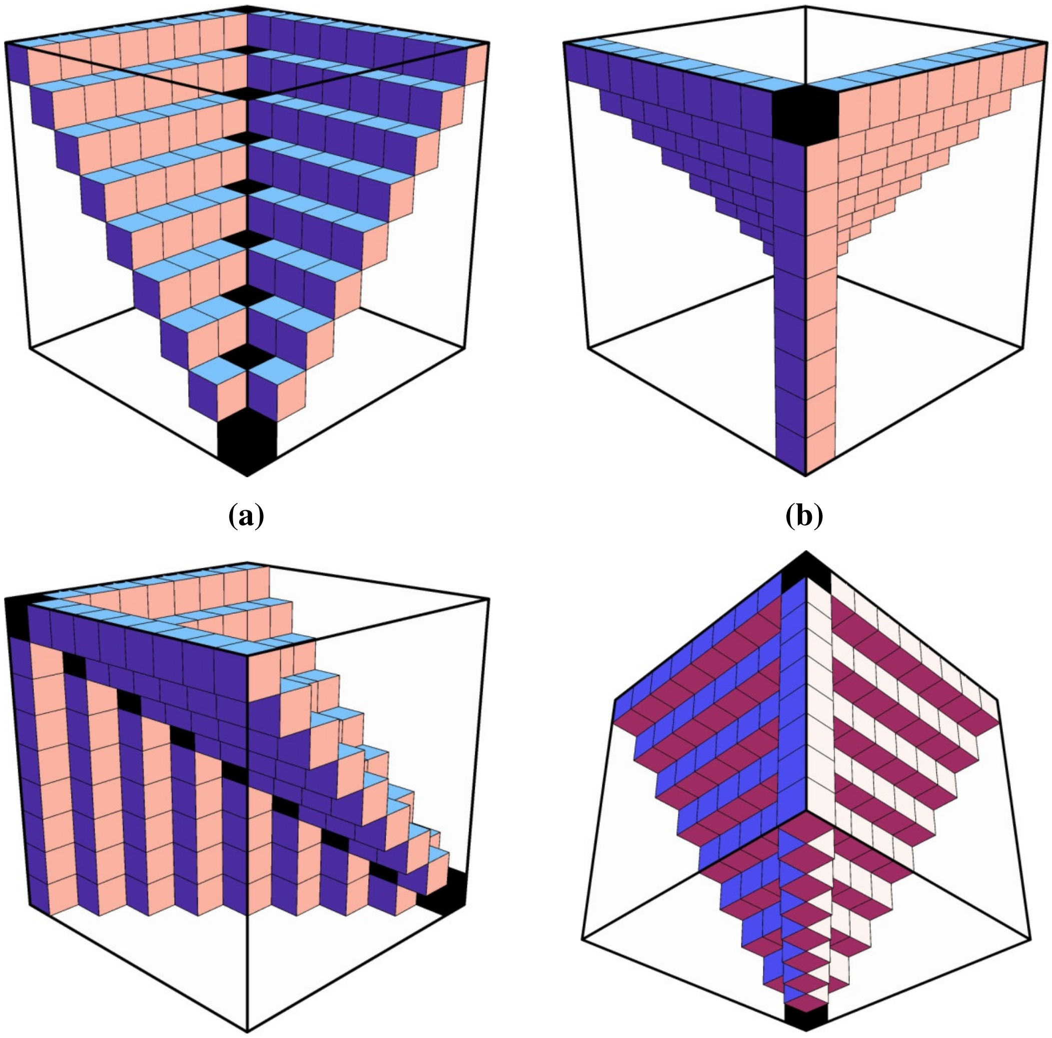
There are not that many variations on the derivations of the formula for sums of integers, unless one counts all the combinatorial questions whose answers are sums of integers. In this section, three geometric derivations will be presented.

2.1.2.1 *Turner's (1980) square completion*

Turner's (1980) derivation is based on the fact that

$$S_1(n) = n^2 - S_1(n-1). \tag{2.14}$$

Coupled with the definition that $S_1(n) = n + S_1(n-1)$, one can solve for $S_1(n)$. Figure 2.5 demonstrates Turner's (1980) geometric interpretation of the former equation, which can also be classified as a heuristic derivation.



(a) (b) (c) (d)

Figure 2.4 Front (a), rear (b), left (c), and rear-bottom (d) views of a geometric version of Paul's (1971) derivation when $p = 2$

2.1.2.2 *Richards' (1984) geometric derivation*

Richards' (1984) derivation is a purely geometric one. In addition, it is similar to the one described in section 1.1.2, with the rectangles stacked to the left rather than in the middle, as shown in Figure 2.6.

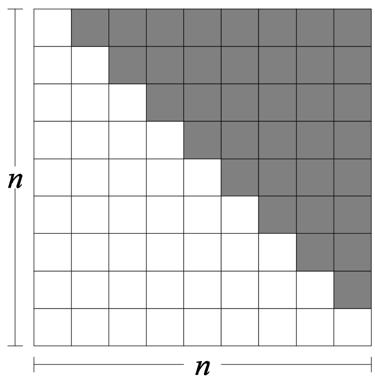


Figure 2.5 Turner's (1980) geometric interpretation of $S_1(n) = n^2 - S_1(n-1)$

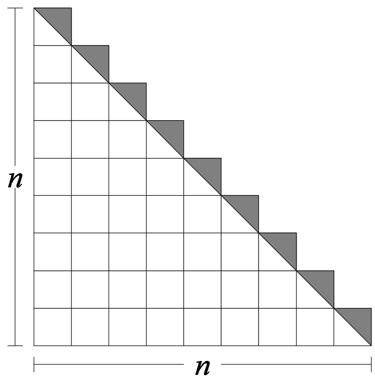


Figure 2.6 Richards' (1984) derivation of the formula for sums of integers

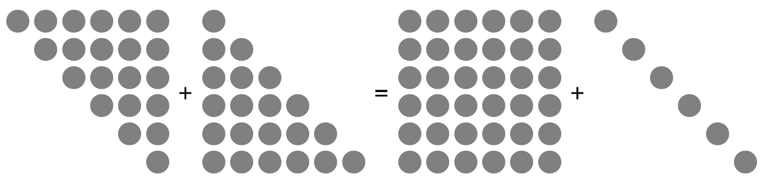


Figure 2.7 Farlow's (1995) derivation of the formula for sums of integers

2.1.2.3 *Farlow's (1995) geometric derivation*

Farlow's (1995) derivation, shown in Figure 2.7, is equivalent to two copies of Richards' (1984) derivation, and, in effect, avoids the use of triangles.

2.1.3 Derivations of formulas for sums of squares

The formula for sums of squares is more complicated than other basic summation formulas, leading to more complicated non-algebraic derivations. One has seen a heuristic/geometric derivation of the formula in Figure 2.4. A few other derivations employ the similar idea of arranging step pyramids to form a cuboid. Two of these derivations will be presented.

2.1.3.1 *Turner's (1977) cube completion*

Turner's (1977) derivation is based on the relation

$$S_2(n) = n^3 - 2S_2(n-1) - S_1(n-1).$$

Her interpretation of the formula can be seen by imagining that all the unit cubes of the steps seen from the front view of Figure 2.4(a) is a part of the upside-down step pyramid. If the sides of the big cube are of lengths n, then the new upside-down pyramid would represent $S_2(n)$ while the other two step pyramids would represent $S_2(n-1)$. In addition, the remaining steps behind the new pyramid would represent $S_1(n-1)$, resulting in the required formula.

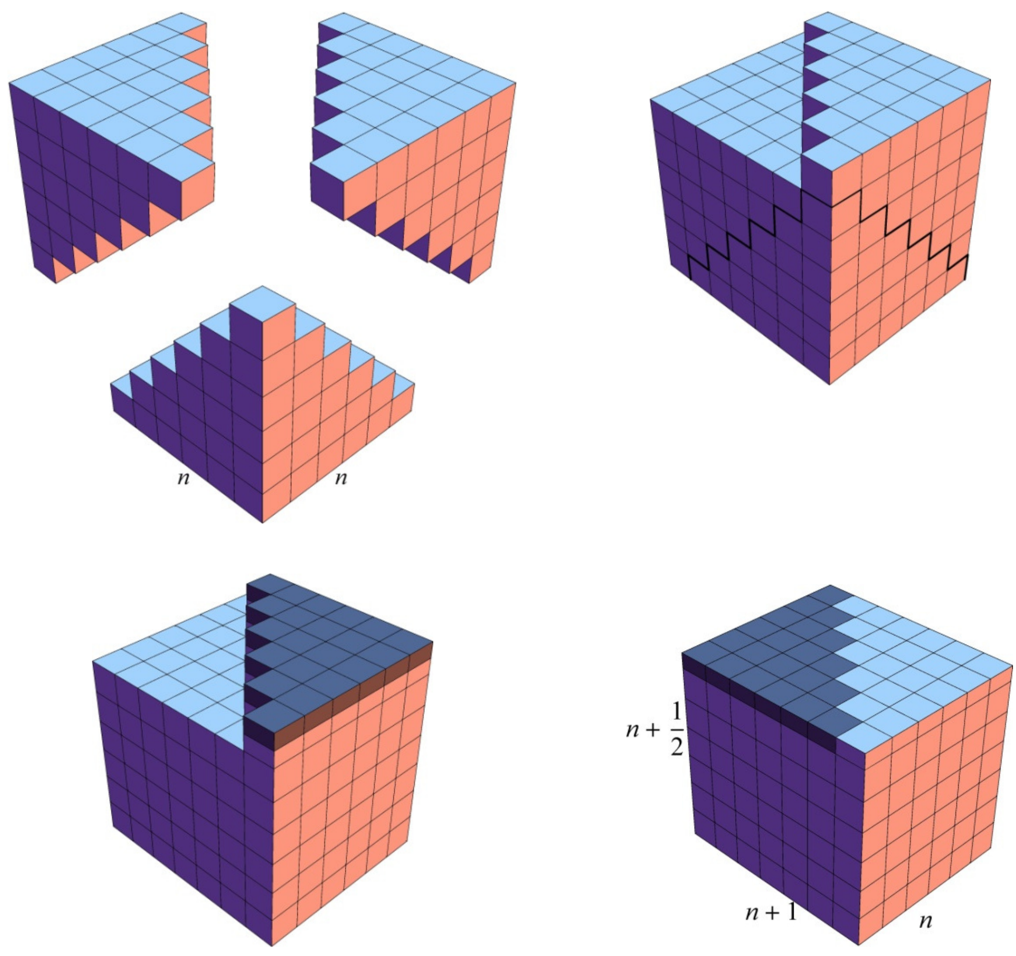


Figure 2.8 Siu's (1984) derivation of the formula for sums of squares

2.1.3.2 *Siu's (1984) cuboid completion*

Siu's (1984) derivation, depicted in Figure 2.8, involves piecing three step pyramids representing $S_2(n)$ together and cutting and pasting to form a cuboid whose sides are of lengths n, $n+1$, and $n+\frac{1}{2}$, resulting in the relation

$$3S_2(n) = n(n+1)\left(n+\frac{1}{2}\right).$$

One can see that the relative positions of the step pyramids are comparable to those in Turner's derivation. This is to be expected for those who have tried this kind of exercise before since the positions correspond to those of square-base right-angle pyramids in forming a cube, which, by the way, are excellent manipulatives in a lesson concerning the volume of a pyramid.

It is hard to think about how to get students to derive the formula for sums of squares using these two proofs or other similar ones, which require rather difficult ways of arranging step pyramids and perhaps other pieces together, especially for Siu's (1984) derivation where one has to halve a tier after the arrangement. As for Turner's (1977) derivation, students may have difficulties transforming the formula from the geometric derivation into a familiar form due to the presence of both $S_2(n)$ and $S_2(n-1)$ with different magnitudes of coefficients.

There are three heuristic/geometric derivations that manage to derive the formula using two-dimensional figures. Two of them employed unfamiliar formulas in the derivations.

2.1.3.3 *Chilaka's (1983) derivation*

Chilaka (1983) employed a complicated relation

$$S_2(n) = S_1^2(n) - 2\sum_{i=1}^{n-1}(i+1)S_1(i)$$

in his derivation. If one looks at the formula carefully, one can see that since $S_1(i)$ contains the term i^2, the summation sign would contain the term i^3 in it, a very strange prerequisite in solving for $S_2(n)$. In fact, Figure 2.9, which appeared in his derivation, is similar to those appearing in the derivations of the formula for sums of cubes.

2.1.3.4 *Chuang's (1989) derivation*

Chuang (1989) utilized a simple-looking relation

$$\sum_{i=1}^{n}\sum_{j=i}^{n} j = S_2(n)$$

and an equally simple-looking diagram as shown in Figure 2.10 in the derivation. However, transforming this formula into a familiar looking one requires a bit of work. In particular, students have to know how to change $\sum_{j=i}^{n} j$ into a function of $S_1(n)$.

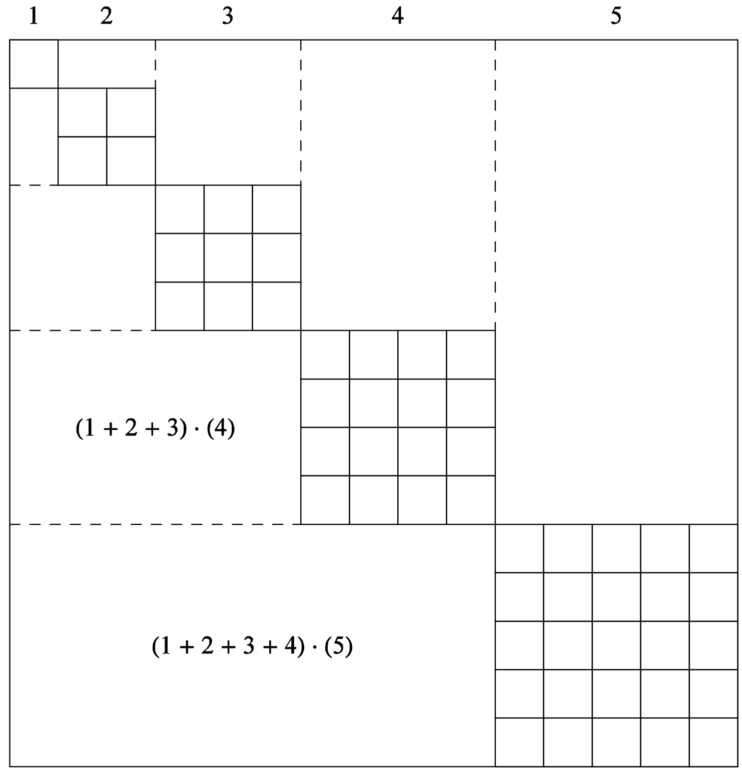


Figure 2.9 Chilaka's (1983) derivation of the formula for sums of squares

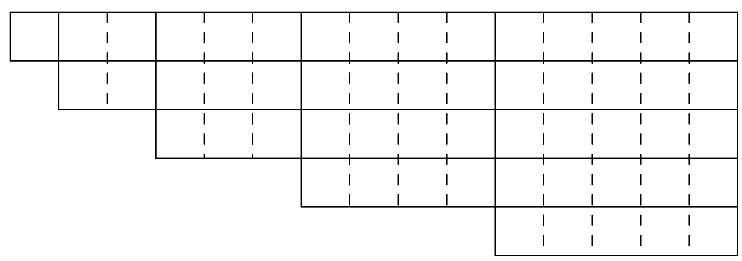


Figure 2.10 Chuang's (1989) derivation of the formula for sums of squares

2.1.3.5 *Kalman's (1991) impressive derivation*

Kalman (1991) was the only one who came up with a two-dimensional heuristic/geometric derivation that does not depend on a peculiar form of the formula. He managed to arrange three copies of a two-dimensional representation of $S_2(n)$ into a rectangle whose sides are of lengths $S_1(n)$ and $(2n+1)$, resulting in the relation

$$(1+2+\cdots+n)(2n+1) = 3(1^2+2^2+\cdots+n^2).$$

His derivation is demonstrated in Figure 2.11.

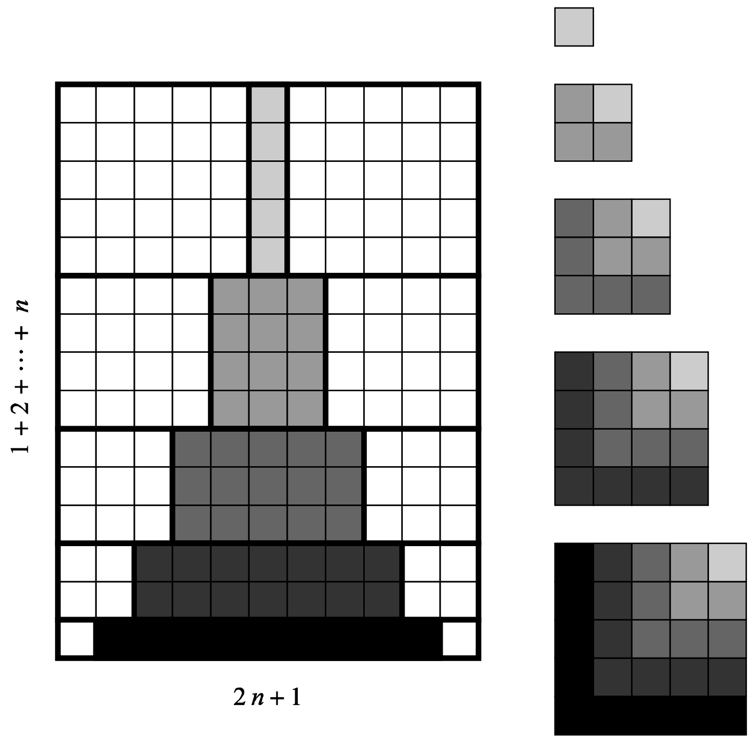


Figure 2.11 Kalman's (1991) derivation of the formula for sums of squares

Kalman's (1991) derivation has potential for being developed into instruction that requires students to derive the formula. For example, they could be provided with the figure of the big rectangle in Figure 2.11 (with all the lines and colors) and asked to derive the formula. The next derivation, which is the only purely geometric one, also has similar potential.

2.1.3.6 *Sakmar's (1997) geometric derivation*

Sakmar's (1997) derivation, demonstrated by Figure 2.12, can be viewed as the extension of the derivation of the formula for sums of integers described in section 1.1.2 (see Figure 1.1). From the figure, one can see that one can turn a square-base pyramid into the enclosing step pyramid by attaching right-triangular prisms to the sides. However, two prisms overlap at the edges of the square pyramid. An overlap is an upside-down square-base right-angle pyramid. The volume of the step pyramid is $S_2(n)$, that of the square-base pyramid is $\frac{n^3}{3}$, that of a prism is $\frac{1}{4}$, and that of an upside-down pyramid is $\frac{1}{12}$. Since the number of prisms on each side is $S_1(n)$ and that of upside-down pyramids is n, the formula for sums of squares is

$$S_2(n) = \frac{n^3}{3} + \frac{4}{4}S_1(n) - \frac{4}{12}n$$
$$= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

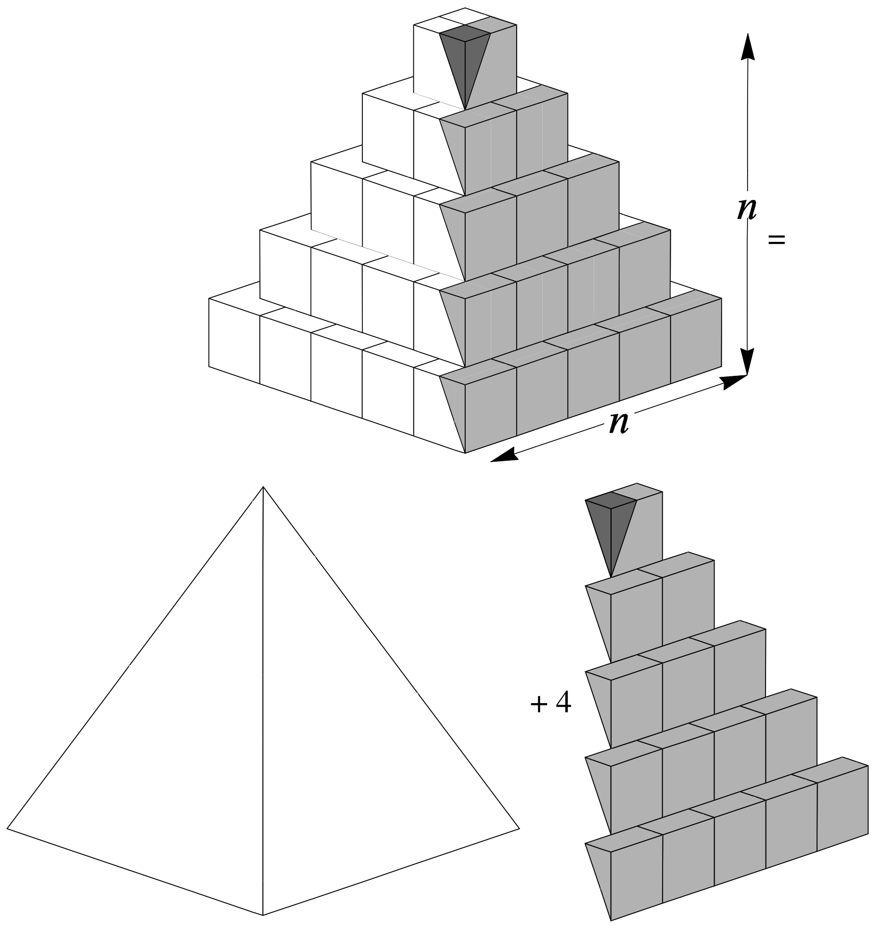


Figure 2.12 Sakmar's (1997) derivation of the formula for sums of squares

In general, a sum of squares can be represented by a stack of objects of increasing size. The objects in Figure 2.12 are square-base cuboids of unit height. If the objects have enough regularity, an inscribed pyramid would be well defined. In Figure 2.12, the differences between the volumes of the step pyramids and those of the corresponding inscribed pyramids form a sequence whose major component is the sequence of sums of integers. In general, if a is the base area of the object at the top of the stack, the base areas of subsequent tiers will be $4a, 9a, \dots, n^2a$. With the height of each tier normalized to one, the volume of the inscribed pyramid when the number of tiers is n will be $\frac{an^3}{3}$ and that of the step pyramid will be $aS_2(n)$. Thus, the difference between the two volumes will be

$$a\left[S_2(n) - \frac{n^3}{3}\right] = a\left(\frac{n^2}{2} + \frac{n}{6}\right) = a\left[S_1(n) - \frac{n}{3}\right]. \tag{2.15}$$

That is, subtracting the volumes of the inscribed pyramids from those of the stacks of objects reduces sums of squares into sums of integers (minus one-third the numbers of tiers), which is the basis for the next derivation.

2.1.3.7 *Somchaipeng, Kruatong, and Panijpan's derivation*

The objects to be stacked in Somchaipeng, Kruatong, & Panijpan's derivation are Plasticine disks of unit height as shown in Figure 2.13. The radius of the top disk is one and it increases by one in each subsequent tiers. So, the base area of the top disk is π and the inscribed pyramid is a cone as shown in Figure 2.14. They let students investigate the differences between the volumes of the step cones and those of the inscribed cones. From equation (2.15), one knows that the sequence of the differences is

$$\frac{2\pi}{3}, \frac{7\pi}{3}, \frac{15\pi}{3}, \dots, \frac{n(3n+1)\pi}{6}.$$

Of course, students cannot be expected to be able to find the general form of the sequence but if they take the differences between subsequent terms, they will see a much simpler sequence. The difference between term n and term $(n-1)$ is

$$\pi\left(\frac{n^2}{2} + \frac{n}{6} - \frac{(n-1)^2}{2} - \frac{n-1}{6}\right) = \pi\left(n - \frac{1}{3}\right).$$

That is, students can use the equation

$$\pi S_2(n) - \frac{\pi n^3}{3} = \frac{2\pi}{3} + \pi\sum_{i=2}^{n}\left(n - \frac{1}{3}\right), \text{ or}$$

$$S_2(n) = \frac{n^3}{3} + \frac{2}{3} + S_1(n) - 1 - \frac{n-1}{3}$$

$$= \frac{n^3}{3} + S_1(n) - \frac{n}{3}$$

to find the formula for sums of squares. The advantage of this derivation is that it concretely demonstrates why sums of squares are polynomials of degrees 3. On the other hand, one has to be careful about the index of the summation sign in the derivation.

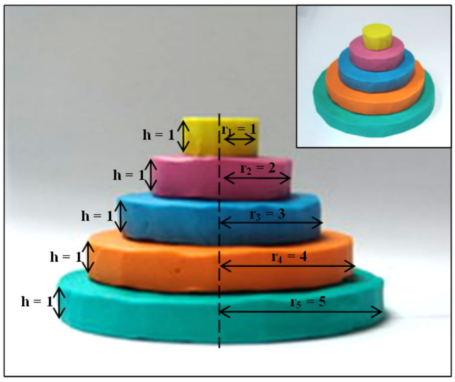


Figure 2.13 A step cone made from of a stack of Plasticine disks

Figure 2.14 A cone inscribed in a step cone

Besides all these heuristic and geometric derivations of the formula for sums of squares, there are a few heuristic/algebraic derivations as well. Two of these derivations will be presented.

2.1.3.8 *Kung's (1989) enlightening derivation*

Extending an algebraic or a geometric derivation to a higher power is one thing, extending a heuristic one is quite another. Kung (1989) managed to do just that: he was able to extend the heuristic derivation of the formula for sums of integers presented in section 1.1.4 to the case of sums of squares. Figure 2.15 demonstrates his arrangement of numbers that facilitates the derivation.

$$
\begin{array}{ccccccc}
n & n & \bullet & \bullet & \bullet & n & n \\
n-1 & n-1 & \bullet & \bullet & \bullet & n-1 & \\
\bullet & \bullet & & & \bullet & & \\
\bullet & \bullet & & \bullet & & & \\
\bullet & \bullet & \bullet & & & & \\
2 & 2 & & & & & \\
1 & & & & & &
\end{array}
\; + \;
\begin{array}{ccccccc}
n & n-1 & \bullet & \bullet & \bullet & 2 & 1 \\
n & n-1 & \bullet & \bullet & \bullet & 2 & \\
\bullet & \bullet & & & \bullet & & \\
\bullet & \bullet & & \bullet & & & \\
\bullet & \bullet & \bullet & & & & \\
n & n-1 & & & & & \\
n & & & & & &
\end{array}
\; +
$$

$$
\begin{array}{ccccccc}
1 & 2 & \bullet & \bullet & \bullet & n-1 & n \\
2 & 3 & \bullet & \bullet & \bullet & n & \\
\bullet & \bullet & & & \bullet & & \\
\bullet & \bullet & & \bullet & & & \\
\bullet & \bullet & \bullet & & & & \\
n-1 & n & & & & & \\
n & & & & & &
\end{array}
\; = \;
\begin{array}{ccccccc}
2n+1 & 2n+1 & \bullet & \bullet & \bullet & 2n+1 & 2n+1 \\
2n+1 & 2n+1 & \bullet & \bullet & \bullet & 2n+1 & \\
\bullet & \bullet & & & \bullet & & \\
\bullet & \bullet & & \bullet & & & \\
\bullet & \bullet & \bullet & & & & \\
2n+1 & 2n+1 & & & & & \\
2n+1 & & & & & &
\end{array}
$$

Figure 2.15 Kung's (1989) derivation of the formula for sums of squares

From the figure, one can see that arranging three copies of the sum of squares in a complementary way results in a constant-sum triangle of numbers. Since the number of numbers in a triangle is obviously $S_1(n) = \frac{1}{2}n(n+1)$, one reaches the conclusion that

$$3(1^2 + 2^2 + \cdots + n^2) = \frac{1}{2}n(n+1)(2n+1).$$

Nelsen (2000, p. 88) demonstrated that one can replace integers by odd integers since

$$n^2 = 1 + 3 + \cdots + (2n-1).$$

The main difference is that there are n ones down to one $(2n-1)$ in a triangle.

2.1.3.9 *Conway and Guy's (1989) heuristic derivation*

Instead of employing three copies of $S_2(n)$, Conway and Guy (2009, p.49) employed $(2n+1)$ copies of $S_1(n)$ to derive the formula. The derivation is illustrated in Figure 2.16.

							1						→	1^2
						1	2	1					→	2^2
					1	2	3	2	1				→	3^2
				1	2	3	4	3	2	1			→	4^2
			1	2	3	4	5	4	3	2	1		→	5^2
		1	2	3	4	5		5	4	3	2	1		
	↙	2	3	4	5				5	4	3	2	↘	
1^2	↙	3	4	5						5	4	3	↘	1^2
2^2	↙	4	5								5	4	↘	2^2
3^2	↙	5										5	↘	3^2
4^2	↙												↘	4^2
5^2														5^2

Figure 2.16 Conway and Guy's (2006) derivation of formula for sums of squares

The sums above the line are obtained using equation (2.14), which states that the sum of two consecutive sums of integers is the square of the number of terms in the larger sum. The sums below the line are obvious. Since there are n columns below the line on both sides and an additional one in the middle, the total number of $S_1(n)$ is $(2n+1)$, and since they sum to three copies of $S_2(n)$,

$$(2n+1)S_1(n) = 3S_2(n).$$

2.1.4 Derivations of formulas for sums of cubes

One of the well-known identities in mathematics is that of the sum of cubes and the square of sums of integers. This identity suggests that it might be easier to come up with a derivation of a formula for sums of cubes than to come up with a derivation of a formula for sums of squares. The literature seems to confirm this since the former predated the latter. The identity also suggests that one can stay in two dimensions and employ squares whose side lengths relate to $S_1(n)$ in a derivation. This is in fact the theme on which several derivations are based. These derivations can be further divided into three groups. The first derivation of each group will be presented.

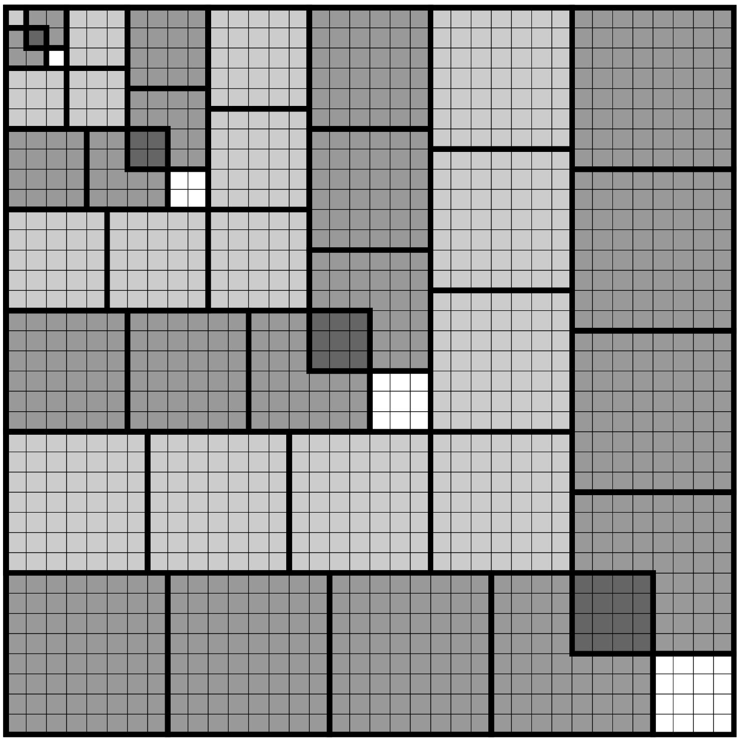


Figure 2.17 Golomb's (1965) derivation of the formula for sums of cubes

2.1.4.1 *Golomb's (1965) derivation*

Golomb's (1965) demonstrated that a square of size $S_1^2(n)$ can be partitioned into n areas of increasing size. The smallest partition, which will be counted as the first partition, contains one unit square. The i^{th} partition contains i squares of size i^2, which means that its area is i^3. So, the sum of the areas of all the partitions is $S_3(n)$, establishing the required identity. His derivation is depicted in Figure 2.17. From the figure, two squares of size i^2 overlap when i is even. However, the overlapping area equals the uncovered area, nullifying each other. Love (1977) avoided the overlaps by halving one of the overlapping squares and rearranging them to fill the uncovered area. Fry (1985) demonstrated the same idea but replaced unit squares with unit cubes to improve the visualization.

2.1.4.2 *Lushbaugh's derivation*

Warren Lushbaugh suggested to Golomb (1965) that one can avoid the overlaps by placing four copies of the partitions together as depicted in Figure 2.18. Arranging the partitions this way also eliminates the need to halve the squares. Cupillari (1989) had the same idea but with better visualization. Figure 2.18 is based on the latter.

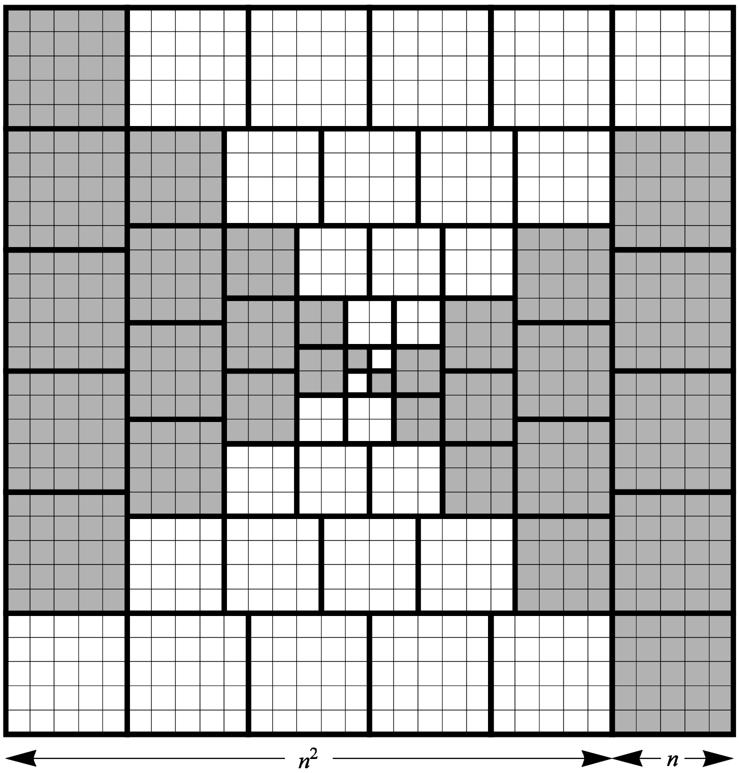


Figure 2.18 Lushbaugh's derivation of the formula for sums of cubes

2.1.4.3 *Shult and Tenenbaum's (1988) derivation*

Shult and Tenenbaum's (1988) inspired students to use different ways to find the volume of a three-dimensional multiplication table in order to rediscover the identity. A three-dimensional multiplication table is a square-base wooden apparatus. The base contains 10×10 unit squares whose heights match the products at the corresponding positions. Thus, the volume of each column equals the product. Figure 2.19 is a graphical representation of a three-dimensional multiplication table. One way of finding the volume of the table is to observe that the volume of the first row is

$$1 + 2 + 3 + \cdots + 10 = 55.$$

Now, the volume of the second row is twice that of the first row, which is 55(2), that of the third row is 55(3), and so forth. So, the total volume is

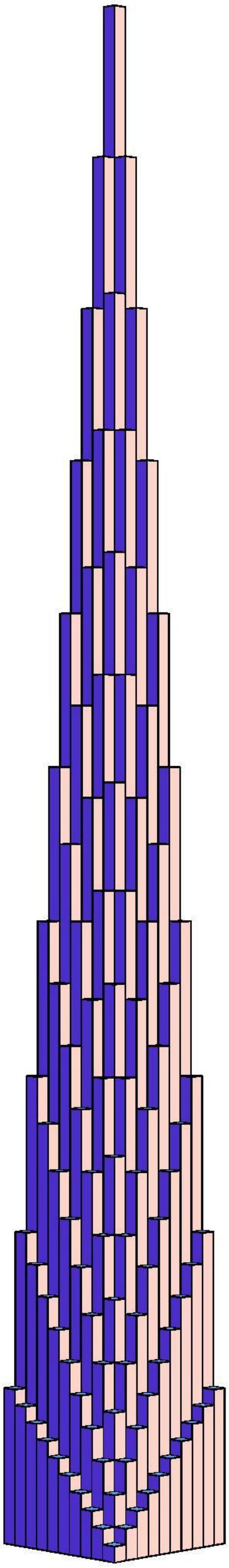


Figure 2.19 A three-dimensional multiplication table

$$55(1) + 55(2) + 55(3) + \cdots + 55(10)$$
$$= 55(1 + 2 + 3 + \cdots + 10)$$
$$= 55^2.$$

Another way to find the total volume is to consider a sequence of multiplication tables, starting with a 1×1 table, whose volume is one. To get the volume of a 2×2 table, one needs to add $2 + 4 + 2 = 8$ to the previous volume. The contribution from a 3×3 table is $3 + 6 + 9 + 6 + 3 = 27$. At this point, students started to notice that the contribution from an $n \times n$ table should be n^3. One can confirm this observation using equation (2.14) as follows:

$$n + 2n + \cdots + n(n-1) + n^2 + n(n-1) + \cdots + 2n + n$$
$$= n[(1 + 2 + \cdots n) + (1 + 2 + \cdots + n - 1)]$$
$$= n[S_1(n) + S_1(n-1)]$$
$$= n \cdot n^2$$
$$= n^3.$$

So, the total volume is also

$$1^3 + 2^3 + 3^3 + \cdots + 10^3.$$

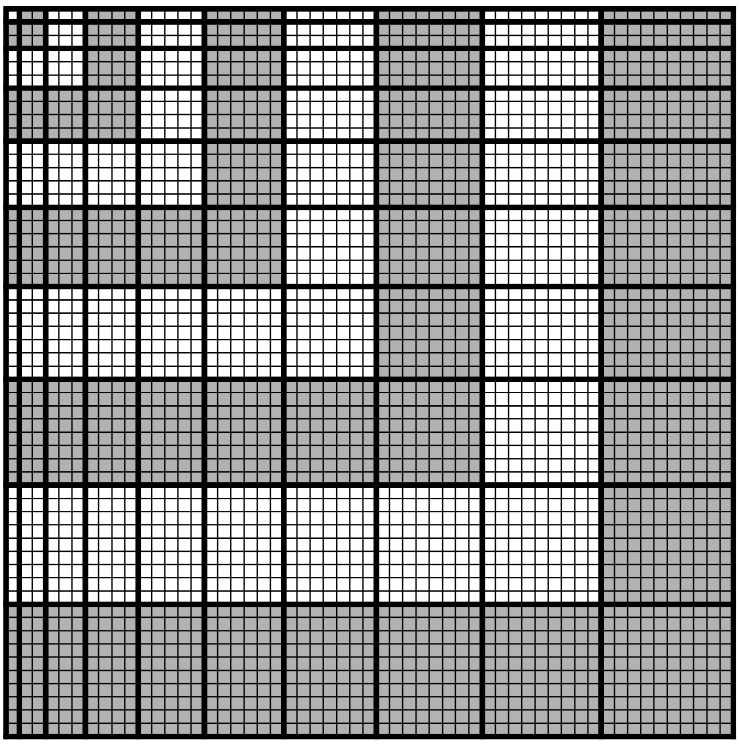


Figure 2.20 Partitioning a square to match entries in a multiplication table

Nelsen (1990) demonstrated a way to partition a square of size $S_1^2(n)$ so that the partitions match the entries in a multiplication table. The partitions are shown in Figure 2.20.

Besides these heuristic/geometric derivations, there is another that does not employ squares so extensively.

2.1.4.4 *Schrage's (1992) derivation*

By now, the arrangement of squares to yield $S_3(n)$ should be familiar. Schrage (1992) noticed that one can construct a triangle whose area equals to the total area of all the squares, as depicted in Figure 2.21. Since the base length of the triangle is $(n^2 + n)$ and the height is $S_1(n)$, the total area of all the squares is

$$S_3(n) = \frac{1}{2}(n^2 + n)S_1(n) = S_1^2(n).$$

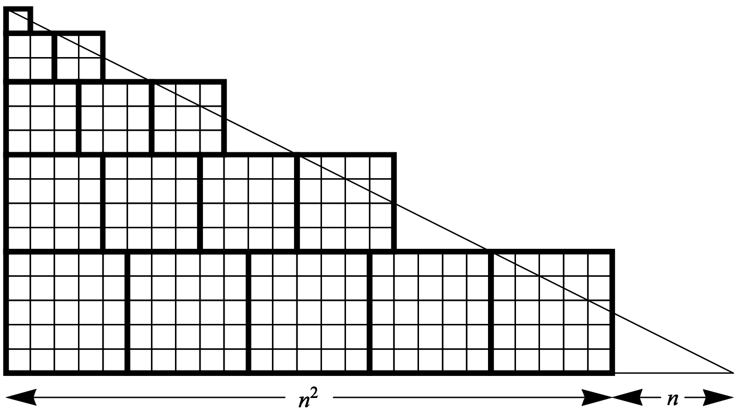


Figure 2.21 Schrage's (1992) derivation of the formula for sums of cubes

The last heuristic/geometric derivation that will be presented is a physical one.

2.1.4.5 *Somchaipeng, Kruatong, and Panijpan's derivation*

Similar to their derivation of the formula for sums of squares, Somchaipeng et al. employed Plasticine balls and disks in their derivation of the formula for sums of cubes. They provided students with Plasticine balls of radius 1, 2, 3, and 4 as shown in Figure 2.22. Students were first asked to make a disk with the height of $\frac{4}{3}$ from the smallest ball. The height was so chosen in order to normalize the radius of the disk to one. Next, students had to use the ball of radius 2 to expand the disk while keeping the height constant. It turned out that the expanded disk had the radius of 3. Continuing in this fashion, students realized that each subsequent ball could expand the radius of the disk by the same amount as the radius of the ball. Equating the total volume of the balls to the volume of disk yielded

$$\frac{4\pi}{3}(1^3+2^3+3^3+4^3)=\frac{4\pi}{3}(1+2+3+4)^2.$$

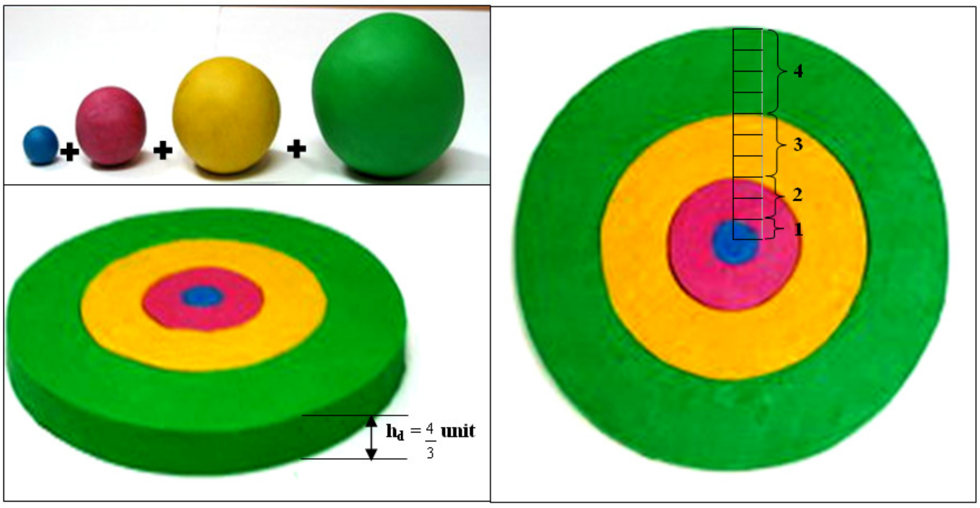


Figure 2.22 Transforming Plasticine balls into rings of a composite disk

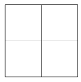
The next derivation employed the familiar square set-up but only its algebraic feature will be presented.

2.1.4.6 *Flores' (1998) derivation*

From Figure 2.17, if one considers expanding the square of size one to the square of size 2^2, three unit squares are needed. The next expansion requires five unit squares. Thus, $2^3=3+5$. Similarly, $3^3=7+9+11$. Since the number of odd integers increases by one for each subsequent cube, the total number of odd integers in $S_3(n)$ is $S_1(n)$, and since the sum of the first n odd integers is n^2, the sum of the first n cubes is $S_1^2(n)$. Flores (1998) also gave the expressions for the first and the last odd integers for each cube.

The identity between sums of cubes and squares of sums of integers suggests that a simple combinatorial derivation can be constructed. Three such derivations will be presented.

2.1.4.7 *Stein's (1971) combinatorial derivation*

Stein (1971) counted the number of rectangles embedded in a ruled square of size n^2. For example, there are nine rectangles embedded in , namely, four unit squares, one big square, and four non-square rectangles. Before delving into his derivation, let us start with two reduced counting problems that can be used to derive the formulas for sums of integers and for sums of squares. The first problem is to count the number of rectangles embedded in a ruled $n \times 1$ rectangle. Consider an $i \times 1$ rectangle. If we designate the vertical line on the extreme left as 0 (assuming that the rectangle is horizontal), such a rectangle would start at 0, 1, … , or $(n - i)$ for the total of $(n - i + 1)$ rectangles. Since the length of a rectangle can range from n to 1, the number of all the rectangles is $(1 + 2 + \cdots + n)$. Now, a rectangle can be uniquely defined by its left and right vertical lines. Since there are $\binom{n+1}{2}$ ways of selecting the two end lines, we get the formula for sums of integers by equating the two ways of counting.

The next problem is to count the number of squares embedded in a ruled square of size n^2. Consider a square of size i^2. If we designate the lower left corner as the point (0, 0), the lower left corner of such a square would have both coordinates in the range of 0 to $(n - i)$ for the total of $(n - i + 1)^2$ squares. Since the size of a square can range from n^2 to 1, the number of all the squares is $S_2(n)$. Now, consider a point (x, y). Such a point is the lower left corner of squares of size 1 to $[n - \max(x, y)]$. That is, all points along  are the lower left corners of the same number of squares. Let $j = \max(x, y)$. Then, there are $(2j + 1)$ points along , each with $(n - j)$ squares associated with it. Furthermore, let $i = n - j$. So, there are $[2(n - i) + 1]$ points along , each with i squares associated with it. Since j ranges from 0 to $(n - 1)$, i would range from 1 to n. Thus, the number of all squares is

$$\sum_{i=1}^{n} i[2(n - i) + 1] = (2n + 1)S_1(n) - 2S_2(n).$$

Equating the two ways of counting yields the formula

$$S_2(n) = (2n + 1)S_1(n) - 2S_2(n) \Rightarrow S_2(n) = \frac{(2n+1)S_1(n)}{3}.$$

Let us now return to the original problem. Again, let (0, 0) denote the lower left corner of the big square. A rectangle can be uniquely defined by its lower left and upper right corners, which correspond to two points on each axis. Since there are $\binom{n+1}{2}$ ways of selecting two points from an axis, the number of all rectangles is $\binom{n+1}{2}^2$. Now, consider the effect of adding one more column to a ruled square of size $(i-1)^2$. Since the new column would result in i more ways of selecting two points from the x axis and there are $\binom{i}{2}$ ways of selecting two points from the y axis, there would be $i\binom{i}{2}$ new rectangles formed. Similarly, adding one more row to this ruled rectangle would result in $i\binom{i+1}{2}$ new rectangles. So, increasing the size of the ruled square from $(i-1)^2$ to i^2 would result in

$$i\binom{i}{2} + i\binom{i+1}{2} = i[S_1(i-1) + S_1(i)] = i^3$$

new rectangles. The latter equality is obtained using equation (2.14). Thus, the total number of rectangles is

$$S_3(n) = \binom{n+1}{2}^2 = S_1^2(n).$$

Stein (1971) pursued a different line of derivation. His derivation is equivalent to the partitioning of a ruled square depicted in Figure 2.20 while this derivation is equivalent to the partitioning in Figure 2.17.

2.1.4.8 *Turner's (1981) combinatorial derivation*

Turner (1981) counted the number of times two sets of lines intersect. Each set of lines consists of all the lines joining two of the points in a set of $(n+1)$ points. All points are in a plane such that no three of the distinct $2(n+1)$ points are collinear and no two lines from different sets are parallel. It turns out that the problem is equivalent to Stein's (1971) and it can be analyzed in the same way. In fact, the derivation presented above is adapted from Turner's (1981), which is more elegant than Stein's (1971).

2.1.4.9 *Benjamin and Orrison's (2002) combinatorial derivation*

Benjamin and Orrison (2002) employed another approach to a combinatorial derivation. They constructed two sets, one having $S_3(n)$ members while another having $\binom{n+1}{2}^2$ members, and established a bijection between the two sets. In particular, the first set S is a set of 4-tuples of integers from 0 to n such that the last component is strictly bigger than the others. That is,

$$S = \{(h, i, j, k) | 0 \leq h, i, j < k \leq n\}$$

This is the same as one of the partitions in Paul's (1971) derivation described in section 2.1.1.8 when $p = k = 3$ (k in that context). One has seen that $|S| = S_3(n)$.

The other set T is a set of pairs of ordered pairs whose elements are also integers from 0 to n. That is,

$$T = \{((x_1, x_2), (x_3, x_4)) | 0 \leq x_1 < x_2 \leq n, 0 \leq x_3 < x_4 \leq n\}.$$

Clearly, $|T| = \binom{n+1}{2}^2$.

The mapping $f: S \to T$ defined as

$$f((h, i, j, k)) = \begin{cases} ((h, i), (j, k)), & \text{if } h < i \\ ((j, k), (i, h)), & \text{if } h > i \\ ((i, k), (j, k)), & \text{if } h = i \end{cases}$$

is a bijection since the case $h < i$ is mapped onto the case $x_2 < x_4$, the case $h > i$ is mapped onto the case $x_2 > x_4$, and the case $h = i$ is mapped onto the case $x_2 = x_4$. Thus, $|S| = |T|$.

2.2 Educational Review

The scaffolded learning unit in this research study employs concrete models to facilitate students' derivation of basic summation formulas. The role of the instructor is to scaffold students' derivation and learning. Thus, two key educational topics are relevant to this research study, namely, scaffolding and the use of models. In addition, even though Wood, Bruner, and Ross (1976) introduced the concept of scaffolding without providing a theoretical basis, Rogoff and Wertsch (1984) soon drew parallels between scaffolding and Vygotsky's (1978) zone of proximal development (ZPD), which were well received in the literature. Since ZPD is based on Vygotsky's holistic approach to teaching, learning, and development, Vygotsky's social constructionism will be reviewed first, followed by scaffolding and the use of models.

2.2.1 Social constructionism

Vygotsky's dissatisfaction with the two prominent schools of thought of his time was the impetus for his undertaking of a new psychology (Vygotsky, 1978, p. 6). Of the two prominent schools, the more prominent one at that time employed a stimulus-response framework, wherein a psychological process, no matter how complex, can be induced by confronting the subject with some kind of stimulus situation, to reduce the study of mind to measuring types and degrees of responses to a set of stimuli and analyzing the data (Wink & Putney, 2002, p. xxi). This school was influenced by the publications of Charles Darwin's *Origin of Species* and two other books (Vygotsky, 1978, pp. 2–4), which provided scientific theories and evidence for the continuation of humans from animals. The other school, which is much more prominent today, believes that a person's development depends on maturation or is preformed and waiting to be triggered. Piagetian constructivism is perhaps the most prominent in this latter school.

The pedagogical implication of the stimulus-response framework is that learning and development are one and the same (Vygotsky, 1978, pp. 80–81), resulting in the traditional way of teaching where a teacher attempts to transmit intended knowledge to learners (Wink & Putney, 2002, pp. 9–10). We might say that the ineffectiveness of this approach has been the reason for the existence of educational journals. The other school's perspective on learning and development is that learning always lags behind development (Vygotsky, 1978, pp. 79–80). Learning *merely utilizes the achievement of development rather than providing an impetus for modifying its course*. If this view is correct, educating young children would be an inefficient use of time and other resources because they would achieve the same level of development regardless of the learning, and since the ability to learn is directly related to the level of development, it would take significantly less time and resources to educate more matured children on the same topics, probably with better results. Would it be better then to compress the first, say, eight years of formal education into just three since children would acquire about the same level of knowledge anyway by the time they enter high schools?

Vygotsky theorized that learning precedes development (Wink & Putney, 2002, pp. 23–25). Not that he rejected Piaget's stage theory completely (Vygotsky, 1978, p. 85), but Vygotsky believed that good learning can and does enhance development, and, in turn, a higher level of development heightens learning ability. By good learning, he meant the kind of learning that takes place in the ZPD, which *is the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers* (Vygotsky, 1978, p. 86). In other words, good instruction contains problems challenging enough so that most students cannot solve them individually but they should be able to solve them cooperatively and/or with guidance from the teacher. Since the teacher cannot be expected to actively interact with each student one by one, group and/or class activity becomes more or less mandatory. This is in accordance with another tenet of Vygotskian theories that emphasizes the socio-cultural aspect of teaching and learning (Wink & Putney, 2002, pp. 60–84). For Vygotsky (1978, pp. 56–57), every development is the result of the internalization of socially rooted and historically developed activity, the process of which takes a long time and may even stay external forever.

The causal relationship between learning and development is not the only major difference between Vygotskian and Piagetian theories. While Piaget believed that individuals construct knowledge internally, Vygotsky, on the other hand, had the view that knowledge is socially constructed before being internalized by each individual (Wink & Putney, 2002, p. 33). This means that from a constructivist's perspective, there is no way to reconcile the knowledge among a group of people since each individual independently constructs the knowledge. So, knowledge is extremely uncertain and there appears to be no way of knowing whether one's knowledge is consistent with another's. On the other hand, since the knowledge is socially constructed in a Vygotskian framework, each individual internalized knowledge tends to be consistent with this socially constructed version. So, reconciliation of knowledge is not only a possibility, but is almost guaranteed. Due to the three key differences between the two theories, the researcher is reluctant to use the term *constructivism* to describe Vygotskian theories. In this document, the term *social constructionism*, which is another term used when referring to Vygotskian theories, which have been generally called *social constructivism*, will be used instead.

Social constructionism consists of three tenets, namely, the relationship between thought and languauge, the socio-cultural aspect of teaching and learning, and ZPD (Wink & Putney, 2002, p. 39).

2.2.1.1 *Thought and language*

Intuitively, one may perceive that one's thought governs one's spoken words. That is, the relationship is mostly unidirectional. However, by observing children's behavior during problem-solving activities, Vygotsky (1978, pp.24–30) noticed that the role of verbal language in the activities was too complicated to be explained by a unidirectional model of the relationship. In general, the youngest children, who were incapable of solving the problem by themselves, tended to direct their spoken words at the experimenter. The oldest children, to whom the activity was not much of a burden, tended to perform the activity silently. Interestingly, those on the verge of being able to solve the problem first directed their speech toward the experimenter and the environment. However, as they faced more obstacles, they tended to use spoken words to plan and direct their action in the problem-solving processes. Intriguingly, an attempt to forbid the use of speech was either futile or the children would freeze up: they seemed to act with less freedom when not allowed to speak. So, language plays a significant role in facilitating one's thought and action. Thus the relationship between thought and language is a reciprocal one, both can and do enhance the other.

Vygotsky described the three phases of speech as *interpersonal* or *socialized*, *intrapersonal* or *egocentric*, and *internalized*. Children's uses of speech tended to depend on their mastery of the language and their mastery of the activity. Only those who possessed the mastery of both domains were able to internalize their speech, i.e., performing the activity silently. The lower the level of mastery, the higher tendency one had to employ interpersonal speech to facilitate problem solving. Thus, by observing the use of language, one can infer the level of development of the subject. Vygotsky proposed that the unit of analysis, or the basic explanatory element, is *word meaning* (Williams, 1989).

Symbols and definitions, especially mathematical ones, can be considered as a form of language. In learning a new mathematics topic that introduces new symbols and definitions, students usually start with a lack of understanding of these symbols and definitions. The lack of understanding is reflected in their use of the *language*. Usually, one can observe that students use the language without knowing its precise meaning. As they develop their understanding of the topic, the meaning becomes more and more precise to them. Only after a complete understanding of the topic is achieved can one's use of the language be at the same level as an expert's.

2.2.1.2 *Socio-cultural teaching and learning*

For Vygotsky, all learning takes place in a social context. Even when one engages in a metacognition process, the process would involve the use of language, which is a socio-cultural artifact (Wink & Putney, 2002, p. 61). Without a socio-cultural artifact, i.e., without the social context, one has no means to learn. Since language is the most important socio-cultural artifact, it determines the level of sophistication of a society. One can observe that the language of an isolated tribe is much simpler than that of a civilized society. Language plays a significant role in facilitating one's thought. Thus, more sophisticated language leads to more sophiticated thought, which, in turn, leads to the extension of the language to explain that thought.

Not only does one utilize language to facilitate one's thought, but one also uses language to communicate with others and learn something from that communication. In other words, one utilizes a socio-cultural artifact in a social activity to make meaning. Learning begins as an *intermental* process—in relation and cooperation with others—and becomes an *intramental* process as one begins to internalize what one has learned through one's interaction with others (Wink & Putney, 2002, p. 61).

2.2.1.3 *Zone of proximal development*

The two aforementioned tenets culminate in Vygotsky's best-known concept of the zone of proximal development. From the socio-cultural aspect of learning, one knows that learning takes place in a social context, mediated by a socio-cultural artifact, which, in most cases, involves language. From the reciprocal relationship between language and thought, one knows that language plays a significant role in facilitating thought. Thus, by interacting with more capable others in a particular context, one learns new words and new meanings, which lead to new thoughts. If the context is set up beyond one's actual developmental level but within one's potential, the newly developed thoughts would also be beyond one's initial developmental level, thus leading one to a higher level. Vygotsky was not specific about the kinds of interactions that should be going on in ZPD in order to maximize the learner's development. This is where scaffolding comes in.

2.2.2 Scaffolding

2.2.2.1 *Wood, Bruner, and Ross (1976)*

Wood, Bruner, and Ross (1976) introduced the term scaffolding as the *process that enables a child or novice to solve a problem, carry out a task or achieve a goal which would be beyond his unassisted efforts* (p. 90). They also emphasized that to benefit from a scaffolding process, *comprehension of the solution must precede production*. Since the latter statement is similar to the concept of ZPD that requires the context to be within the learner's potential, these two statements, in effect, define scaffolding as a possible process in ZPD.

In order to characterize a scaffolding process, Wood et al. (1976) set up a task for a child aged three to five and observed the completion of the task under a tutor's guidance. Incidentally, the task was to construct a six-tier step pyramid from twenty-one wooden components that fit together via pegs and holes. They required that the task be fun, interesting, challenging, within reach of a child's skill, and continously yielding knowledge. These requirements can serve as guidelines on the design of an activity for students. In addition, they instructed the tutor to allow the child to do as much as possible for oneself and to prefer verbal instructions to more direct interventions. Finally they characterized the following scaffolding functions (p. 98).

Recruitment

First, the tutor had to engage a child in the task.

Reduction in degrees of freedom

The task had to be within the child's comprehension.

Direction maintenance

This included keeping the child motivated and gearing the child toward the objective when the child became distracted.

Marking critical features

If the child seemed to be heading away from the solution, the tutor had to accentuate the point of discrepancy.

Frustration control

If the child ran into difficulties that could not be overcome, the tutor had to simplify the task, but not to the extent that the child depended too much on the tutor.

Demonstration

Demonstrating a solution to a task was acceptable, especially when the child had already partially constructed the solution.

2.2.2.2 *Anghileri (2006)*

Anghileri (2006) extended Wood et al.'s (1976) list of scaffolding functions according to the practice in mathematics classrooms and classified them into three levels where a higher level corresponded to a higher form of scaffolding. The three levels, together with selected associated scaffolding functions, are described below.

Level 1: Environmental provisions

The first level of scaffolding does not involve student-teacher interactions but aims to facilitate and enhance higher-level scaffoldings. The provisions may include the selection of social *artifacts* (e.g., manipulatives), the *task*, *grouping*, and *emotive feedback* (e.g., approval and encouragement).

Level 2: Explaining, reviewing, and restructuring

This level of scaffolding involves direct student-teacher interactions related specifically to the mathematics being considered, as opposed to broader or deeper mathematical knowledge. Explaining may not be considered as a scaffolding function, but so is demonstratoin in Wood et al.'s (1976) characterization.

Reviewing

Students may not be able to continue after making some progress toward the solution. Reviewing scaffoldings can be employed to help them reflect on what they have done and probably see what they need to do. Reviewing scaffoldings are described below.

• *Verbalizing*

This scaffolding function is a direct consequence of the intertwinement of thought and language. The function involves asking students to verbalize their thoughts. By thinking aloud, students are forced to reflect on their thinking, which may lead them to notice a flaw in their reasoning or calculation. Verbalizing is most useful when students are slightly off track.

• *Prompting*

The teacher can ask prompting questions that successively lead the students toward a predetermined solution. This funneling pattern of interaction may cause the teacher to misjudge the levels of understanding in students as they may simply pick up clues from the questions. A typical worksheet in mathematics classrooms usually contains this kinds of questions, which are at the lowest end of scaffolding functions and require the least amount of time. Since most mathematics classes have time constraints, prompting may be the only practical form of scaffolding. The funneling learning unit in this study employs prompting almost exclusively.

• *Probing*

Unlike prompting questions, probing questions intend to get students to expand on their own thinking. This is useful when students are on the verge of a discovery but require more thorough mathematical understanding to continue.

• *Interpreting students' actions and talk*

If probing questions fail to elicit the intended understanding from students, the teacher may make explicit the mathematical meaning of students' work up to that point. This scaffolding is also useful when the teacher asks students to share their solutions with the class, as it would enhance other students' understanding of the solutions.

• *Parallel modeling*

If all the interactions identified above cannot lead to a solution, instead of explaining, the teacher can demonstrate the solution to a different task that shares some characteristics with the original task. This way, students retain the ownership of their solutions.

• *Students explaining and justifying*

This is similar to verbalizing but it is carried out after students have found a solution. So, students can reflect on the overall strategy used to accomplish the task. In addition, they have to answer questions raised by others.

Restructuring

If students show no progress toward a solution, more drastic measures are needed. Restructuring scaffoldings can be employed to assist students in finding the direction to pursue. Restructuring scaffoldings consist of the functions described below.

• *Identifying meaningful contexts*

When students struggle to understand an abstract mathematical concept, the teacher can provide a meaningful context to which the concept can relate. Students will come to identify key characteristics of the task and relate them to the familiar context as they develop their own understanding of the abstract links.

• *Simplifying the problem*

If students cannot accomplish a task, the teacher can simplify the task so that the simplified task is within reach of students and revert back to the original task once students have success with the simplified task. The teacher may also focus students' attention to a part of the original task that is key to the accomplishment of that task.

• *Rephrasing students' talk*

This is closely related to *interpreting students' actions and talk*, but here students' difficulties with the task are more extensive and students seem to give up on the correct line of reasoning. It also includes introducing and extending more formal language of mathematics, which, according to the relationship between thought and language, can help guide the cognitive construction of students.

• *Negotiating meanings*

It is through a struggle for shared meaning that a process of cooperatively figuring things out determines what can be said and understood by both teacher and students and this is what constitutes real mathematics learning in the classroom. This is in accordance with the notion of socially constructed knowledge in social constructionism.

Level 3: Developing conceptual thinking

This level of scaffolding aims at extending students' understanding beyond the mathematics directly learned from a specific task, especially through *abstraction*, *extension* or *extrapolation*, and *generalization*. It can be identified with the highest end of ZPD. Level 3 scaffoldings are described below.

Developing representational tools

Representational tools revolve around language, both the informal language used by students to describe models and the formal mathematical language introduced by the teacher and later used by students as they gain conceptual understanding of the topic. It is related to *rephrasing students' talk*, but while the latter aims at assisting students in accomplishing the task, the former aims at extending students' understanding gained from that accomplishment. Representations also include the structuring of practical activities to provide powerful visual imagery of the underlying concept and/or to provide a framework for the structuring of knowledge. They can also involve *notating* students' interpretations and solutions so that these symbolizations would then constitute a resource that students can use to express, communicate, and reflect on their mathematical activity.

Making connections

A myriad of connections exist in mathematics. Taking the four basic arithmetic operations for example, connections exist between each and every possible pair of operations. One might argue that students who do not understand these connections thoroughly also do not understand arithmetic operations, even though they may be fluent at carrying out the operations. If one asks elementary-school students to find the answer of $\frac{274+274+274}{3}$, many of them, who are fluent in all the four operations, will solve the problem by adding all the numbers in the numerator and then performing the division (Schoenfeld, 1988). Clearly, they lack the understanding of the connection between addition and multiplication, or between multiplication and division, or both. So, to develop conceptual thinking, the teacher needs to find opportunities to help students discover mathematical connections that are relevant to the topic being studied.

Generating conceptual discourse

The teacher can generate a conceptual discourse by posing a problem that can stimulate long and meaningful discussion, which requires students to employ mathematical thinking to understand and participate in the discussion. This is the highest form of scaffolding. Students who regularly participate in this kind of discourse will develop their mathematical thinking to the fullest potential.

Anghileri's (2006) characterization can serve as a framework for mathematics teachers to reflect and improve on their own teaching. As the study by Bliss, Askew, and Macrae (1996) indicated, proper scaffolding is extremely illusive. Memorizing all the scaffoldings in this list does not improve one's scaffolding practice either. However, given time, the list can serve to scaffold one's scaffolding from a novice level to an expert level.

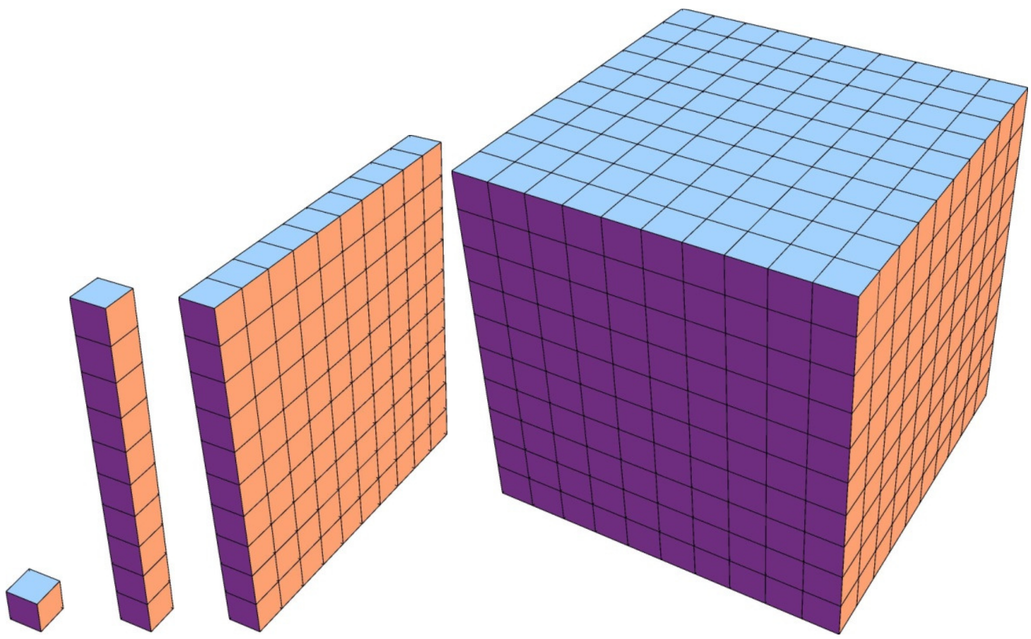


Figure 2.23 Multi-base blocks for base ten

2.2.3 The use of models

Zolton Paul Dienes (1995), the inventor of the *multi-base blocks*, depicted in Figure 2.23, comprehensively described the use of models in his theory of mathematical learning, which concerns abstraction and generalization of mathematical concepts. He characterized six stages of abstraction wherein models play crucial roles in the second to the fourth stages. In the second stage, a game that possesses the intended mathematical concept is played by students, without introducing any formal mathematics. The role of the model is to provide a concrete situation which students can understand, without the abstraction of mathematical symbols. The third stage involves at least two more games possessing the same concept, although not transparently, but employing different models. Toward the end of this stage, students should notice the similarity between these games, and this similarity gives rise to the initial idea of the abstraction. In the fourth stage, another model is used to represent the similarity. This last model is usually more compact that a previous model. The fifth stage is when symbols are introduced to represent the abstraction. So, symbols are introduced into an experience-rich context wherein students have clear images of what the symbols stand for. Since several models are used, one particular model does not have a dominant role starting from the end of the third stage. In addition, students understand the mathematical concept that the models represent.

Contrary to the teaching approach suggested by Dienes (1995) above, most teachers employ only one model in a particular topic. Also, they never follow Dienes' (1995) stages. Most of the time, the model is introduced as a tool to simplify mathematics operations. So, students cannot make the proper connection between the model and the mathematics they are supposed to learn. Thus, the findings about the effectiveness of models in the literature are mixed. The most striking was the result that the student who performed the best with multi-base blocks performed the worst in the written subtraction problems (Uttal, Scudder, & DeLoache, 1997). Uttal et al. (1997) explained this obvious separation of mathematics in the model and traditional mathematics using Deloache's (1995) dual representation hypothesis. The hypothesis implies that students may see the model as an object in its own right, not as a representation of the underlying mathematics.

The hypothesis arose after a series of experiments on children's understanding of the relation between a scaled model and the room that it represented. In the experiments, children were allowed to see where a small toy was hidden in a scaled model of a room in which all the miniature furnitures matched the real furnitures, both in their appearances and in their positions. Children were then brought to the real room and instructed to find the real toy that was hidden in the same place. Surprisingly, most children before the age of three could not infer the position of the toy from this apparently transparent symbol-referent relationship. Before the experiment, children were oriented about the similarities between the model and the room. If the orientation was withdrawn, even the three year olds could not find the toy. However, if children were made to believe that the model and the room were one and the same, even those before the age of three could accomplish the task. In order to instill this belief, children were allowed to witness the room being *shrunk* into a model-sized room before the experiment. The interpretation was that very young children saw the model as an object in its own right, having nothing to do with the room, but the shrinking made them think that the model was the room, not a separate object.

The implication of the dual representation hypothesis is that one needs to make sure that students know what models represent in a particular situation. Otherwise, the solution in the model world may stay there forever.

CHAPTER III
METHODOLOGY

Overview

This chapter describes the methodology and methods used to conduct the research to answer the research questions posed earlier. It comprises mathematical and educational aspects. The development of the instructional units used for conducting this research and the research methodology are described. Next, the data collection and analysis used for the evaluation of the instructional units are specified.

3.1 Study Design

The study design was divided into mathematical and educational aspects. The mathematical aspect was used to develop a family of geometric derivations of general summation formulas wherein derivations of the formulas for sums of integers and for sums of squares are manageable for high-school students and pre-service mathematics teachers. Two instructional units were developed: the funneled instructional unit employing the newly developed derivations aimed at comparing the effectiveness of the derivations on high-school students' achievement in and retention of basic summation formulas to that of algebraic derivations employed in traditional instruction, while the scaffolded (Wood, et al., 1976) instructional unit aimed to facilitate students' derivation of the formulas. The former instructional unit was designed in such a way that students had opportunities to participate in the derivations via class discussion while the latter was designed to encourage students to work cooperatively to investigate and derive the formulas by themselves. The teacher helped scaffold students' learning.

The development of the instructional units involved the following steps:

1. Develop the two instructional units.
2. Consult with curriculum experts to improve the instructional units.

3. Revise each unit according to suggestions of the experts.
4. Pilot the instructional units with high-school students and pre-service mathematics teachers.
5. Revise the instructional units according to comments and suggestions of the teachers, students, and experts.

3.2 Pilot Studies

The pilot study of the funneled instructional unit was conducted with 25 grade-twelve students, aged between 17 and 18 years from a school in the city area. The scaffolded instructional unit was tried out on 40 second- and third-year pre-service mathematics teachers from a public university in the city area of the country. The students were assessed for their learning outcomes by using various instruments, i.e., classroom observation, written work, and interview.

For the funneled instructional unit, the participants had already studied the topic in the previous year in a traditional way. This did not conform to the research design in two ways. First, since the study intended to compare the effectiveness of the instructional unit with that of a traditional one, one needed to be able to divide the participants into two or more groups but this was not possible because all the participants were in the same class which meant that they had to follow the same schedule (all day and everyday). This was the standard arrangement in many schools. Second, the participants should have no prior knowledge about the topic. Nevertheless, this trial was treated as a pilot study and the participants were administered the pre-test. Surprisingly, practically no one could recall any formula in the pre-test. Given that these formulas were not that complicated, this result was probably due to the combination of ineffective traditional mathematics instruction and underachieving participants. Still, it did provide one with an opportunity to see whether these students could benefit from the instruction. It turned out that they correctly restated 74.7 per cent of the formulas in the post-test. This was rather satisfactory providing that they probably classified themselves as nonmathematical.

The results from the pilot study of the scaffolded instructional unit showed that students found it difficult to follow the instruction and to meet the expectation in terms of derivations of the formulas while looking at the illustrations, especially those about sums of squares. After some higher-level scaffolding by the instructor, about forty per cent of the students could do the derivation. The rest could do better after the instructor resorted to lower-level scaffolding. The experience led to modification of the instructional process whereby students were guided into the lesson more slowly and with the assistance from extra illustrations. The slowdown in the preliminary guiding process on the summation of integers plus additional illustrations help students to derive the formula for sums of squares based on the physical models better. Regarding attitude toward the instructional units, the students expressed their willingness to collaborate in all activities of both units.

3.3 Mathematical Aspect

This section describes the methods used to develop the mathematical results.

3.3.1 Family of geometric derivations of general summation formulas

To develop a family of geometric derivations of general summation formulas, the researcher began by exploring the characteristics of geometric derivations of the formulas for sums of integers and sums of squares, especially their extendibility and the assemblage of their components. The former was to ensure that the derivation can be generalized and the latter was to ensure that it can be used to facilitate students' derivation of the formula for sums of squares. Once proper derivations were found, the researcher attempted to generalize the derivations to higher dimensions and to look for their connections with other forms of derivations. The resulting derivations and their connections to other derivations are discussed in section 4.1.1.

3.3.2 Another geometric derivation of the formula for sums of cubes

To develop a geometric derivation of the formula for sums of cubes that does not involve a square, the researcher investigated different shapes that can form a lattice. Once a proper shape was found, the researcher attempted to arrange copies of the shape in a way that can elicit the identity between sums of cubes and squares of sums of integers. The resulting derivation is discussed in section 4.1.2.

3.3.3 Procedure for the generation of general summation formulas

To develop a procedure for the generation of general summation formulas that can be easily implemented and converges quickly to the true sums, the researcher observed Witmer's (1935) coefficients from Table 2.9 and attempted to uncover the underlying relations among these coefficients. The resulting relations are discussed in section 4.1.3.

3.3.4 Applications

From the purely mathematical work (with cognizance of contributions from others) dealing with visualization of basic summations, the two instructional units, described in sections 4.2.1 and 4.2.2, were developed.

3.4 Educational Aspect

This section describes the methodologies for assessing the two instructional units that employed parts of the mathematical results to scaffold students' learning.

3.4.1 Funneled instructional unit

3.4.1.1 *Participants*

The participants were 66 grade-eleven students aged between 16 and 18 years from two classes in a high school in northern Thailand.

3.4.1.2 *Implementation of the instructional unit*

The implementation began by administering the pre-test to the participants in order to assess their prior knowledge about basic summations and their formulas. The time allowed for this phase was ten minutes. The developed instruction, which consisted of four activities, was then implemented. Students were given worksheets that helped guide them through the learning process intended by each activity. Throughout the four activities, class discussion was encouraged with the instructor providing the funneling questions. The objective of the first activity was to establish the fact that the number of smaller triangles in the larger one equaled the square of the number of tiers (see the large triangle in Figure 4.8). This knowledge was used in the second activity which showed the equality of sums of cubes and squares of sums of integers. The first activity also yielded the formula for sums of integers, which was also the aim of the third activity. The fourth activity shared the same concept with the third one in order to facilitate the transfer of knowledge. Students were expected to come up with the formula for sums of squares after the fourth activity. The remaining time was used to let students practice applying the formulas in an exercise. The instruction phase was allotted 90 minutes, followed by the post-test with questions parallel to those in the pre-test. As evidenced by students' comments, the instruction could have been better had more time been allotted, but that was not our choice to make because we had to follow the preset schedule in the curriculum. Fortunately, the instruction was spread over two sessions (on separate days) which coincided with the natural break in the instruction (after the activity for sums of cubes), affording the students to contemplate the learned lesson and engage in self-organized group discussion. Afterwards, students were asked to answer the questionnaire, which was used to probe students' opinions about the instruction.

The participating students had never studied the topic. The result of the pre-test confirmed that they had no prior knowledge about summation formulas. They were divided into two groups according to their regular class arrangement. It was impossible to group them differently because of the resulting conflict in their class schedules. Unfortunately, the average mathematics scores from the previous semester of both groups were significantly different, with the p-value (1 − significant level) less than 0.001 (see Table 3.1), because this school grouped students according to their academic achievements. So, the funneled instructional unit was tried on the weaker group, while the other group was taught in a traditional way (this was perhaps the only alternative). Let us call the former the treatment group, and the latter the control group. The control group consisted of 38 students (29 females and 9 males) while the number of students in the treatment group was 28 (9 females and 19 males). For the control group, Pascal's derivations of basic summation formulas were used instead of the newly developed derivations. Since the traditional instruction took less time to complete, students in the control group had more opportunities to apply the formulas in the exercise. Both groups were administered the retention test three months after the instruction.

Table 3.1 Previous-semester mathematics scores (funneled)

Group	Size	Mean (%)	SD	SE	t statistics	p-value
Control	38	78.9	8.11	1.32	4.71	0.000
Treatment	28	68.4	9.57	1.81		

3.4.1.3 *Data collection*

Students' achievement was assessed by the pre-test and the post-test. Students' reflection was also used for evaluating students' understanding.

Pre-test and post-test

The pre-test and the post-test consisted of three questions on basic summation formulas and three parallel questions on their applications. The pre-test scores were used to assess students' prior knowledge whereas the post-test scores were used to follow students' achievement after the intervention.

Retention test

The retention test contained only the questions on the formulas. The test was given three months after the post-test.

Students' reflection

Students' reflection was used to provide in-depth data on students' understanding of the derivations of basic summation formulas. After going through each concept, they were asked to write what they knew, what they did not understand, and factors affecting their learning. It was also used to probe students' perception of the instructional unit.

3.4.1.4 *Data analysis*

The data were analyzed using descriptive statistics, namely, means and standard deviations (SD), for the pre-test, the post-test, and the retention test. The t-test was employed for the comparison between the mean score of the control group and the corresponding mean score of the treatment group. The mean scores in consideration were the mean scores from the pre-test, the post-test, and the retention test.

3.4.2 Scaffolded instructional unit

3.4.2.1 *Participants*

The participants were high-school students and pre-service mathematics teachers. The 42 grade-eleven students were from a competitive school in a big city. The students, aged between 16 and 18 years, consisted of 32 females and 10 males. The 29 second-year pre-service teachers were from a public university in a medium-sized city. The pre-service teachers, aged between 19 and 21 years, consisted of 24 females and 5 males.

3.4.2.2 *Implementation of the instructional unit*

The implementation began by administering the pre-test to the participants in order to assess their prior knowledge about basic summation formulas. The time allowed for this phase was ten minutes. The developed instruction, which consisted of four activities, was then implemented. Unlike the funneled instructional unit, students were given worksheets that contained only the illustrations, which are discussed in section 4.2.2, on which the intended derivations were based. Throughout the four activities, peer collaboration was the primary mode of operation. The instructor monitored each group's progress and scaffolded when necessary. The scaffolding level progressed from high to low. The objective of the first activity was to derive the formula for sums of integers from Figure 2.6, which was similar to the figure used in the next activity to derive the same formula, whose extension to three dimensions was the basis for the derivation of the formula for sums of squares in the third activity wherein students were also given concrete models from which the concrete three dimensional object represented by the illustration could be built. The fourth activity required them to derive the formula for sums of cubes using Figure 4.7. The remaining time was used for debriefing. The instruction phase was allotted 150 minutes, followed by the post-test which was the same as the pre-test. Afterwards, students were asked to answer the questionnaire, which was used to probe students' opinions about the instruction.

Table 3.2 Average pre-test scores (scaffolded)

	Item	Mean ± SD (%) High-school (42)	Mean ± SD (%) Pre-service (29)	*t* statistics	*p*-value
S_1	Restatement	51.2 ± 50.0	6.9 ± 25.8	4.88	0.000
	Derivation	14.3 ± 35.4	0.0 ± 0.0	2.61	0.012
S_2	Restatement	4.8 ± 21.6	0.0 ± 0.0	1.43	0.160
	Derivation	0.0 ± 0.0	0.0 ± 0.0	–	–
S_3	Restatement	4.8 ± 21.6	0.0 ± 0.0	1.43	0.160
	Derivation	0.0 ± 0.0	0.0 ± 0.0	–	–
Total	Restatement	20.2 ± 24.3	2.3 ± 8.6	4.40	0.000
	Derivation	4.8 ± 11.8	0.0 ± 0.0	2.61	0.012

Table 3.2 shows the average pre-test scores of high-school students and pre-service mathematics teachers. Even though the high-school students had never studied the topic in school, they significantly outperformed the pre-service teachers, who had already studied the topic in high school. About half the high-school students could restate the formula for sums of integers while practically none of the pre-service teachers could. This information reflects the state of mathematics education in this country that most mathematics teachers have very poor understanding of mathematics even though they are required to take many advanced mathematics courses. That is, getting a bachelor degree is no guarantee of understanding. There should be more emphasis on pre-service mathematics teachers' understanding of topics they are going to teach, rather than on their exposure to more advanced topics they will never use.

3.4.2.3 *Data collection*

Students' achievement was assessed by the pre-test and the post-test. Students' reflection was also used for evaluating students' understanding.

Pre-test and post-test

The pre-test and the post-test were the same, consisting of three questions on basic summation formulas and three questions on their derivations. The pre-test scores were used to assess students' prior knowledge whereas the post-test scores were used to follow students' achievement after the intervention.

Students' worksheets

The worksheets containing students' derivations from the four activities were collected, one worksheet per group per activity. It was used as a formative assessment of the group's collective understanding of the derivation of the formula.

Students' reflection

Students' reflection was used to provide in-depth data on students' understanding of the derivations of basic summation formulas. After going through each concept, they were asked to write what they knew, what they did not understand, and factors affecting their learning. It was also used to probe students' perception of the instructional unit.

3.4.2.4 *Data analysis*

The data were analyzed using descriptive statistics for the pre-test, the post-test, and the worksheets. The t-test was employed for the comparison between the mean scores of the two groups of participants. The mean scores in consideration were the mean scores from the pre-test, the post-test, and the worksheets. In addition, the paired-sample t-test was employed for the comparison between the mean score from the pre-test and that from the post-test.

Summary of the Chapter

The methods for both mathematical and educational aspects were described in this chapter. The mathematical aspects consisted of three different topics, some of which were partially employed in developing the instructional units to promote students' understanding of basic summations. The educational aspects comprised two units. First was the funneled instructional unit and second was the scaffolded instructional unit. These instructional units were implemented on the high-school students from competitive schools in a big city and only the second unit on the pre-service mathematics teachers from a public university in a medium-sized city. Finally, the research instruments and data analysis were described.

CHAPTER IV
RESULTS

Overview

This chapter reports the family of geometric derivations of general summation formulas, together with two other mathematical results. The educational aspect is about the newly developed instructional units that employed the newly developed geometric derivations to scaffold students' learning. The results from students' conceptual test and students' reflection are described.

4.1 Mathematical Aspect

This section describes the three mathematical results.

4.1.1 Family of geometric derivations of general summation formulas

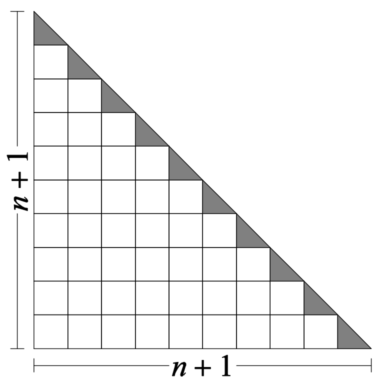


Figure 4.1 Subtracting excess from enclosing triangle yields sum of integers.

4.1.1.1 *Sums of integers*

Figure 4.1 shows that the total area (the number) of all the squares is equal to the difference between the area of the enclosing right triangle and the areas of all the small right triangles. That is,

$$S_1(n) = \frac{(n+1)^2}{2} - \frac{n+1}{2}. \quad (4.1)$$

The derivation is similar to Richards' (1984) depicted in Figure 2.6, but here, one employs the enclosing triangle instead of the inscribed triangle.

4.1.1.2 *Sums of squares*

Figure 4.2, the three-dimensional extension of Figure 4.1, illustrates that the total volume of all the cubes can be obtained by subtracting the excess from the enclosing right-angle square pyramid. That is,

$$S_2(n) = \frac{(n+1)^3}{3} - \left[\frac{n+1}{3} + \frac{2S_1(n)}{2}\right]. \quad (4.2)$$

4.1.1.3 *Sums of cubes*

The left-hand side of Figure 4.2 is a step pyramid whose i^{th} layer (counting down from the top) contains $i \times i$ unit cubes, forming a square-base box of unit height. This step pyramid is enclosed by the pyramid on the right-hand side, which has an $(n+1) \times (n+1)$ square base and comes to a point at the top, $(n+1)$ units above the base. By analogy, extending the step pyramid to the fourth dimension yields a hyper step pyramid whose i^{th} layer is a unit-height hyper box with an $i \times i \times i$ cubical base. This hyper step pyramid is enclosed by a hyper pyramid that has an $(n+1) \times (n+1) \times (n+1)$ cubical base and a point at the top. To facilitate our discussion, let the orientation of this enclosing hyper pyramid be the frame of reference for the directions *base* and *top*.

For sums of squares, the basic pieces whose volumes have to be subtracted off come in two shapes. Both shapes have unit-square bases (2-dimensional), but the tops of triangular prisms are unit-length edges (1-dimensional) while the tops of small pyramids are points (0-dimensional). For sums of cubes, the basic pieces whose hyper volumes have to be subtracted off will come in three shapes: all have unit cubical bases and their tops will be unit squares, unit-length edges, or points. Let W_p^k denote a basic piece whose base and top are of dimensions p and k respectively. For example, a small pyramid and a triangular prism in Figure 4.2 can be denoted by W_2^0 and W_2^1. Note that the hyper volume of a W_p^k is $\frac{1}{|p-k|+1}$ (it can be found by, for example, integration).

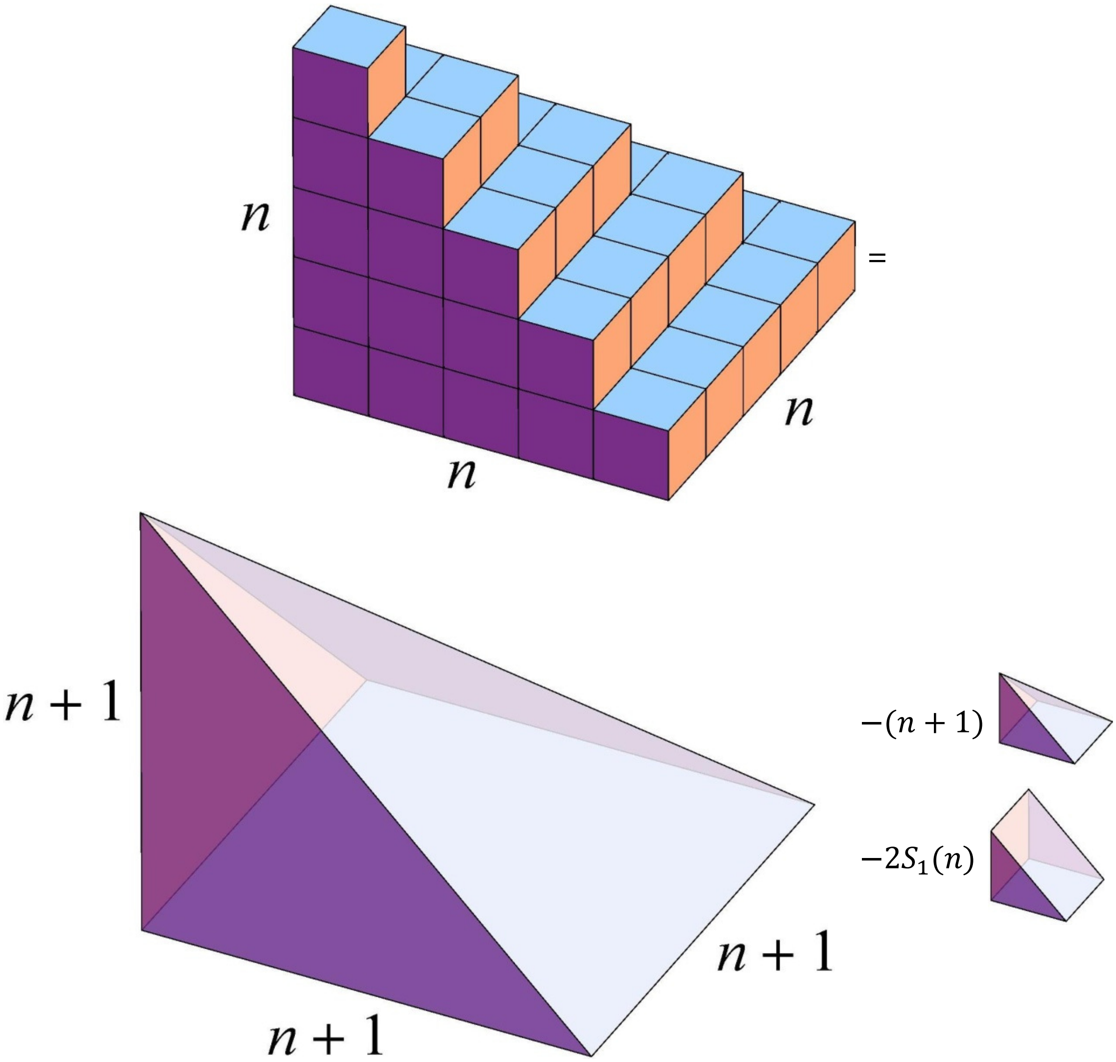
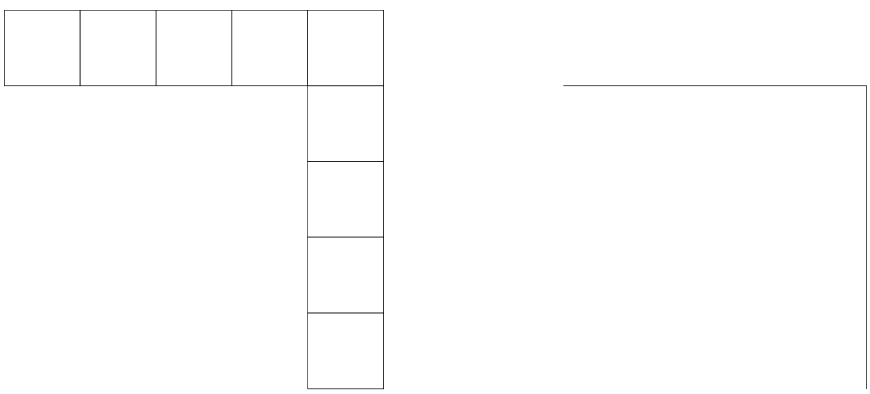
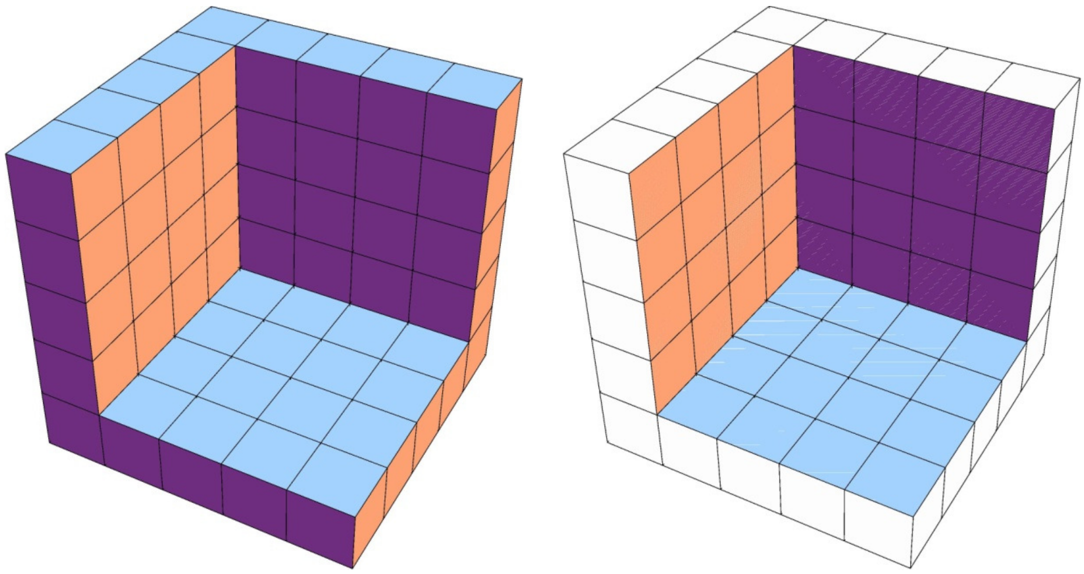


Figure 4.2 Subtracting excess from enclosing pyramid yields sum of squares.



(a) **(b)**

Figure 4.3 The base (a) and the top (b) of the fourth-layer L-shaped wedge



(a) **(b)**

Figure 4.4 The base (a) and the top (b) of the fourth-layer excess region

In Figure 4.2, if one turns a step pyramid with an $n \times n$ base into one with $(n+1) \times (n+1)$ base by adding unit cubes to each square layer along two adjacent sides of the square and adding a unit cube at the top, the intersections of the enclosing pyramid with these added cubes in a single layer would form an L-shaped wedge whose base and top are depicted in Figure 4.3. By analogy, for sums of cubes, unit hyper cubes would have to be added along three adjacent sides of each cubical layer and at the top to make a hyper step pyramid one unit larger along each edge, and it is the intersections of the enclosing hyper pyramid with these added hyper cubes that give the W_3^ks. The union of the W_3^ks from a single layer yields a region whose base is a three-sided wall of a hollow cube with unit thickness (Figure 4.4(a)) and whose top is the inside surface of a similar wall (Figure 4(b)). Looking at the base of this region from the i^{th} layer, one would see i^2 bases of W_3^2s along each side of the wall. Thus there are $3S_2(n)W_3^2$s. Similarly, along each edge of this wall are i bases of W_3^1s and there are three edges per wall (Figure 4.4) for the total of $3S_1(n)W_3^1$s. In addition, there is one W_3^0 at the vertex of each layer and at the top. So, the equation becomes

$$S_3(n) = \frac{(n+1)^4}{4} - \left[\frac{n+1}{4} + \frac{3S_2(n)}{2} + \frac{3S_1(n)}{3}\right]. \tag{4.3}$$

4.1.1.4 *General summations*

In order to determine the number of hyper edges for each hyper wall in higher dimensions, note that the wall in Figure 4.4(b) contains 3 (1-dimensional) edges because 2 out of 3 (2-dimensional) sides are needed to form an edge, for the total of $\binom{3}{2}$ edges with i^1 W_3^1s along each edge. For a general summation, k out of p, $(p-1)$-dimensional sides are required to form a $(p-k)$-dimensional edge for the total of $\binom{p}{k}$ such edges with i^{p-k} W_p^{p-k}s along each edge. Thus, the total hyper volume of all W_p^{p-k}s is

$$\frac{\binom{p}{k}}{k+1} S_{p-k}(n),\ k = 1, \dots, p-1.$$

In addition, all $(p-1)$-dimensional sides meet at the vertex of each layer to form the top of a $(p+1)$-dimensional small pyramid. A similar excess can be found at the top of the $(p+1)$-dimensional enclosing pyramid for the total of $(n+1)$ small hyper pyramids. Subtracting the excess from the enclosing hyper pyramid yields Pascal's formula (2.8). This means that the derivation can be perceived as a geometric interpretation of Pascal's general summation formula. Notice that equations (4.1)–(4.3) correspond to equation (2.8) when $p = 1$, 2, and 3.

4.1.1.5 *Pascal's formula with alternating signs*

It is straightforward to verify that applying the telescoping-sum technique to the binomial expansion of $[i^{p+1} - (i-1)^{p+1}]$ yields the following general summation formula:

$$S_p(n) = \frac{n^{p+1}}{p+1} + \sum_{k=1}^{p} \frac{(-1)^{k+1}\binom{p}{k}}{k+1} S_{p-k}(n). \tag{4.4}$$

This formula is more appealing to some because the first term on the right is a power of n, not $(n + 1)$. Its geometric interpretation is very similar to that of formula (2.8), only slightly more complicated. Figure 4.5, which is similar to Sakmar's (1997) derivation depicted in Figure 2.12, illustrates the differences between the two interpretations for sums of squares. Instead of subtracting the wedges (triangular prisms) from the enclosing pyramid, one needs to add the inverted wedges to the inscribed pyramid. However, where two edges meet at a corner, the two inverted wedges overlap to form an inverted pyramid, which has to be subtracted off. These inverted wedges are the parts of unit cubes in the step pyramid that lie outside the inscribed pyramid. In summary, looking at a single layer, instead of subtracting the union of all non-overlapping W_2^ks from the enclosing pyramid, one has to add the union of all W_1^2s, some of which overlap, to the inscribed pyramid.

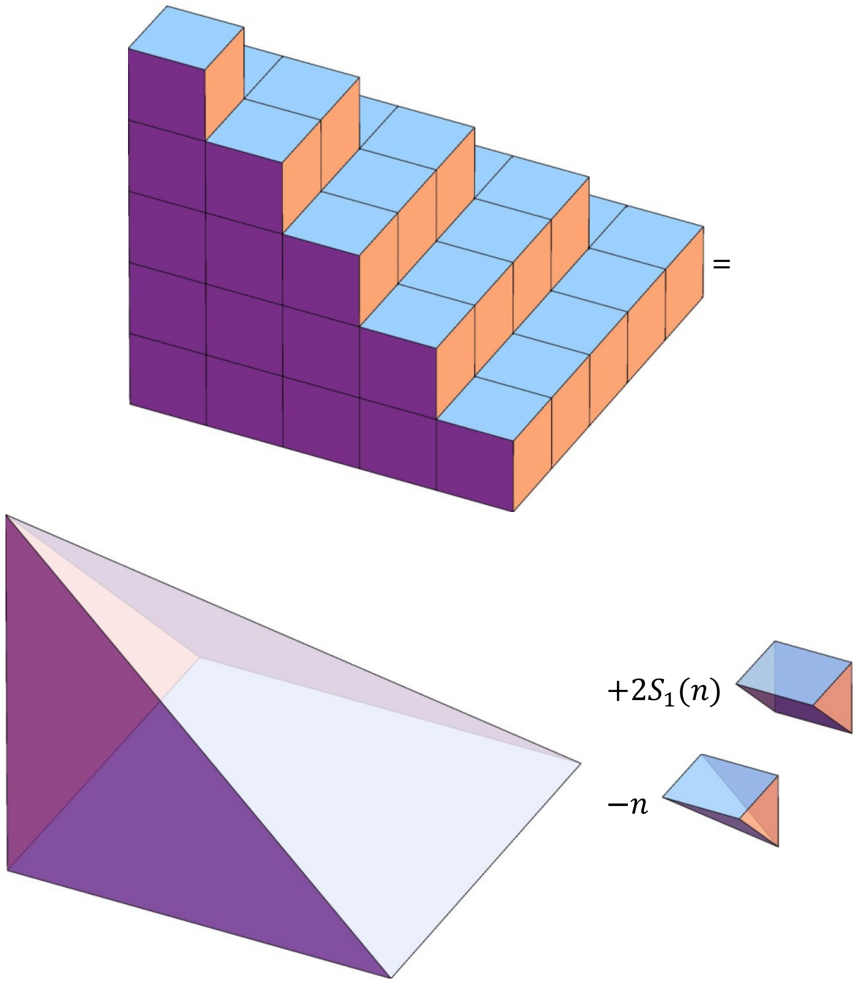


Figure 4.5 Adding excess (and subtracting overlaps) to inscribed pyramid yields sum of squares.

Analogously, for sums of cubes, Figure 4.4(a) would depict the top of the union of all W_2^3s from the *fifth* layer, and the corresponding base would be similar to its *outside* surface. For each unit along the edges of the wall, two W_2^3s overlap to form a W_1^3, which has to be subtracted off. At the vertex of each layer, three W_2^3s and three W_1^3s overlap (see Figure 4.4), nullifying each other. Thus, their overlap, W_0^3, has to be added back. A third-level overlap (in yet a higher dimension) involves $\binom{4}{1} W_{p-1}^p$s, $\binom{4}{2} W_{p-2}^p$s, and $\binom{4}{3} W_{p-3}^p$s with alternating signs. So, one would have already added $4 - 6 + 4 = 2$ copies of the overlap and thus have to subtract one W_{p-4}^p. In general, a j^{th}-level overlap involves $\sum_{i=1}^{j}(-1)^{i+1}\binom{j+1}{i} W_{p-i}^p$s. Since the sum of all, alternating-sign, binomial coefficients from an expansion is zero (it can be verified by expanding $(x-x)^n$), the sum of the numbers of W_{p-i}^ps is 2 for odd js and 0 for even js, which means that one has to alternately subtract and add consecutive levels of overlaps, in order to retain exactly one copy, resulting in equation (4.4).

For both derivations, the (hyper) step pyramids can also be made symmetrical if only for aesthetics, with W_p^ks and W_k^ps being evenly split among opposite sides, edges, or corners, without changing their total volumes.

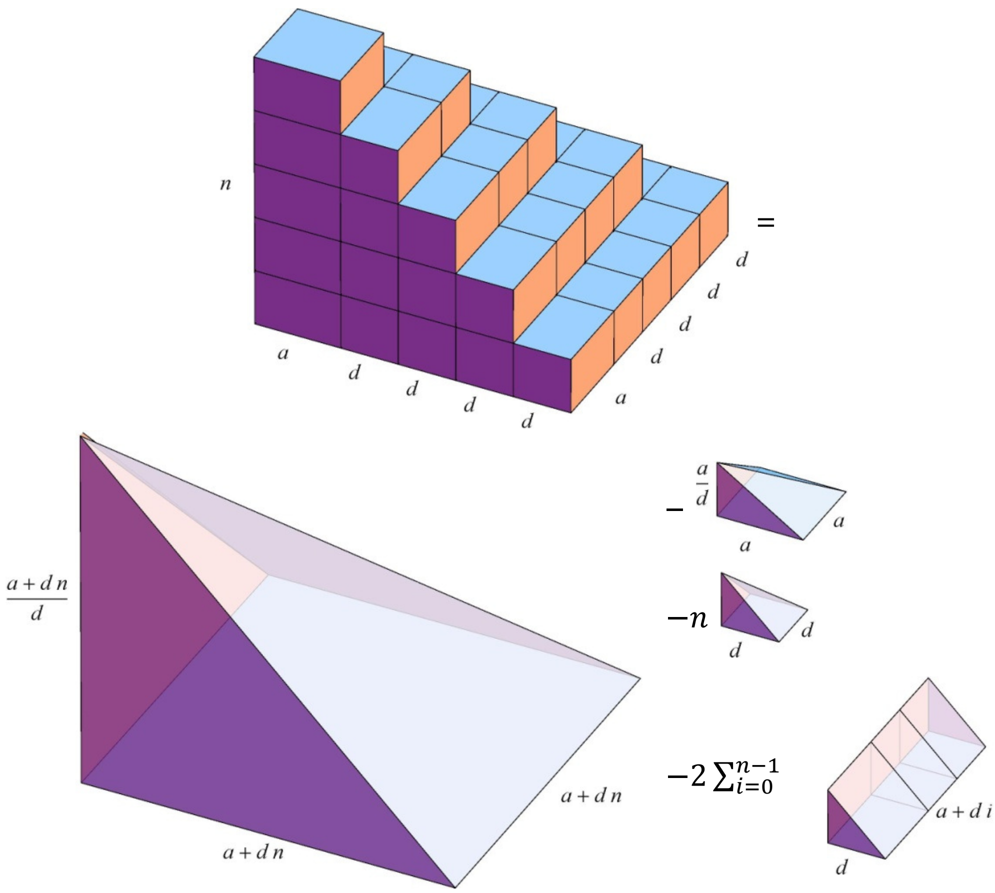


Figure 4.6 A geometric derivation of Pascal's progression summation formula

4.1.1.6 *Progression summations*

The geometric derivation can be applied to Pascal's progression summation formula (2.9). For arithmetic series, the width of the first column of the stack of rectangles in Figure 4.1 is replaced by a while that of other columns is replaced by d. The height of each layer is still one. Thus, the enclosing triangle has the width of $(a + nd)$ and the height of $\left(\frac{a}{d} + n\right)$. Figure 4.6 illustrates the derivation when p = 2. It is straightforward to verify that extending this construction to higher dimensions yields formula (2.9).

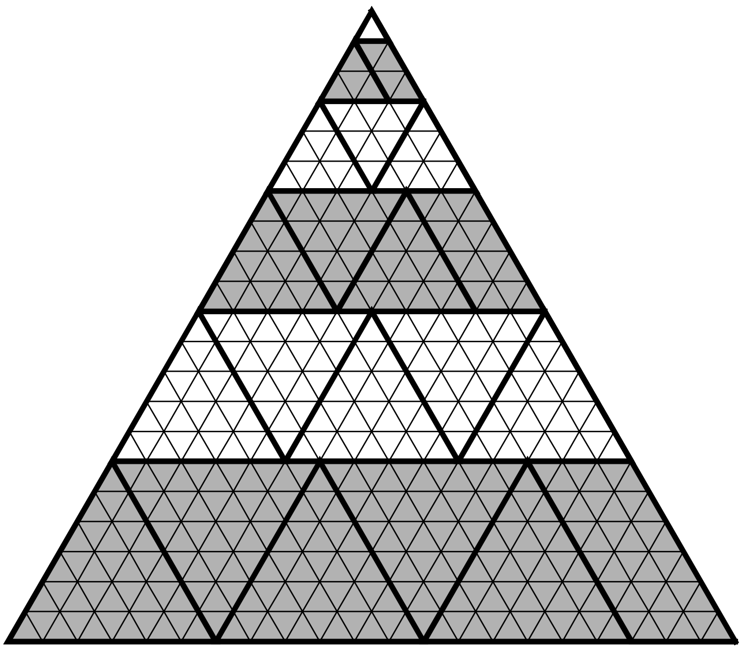


Figure 4.7 Another geometric derivation of the formula for sums of cubes

4.1.2 Another geometric derivation of the formula for sums of cubes

Instead of using squares, it turns out that triangles can also be arranged to elicit identity between sums of cubes and squares of sums of integers. The arrangement is depicted in Figure 4.7. From the figure, one may notice that the number of small triangles in the top tier is 1, in the next tier is 3, and in tier i is $(2i - 1)$. Thus, the total number of small triangles if there are i tiers is

$$1 + 3 + \cdots + 2i - 1 = i^2.$$

The tiers are grouped so that the i^{th} group contains i tiers. Thus, if there are n groups, the total number of tiers will be $S_1(n)$, which means that the total number of small triangles will be $S_1^2(n)$. Now, group i contains i triangles with thick borders, and each such triangle contains i tiers. Thus, the number of small triangles in group i is $i \times i^2 = i^3$, which means that the total number of small triangles is also $S_3(n)$.

4.1.3 Procedure for the generation of general summation formulas

For the third mathematical result, observe that Witmer's (1935) coefficients from Table 2.9 exhibit the same regularity as Bernoulli's if the constant terms are not considered. The regularity is expressed by equation (2.11). In addition, the sum of the non-constant coefficients for $S_p\left(n+\frac{1}{2}\right)$ is $\frac{1}{2^p}$. The constant term can be found by observing that $S_p\left(0+\frac{1}{2}\right)=0$. Let b_p^k denote the coefficient of $\left(n+\frac{1}{2}\right)^k$ in $S_p\left(n+\frac{1}{2}\right)$. Then, using Bernoulli's relation (2.12), one can write

$$S_p\left(n+\tfrac{1}{2}\right) = \sum_{i=0}^{p+1} b_p^{p+1-i}\left(n+\tfrac{1}{2}\right)^{p+1-i}$$

$$= \frac{1}{p+1}\sum_{i=0}^{p}\binom{p+1}{i} b_i^1 \left(n+\tfrac{1}{2}\right)^{p+1-i} + b_p^0. \qquad (4.5)$$

This is analogous to Bernoulli's equation (2.13). The coefficients b_p^1 can be recursively defined by the two equations:

$$b_0^1 = 1 \text{ and}$$

$$b_p^1 = \frac{1}{2^p} - \frac{1}{p+1}\sum_{i=0}^{p-1}\binom{p+1}{i} b_i^1, \ p = 1, 2, 3, \ldots$$

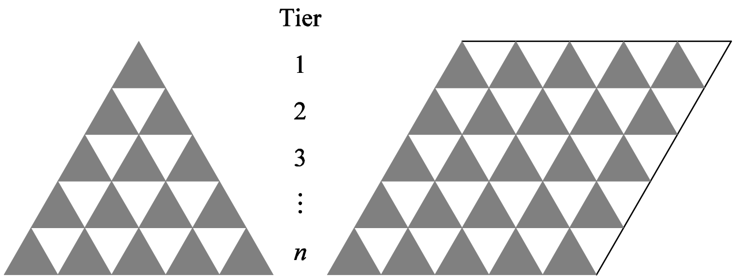
The coefficients b_p^0 can then be defined by the equation:

$$b_p^0 = -\sum_{k=1}^{p+1}\frac{1}{2^k} b_p^k, \ p = 1, 2, 3, \ldots$$

In fact, these are the equations used to create Table 2.9.

4.2 Educational Aspect

This section describes the two instructional units, together with the results of their implementation.



(a) **(b)**

Figure 4.8 A large triangle composed of smaller ones (a) and two large triangles put together (b)

4.2.1 Funneled instructional unit

Activity 1: Sum of integers

The instructor began the lesson by showing students an arrangement of triangles as in figure 4.8(a) and then asked them for the number of triangles in each tier. Students could see that tier i consisted of $(2i-1)$ triangles. The instructor then apposed two large triangles (containing smaller triangles) together as in figure 4.8(b) and asked them to count the number of right-side-up (gray) and upside-down (white) small triangles. They could see that there were n^2 triangles in each case for a total of $2n^2$ triangles. Thus, they could conclude that

$$2\sum_{i=1}^{n}(2i-1) = 2n^2 \Rightarrow \sum_{i=1}^{n}(2i-1) = n^2.$$

In other words, the number of small triangles equaled the square of the number of tiers. One could exploit this opportunity to remind students about the relationship between the ratio of sides and the ratios of areas and volumes, a simple fact that is often lost on students.

Once students saw that the number of smaller triangles in figure 4.8(a) is n^2, the instructor pointed out that it was the sum of right-side-up and upside-down triangles and asked about the number of tiers of up and down triangles, which were n and $(n-1)$ respectively. It became obvious that the number of up (and down) triangles was the same as the sum of integers up to the number of tiers of the triangles. Thus, we had $S_1(n) + S_1(n-1) = n^2$, and, by definition, $S_1(n) - S_1(n-1) = n$. Adding the two equations yielded the formula for sums of integers.

One could also ask students to derive the formula for sums of integers directly from the formula for sums of odd integers,

$$\sum_{i=1}^{n}(2i-1) = 2S_1(n) - n = n^2,$$

and solve for $S_1(n)$. From the researcher's experience, some students had difficulty expanding the summation even in this simple form. So, one should provide them with the opportunity to become more familiar with the expansion of summations.

Activity 2: Sum of cubes

Armed with the fact that the number of smaller triangles in an n-tier triangle is n^2, the instructor now guided students to find the formula for sums of cubes. Start with one one-tier triangle and two two-tier triangles as in Figure 4.9.

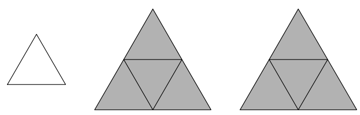


Figure 4.9 One one-tier and two two-tier triangles

The number of smaller triangles in one one-tier triangle can be represented by $1 \times 1^2 = 1^3$ and in two two-tier triangles by $2 \times 2^2 = 2^3$. Put together the three triangles as in Figure 4.10.

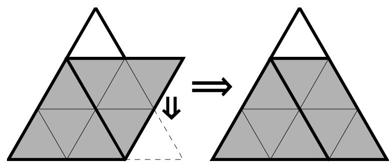


Figure 4.10 Putting one one-tier and two two-tier triangles together

Notice the realignment of a small triangle (jutting out). The instructor should prepare by attaching the small triangle to the bigger piece with sticky tapes so that it can be detached and reattached easily. These triangles could be made from corrugated plastic boards for durability and reusability.

Students already knew that the resulting triangle represents 3^2. Thus, they should see that $1^3 + 2^3 = 3^2 = (1+2)^2$. Add three triple-tier triangles (Figure 4.11) and attach them as in Figure 4.12.

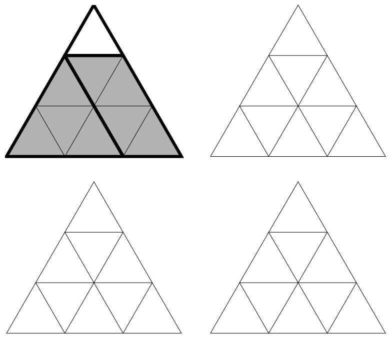


Figure 4.11 Adding three triple-tier triangles

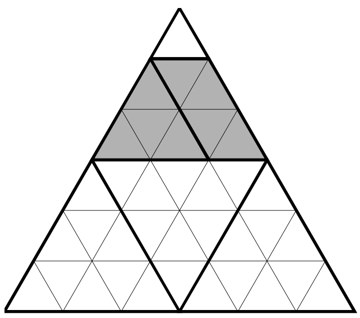


Figure 4.12 Attaching three triple-tier triangles to the bottom

By now students should be able to conclude that $1^3 + 2^3 + 3^3 = (1 + 2 + 3)^2$ and arrive at the formula for sums of cubes.

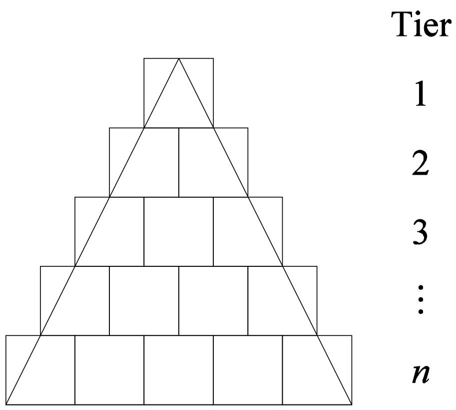


Figure 4.13 Stacked rectangles with the inscribed triangle

Activity 3: Alternative way to find the formula for sum of integers

To prepare students for finding the formula for sums of squares, the instructor used a similar approach to find the formula for sums of integers. Start by arranging unit squares as in Figure 4.13 and creating a triangle whose base was the same as the base of the squares and whose apex was at the middle of the top of the squares (the inscribed triangle). From Figure 4.13, it was clear that, for a given number of tiers, the number (area) of squares was equal to the sum of the area of the isosceles triangle and the areas of the right triangles. Since there were two right triangles in each tier and the area of a right triangle was $\frac{1}{4}$, we had

$$S_1(n) = \frac{n^2}{2} + \frac{2n}{4} = \frac{n(n+1)}{2}.$$

This derivation is similar to that of Richards (1984). However, here the instructor asked students to complete Worksheet 1 based on Figure 4.13.

Worksheet 1 The difference between the area of squares and the area of triangle

No. of tiers	**Area of squares (A_1)**	**Area of isosceles triangle (A_2)**	$A_1 - A_2$
1	1	$\frac{1}{2}$	$\frac{1}{2}$
2	$1 + 2 = 3$	2	1
3	$1 + 2 + 3 = 6$	$\frac{9}{2}$	$\frac{3}{2}$
n	$1 + 2 + \cdots + n$	$\frac{n^2}{2}$	$\frac{n}{2}$

Generally, students could easily complete the worksheet and reach the correct conclusion about the formula for sums of integers.

Activity 4: Sum of squares

Extending Figure 4.13 three dimensionally, squares would become cubes and the isosceles triangles would become square pyramids (see Figure 4.14). Students were asked to complete Worksheet 2. They could find the general forms of V_1 and V_2 easily but many required help in deriving the term for $(V_1 - V_2)$. For two tiers, $(V_1 - V_2)$ was $\frac{7}{3} = \frac{2}{3} + \frac{5}{3}$, and for three tiers, $(V_1 - V_2)$ was $5 = \frac{2}{3} + \frac{5}{3} + \frac{8}{3}$. Thus, the formula for $(V_1 - V_2)$ should be within reach of students. Armed with the general forms, the formula for sums of squares could be obtained as follows (once provided with the formula for $(V_1 - V_2)$, most students had no trouble deriving this):

$$S_2(n) = \frac{n^3}{3} + \frac{1}{3}\sum_{i=1}^{n}(3i-1)$$
$$= \frac{1}{3}\left(n^3 + \frac{3}{2}n^2 + \frac{3}{2}n - n\right)$$
$$= \frac{1}{6}(2n^3 + 3n^2 + n)$$
$$= \frac{n(n+1)(2n+1)}{6}.$$

Worksheet 2 The difference between the volume of cubes and that of pyramid

# of tiers	Volume of cubes (V_1)	Volume of pyramid (V_2)	$V_1 - V_2$
1	1	$\frac{1}{3}$	$\frac{2}{3}$
2	$1+4=5$	$\frac{8}{3}$	$\frac{7}{3}$
3	$1+4+9=14$	9	5
n	$1^2+2^2+\cdots+n^2$	$\frac{n^3}{3}$	$\frac{1}{3}\sum_{i=1}^{n}(3i-1)$

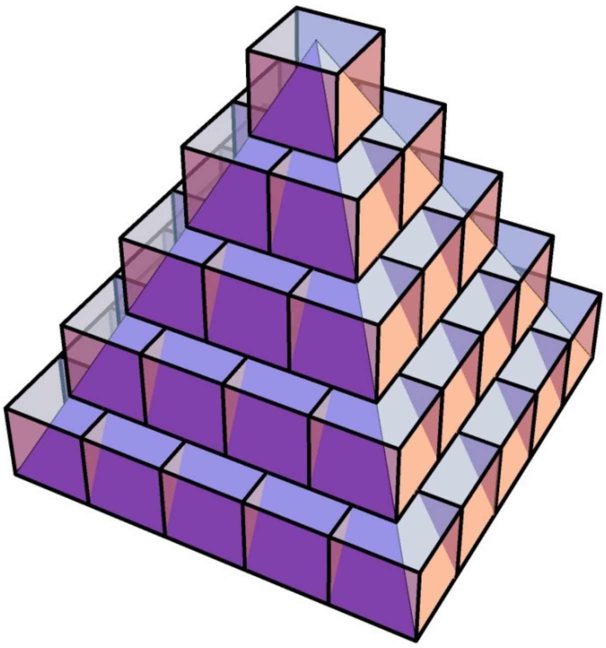


Figure 4.14 Stacked cubes with the inscribed pyramid

4.2.1.1 *Students' performance on recall and retention*

Generally almost all of the students, especially those in the treatment group, restated all the formulas correctly in the post-test, and when they did so, they usually had no trouble applying the formulas. Given that the average previous-semester mathematics score of the control group was significantly higher than that of the treatment group, this result indicated that the funneled instructional unit could help average students to perform at the level of high-achieving ones. As the results from the pilot study illustrated, the ability to correctly remember the formulas could not be taken for granted. Table 4.1 shows that, while the treatment group seemed to outperform the control group in the post-test, the difference was not significant, with the p-value of 0.24. This was to be expected since both groups performed close to perfection, leaving no room for statistical significance. The ostensibly large discrepancy in the standard deviations was due to the fact that practically all the students in the treatment group recalled the formulas perfectly, resulting in an unusually low standard deviation.

Table 4.1 Post-test accuracy in restating the formulas (funneled)

Group	Size	Mean (%)	SD	SE	t statistics	p-value
Control	38	94.7	19.8	3.21		
Treatment	28	98.8	6.3	1.19	1.19	0.240

Table 4.2 Retention-test accuracy in restating the formulas (funneled)

Group	Size	Mean (%)	SD	SE	t statistics	p-value
Control	38	57.9	46.9	7.60		
Treatment	28	81.0	36.8	6.93	2.24	0.029

The merit of this instructional unit was much more obvious in the retention test, which is shown in Table 4.2: the students in the treatment group significantly outperformed those in the control group, with the p-value of 0.029. The standard deviations were rather high because the majority of students could still remember all the formulas. Given that the average previous-semester mathematics score of the control group was higher, the significance of this result was even more pronounced. In general, one's performance in mathematics was related to one's attitude toward mathematics (Ma & Kishor, 1997). Thus, this result also reflected students' attitude toward the developed instruction.

4.2.1.2 *Students' perception*

The questionnaire was administered to the participants in the treatment group. It was open-ended, asking for students' positive experiences, negative experiences, and suggestions. Out of 28 participants, 16 and 12 participants commented on their positive and negative experiences respectively (not exclusively), while only 4 participants provided some suggestions. Of those sixteen students who commented on their positive experiences, seven specifically used the word *understand*. For example:

- [The instruction] *made me understand better.*
- [I] *understood* [the instruction] *and* [it was] *practical.*
- [It was an] *enjoyable learning experience, not stressful, and* [I] *understood* [the instruction].

Three of them mentioned *know*:

- [The instruction helped me] *know the formula and its application: great for exams.*
- [The instruction] *made me know more.*
- [The instruction] *made me know about powers.*

Two students were impressed by the *techniques* in the derivations:

- [I] *acquired new techniques in problem solving.*
- [I] *learned several techniques.*

Other positive experiences included:

- [The instruction helped me] *apply the formula more fluently.*
- *Um, good.*
- *An eye opener.*

There were other comments that were just as positive as the previous one but they were provided as negation to negative experiences:

- *None: everything was perfect.*
- *None, because all the activities were very good.*

- *I think* [the instruction was] *flawless.*

Two students blamed themselves, not the instruction:

- *Almost none: only some miscalculations.*
- *Everything was okay, but I was too sleepy.*

Three negative comments were the result of the limitations on the amount of time allotted for the instruction:

- *There was too little time for the instruction.*
- *There was some time-wasting during the instruction* [due to an interruption].
- *The instructor taught too fast. Sometimes I understood but sometimes I was confused.*

The problem of time limitation was not under our control: we had to follow the national curriculum, which was rigid and specified in detail how long each topic should be taught. In order to cover all the topics in the curriculum, most instructors gave priority to the width, instead of the depth, of knowledge, with the side effect of promoting rote learning, instead of learning with understanding. The priority issue is universal, affecting even (or especially) the most developed countries, and it is not limited to mathematics. Major curriculum reform is needed for learning with understanding to become a norm rather than an exception.

Naturally, there were some valid negative experiences:

- [The instruction] *was a little complicated:* [it] *confused me.*
- *Difficult.*
- [I] *still do not know as much as I should.*
- [The instructor] *should have provided more examples.*

The four suggestions overlapped with some positive and negative comments. They were:

- [The instructor] *should slow down: I could not follow the instruction.*
- [The instructor] *should provide more exercises so that I could practice until I became fluent, and did not easily forget the formulas.*
- [I] *wish that the instructor would teach here regularly.* [It was] *enjoyable.*
- [The instructor] *should teach other topics, for example, trigonometry, so that they would be more understandable.*

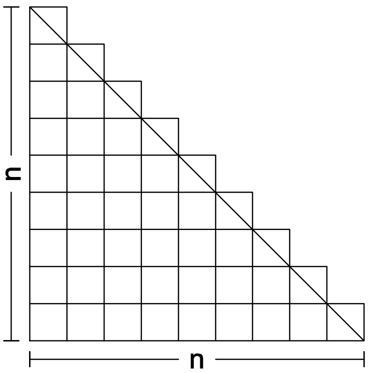
4.2.2 Scaffolded instructional unit

The structures of all four activities in this instructional unit were similar. First, the instructor handed out a worksheet that contained nothing but an illustration and an instruction. The instruction was to find the formula for one of the three basic summations using the illustration provided. In the third activity, students were also provided concrete models that could be used to physically construct the three-dimensional object in the illustration as an aid in finding the formula. The illustrations in other activities represented two-dimensional objects. Throughout this section, scaffolding functions will be identified using Anghileri's (2006) framework by parenthesizing the functions.

Students were encouraged to work in groups and were free to talk to those in other groups or to spy on them (*Level 1: environmental provisions*). The instructor monitored the progress of each group and responded to any request for help by using an appropriate scaffolding function described in section 2.2.2.2. The request for help might not be explicit, e.g., blank faces, playing with the models, and signs of discouragement. Also, if the instructor observed that a group employed incorrect reasoning, immediate attention should be given. The scaffolding could also be performed by distributing appropriate clues. The clues were prepared in advance by the researcher. However, the researcher found that students responded better to the instructor than to the clues. Since there were several groups in a class, it was advisable to have an assistant or two when conducting this kind of classroom.

Activity 1: Sum of integers

The instruction started by dividing students into groups of three to five and posing the following problem:



Use the figure on the left to find the formula for

$$1 + 2 + 3 + \cdots n.$$

Please record all your thoughts and ideas, regardless of whether they work or not, on this worksheet. Please also identify whether any clue was useful and how it was useful in finding the formula.

This is Richards' (1984) geometric derivation depicted in Figure 2.6 and explained in sections 1.1.2 and 4.2.1. It was selected as the first activity due to its simplicity and its relevancy to the next activity, which, in turn, relates to the third activity. In the pilot study, the researcher found that starting with the second activity was not a good idea: students would take hours to find the formula. The clues for this activity were as follows:

1. Let the lengths of the sides of □ be 1. Which part of the figure can be represented by $1 + 2 + 3 + \cdots n$?

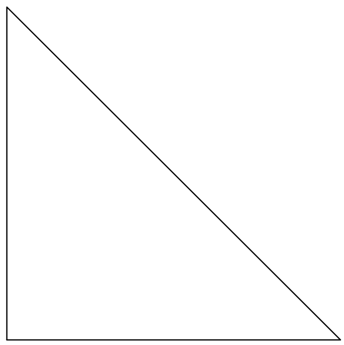
This clue should help to *simplify the task* (*Level 2*) for students. Actually, all clues could be classified as such. It also helped them connect the mathematical symbols to the image (*Level 3: developing representational tools*). They could now focus their attention on this easier task. However, before distributing a clue, the instructor had to be certain that the group had not grasp the idea underlying that clue. This could be achieved by looking at the worksheet, listening to their conversation, or simply asking them (which would be equivalent to giving out the clue verbally). If any group requested further help on this clue, the instructor could distribute its sub-clue:

1.1. How many □ are there and what is their total area?

This clue was much more specific and fell somewhere between *prompting* and *probing* (*Level 2*) but the prompting was done only to get the answer to clue 1: students still had to employ mathematical thinking to find the formula. If a group still had difficulties with this clue, the instructor had to attend to the group personally, and probably had to employ more prompting questions, for example, asking them for the number of squares in each tier, starting from the top. This question could also be made into a clue, but be reminded not to use any indexing variable because it would confuse students as the researcher found out in the pilot study. Then, the instructor might need to ask them about the sum of these numbers. From the researcher's experience, students had the impulse to give the result of an arithmetic operation or operations when asked even though the operation(s) could not be carried out. In this case, a lot of them replied n when the instructor asked for the sum, leading to more scaffolding.

2. How can we find the area of the steps in clue 1 by using the areas of other shapes besides that of □?

This clue was the main idea behind a geometric derivation: using objects whose areas or volumes can be found in order to find the required unknown area. Similar ideas are pervasive in mathematics (*Level 3: making connections*). Obviously, there were sub-clues.

2.1. What is the area of  ?

2.2. How many  are there and what is their total area?

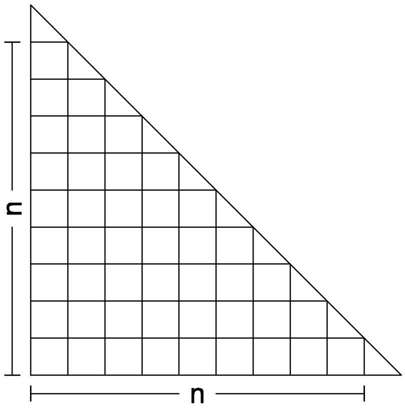
The researcher did not expect students to have difficulties answering these two questions. However, far too many students never understood the concept of generalization in mathematics. They tended to answer with a number (40.5 for the clue 2.1) even though the figure was design to discourage them from counting the exact number of squares. It seemed that they had never been required to think mathematically during their dozen plus years of mathematics lessons. The last clue was there to help them put two and two together, in case they needed it:

3. How can the relation between the areas from clue 1 and clue 2 be described by and equation? (The area from clue 1 should be on the left side of the equation.)

Even though the task seemed so simple, a group of students usually took more than twenty minutes to find the formula. This should serve as a telltale sign of the failure of traditional instruction in mathematics. Personally, the researcher found that students' work in the next activity not only helped confirm this failure, but also strengthened it.

Activity 2: Another sum of integers

The instructor posed the following problem as a continuation of the first activity and as a precursor to the next activity.



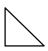
In the previous activity, we used the triangle inscribed inside the steps to find the formula for

$$1 + 2 + 3 + \cdots + n.$$

If we replace the inscribed triangle with the triangle that surrounds the steps, we will get the figure on the left. *Use the figure on the left to find the formula for*

$$1 + 2 + 3 + \cdots + n.$$

This activity employed the newly developed geometric derivation described in section 4.1.1.1. Despite the seemingly transparent relationship between the two activities and despite the researcher's attempt to draw the similarity between them in posing the problem, it was almost as if students had never finished the first activity (even though the instructor tried to make sure that they did) before attempting this activity. Far too many of them did not notice any difference between the two figures even though the difference was practically handed to them. Even more surprising was the fact that some groups who accomplished the first task all by themselves (without any clue) failed to see the connections between the two tasks and could not find the formula with the aid of the clues. Perhaps, what they did in the first activity was trying to put different combinations of known formulas together to see whether any combination yielded the formula for sums of integers that they already knew. Some group could not even give the correct answer for the length of a side of the big right triangle.

This derivation belongs to the family of derivations described in section 4.1.1 and the researcher has never seen this derivation anywhere. The clues were almost identical to those for the first activity except that one had to replace  with  in clue 2.2.

After accomplishing the task, students were asked to construct the figure with the concrete models designed for the next activity. The concrete models were the component pieces in Figure 4.2 (unit cubes, small triangular prisms, and small right-angle square pyramids). They could be constructed using the folding diagrams in

Figures 4.15–4.17. For better learning, different kinds of objects should be in different colors (*Level 3: developing representational tools*).

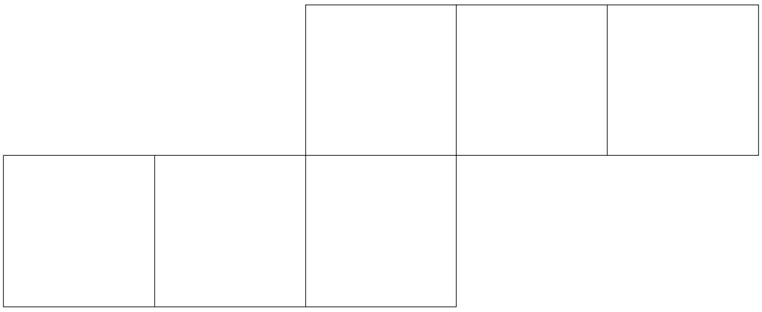


Figure 4.15 A folding diagram of a unit cube

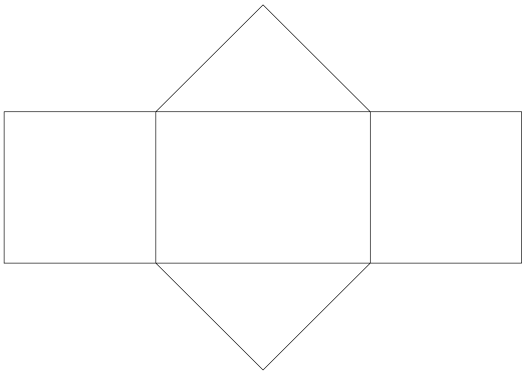


Figure 4.16 A folding diagram of a triangular prism

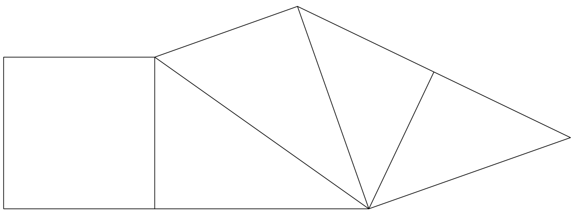


Figure 4.17 A folding diagram of a right-angle square pyramid

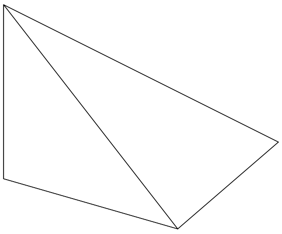
The next activity was key to this instructional unit in particular and to this research study in general. From the literature review, one can see that there is only one purely geometric derivation of the formula for sums of squares, namely, Sakmar's (1997) derivation described in section 2.1.3.6 and depicted in Figure 2.12. Unfortunately, it is hard to construct a concrete model of the derivation because it would involve overlaps. If one were to employ the model to scaffold students' derivation of the formula, one would have to show them Figure 4.14 and hope that somehow they could get to Figure 2.12 and could then derive the formula, which would be too much to hope for. Besides, it would be very difficult for the instructor to scaffold students' derivation from Figure 4.14 without showing them Figure 2.12.

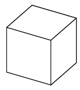
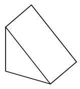
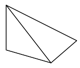
Other models that may be employed to scaffold students are Kung's (1989), Kalman's (1991), and Conway and Guy's (2006). Kung's (1989) derivation, described in section 2.1.3.8 and depicted in Figure 2.15, employs an arrangement of three copies of numbers in a triangular form so that the sums of the three copies are equal across the triangle. It would be difficult to design an activity based on this derivation without revealing too much to students. However, it is an excellent alternative for a demonstration. Conway and Guy's (2006) derivation, described in section 2.1.3.9 and depicted in Figure 2.16, shares these characteristics with Kung's (1989). Kalman's (1991) derivation, described in section 2.1.3.6 and depicted in Figure 2.11, due to its geometric nature, has potential as a basis for an activity. However, the result seems like a magic, thus, does not provide enough insights into the formula.

As a result, the researcher developed a geometric derivation of the formula for sums of squares that is three-dimensional and can be concretely constructed. The derivation is described in section 4.1.1.2 and depicted in Figure 4.2. Since the figure is two dimensional, it does not compare with the concrete model. Some curriculum experts suggested that it would be hard for students to construct the pyramid using the figure from the next activity, which provides less detail than does Figure 4.2, because most students do not have the developed visualization skills to follow the two-dimensional picture provided in the activity without the three-dimensional object. From the researcher's experience, a group of students usually took about ten to fifteen minutes to construct the pyramid. It was the fun part of the activity and never was a problem. The difficulties came after the construction.

Activity 3: Sum of squares

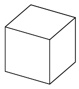
The instructor posed the following problem.



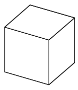
Construct the object on the left using as many  *and as few*  *and*  *as possible. Use the constructed model to find the formula for*

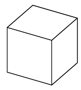
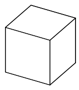
$$1^2 + 2^2 + 3^2 + \cdots + n^2.$$

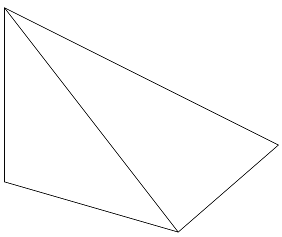
This is the three-dimensional extension of the preceding activity as it is the second member of the family of derivations described in section 4.1.1. In retrospect, perhaps this relationship should be the first clue of this activity because it exemplifies one of the most important concepts in mathematics, namely, *extension* (*Level 3*). Instead, the first clue for this activity was analogous to that of the previous activities.

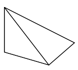
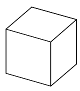
1. Let the lengths of the edges of  be 1 and their number of layers in the constructed model be n. Which part of the figure can be represented by $1^2 + 2^2 + 3^2 + \cdots + n^2$?

Other clues for this activity, which are presented below, were also analogous to those in the previous activities.

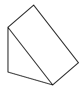
1.1. How many  are there and what is their total volume?

2. How can we find the volume of the step pyramid (all the  in the constructed model) by using the volumes of other shapes besides that of ?

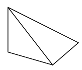
2.1. What is the volume of  ?

2.1.1. How many  does it take to make ?

Being used to symmetrical pyramids, students might not recognize the pyramids in the model, or they might forget the formula for the volume of a pyramid. This clue should help them realize that the volume of this kind of pyramid can be found physically, comparable to apposing two right triangles together in order to find the area of a right triangle (*Level 3: making connections*).

2.2. How many  are there and what is their total volume?

This was one of the most difficult steps for students even though they had just completed two activities on sums of integers. Looking at Figure 4.2, one might require some visualization to see that the number of triangular prisms in a layer is twice the length of that layer. However, this fact was plainly obvious in the constructed model. Students' difficulties probably stemmed from the fact that they never saw the connections between classroom mathematics and real-world situations that required the same mathematics (*Level 3: developing representational tools*). In retrospect, a neat scaffolding activity for this clue would be to have students look at the constructed model from the front view facing one of the two sides having the triangular prisms and ask whether they recognize what they see. The view would be the same as the figure in Activity 2.

2.3. How many  are there and what is their total volume?

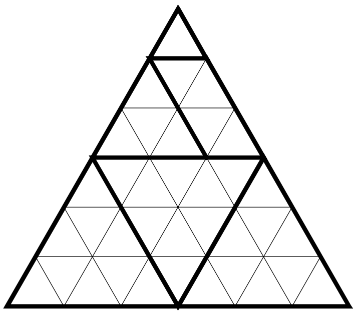
The instructor might need to remind students of clue 2.1.1 in order to help them find the volume of a pyramid. The most common mistake, however, was the fact that students tended to see only n such pyramids.

3. How can the relation between the volumes from clue 1 and clue 2 be described by and equation? (The volume from clue 1 should be on the left side of the equation.)

The resulting equation was not that easy to simplify. However, the instructor should emphasize the thought process, not the simplification of the end result. In addition, students could usually correct their own mistakes regarding simplification.

Activity 4: Sum of cubes

The instructor posed the following problem.



Small triangles are the ones with thin borders.

Big triangles (and parallelograms) are the ones with thick borders.

Small tiers are between thin horizontal lines.

Big tiers are between thick horizontal lines.

If there are n big tiers, use the figure on the left to find the formula for

$1^3 + 2^3 + 3^3 + \cdots + n^3$.

This is the geometric derivation described in section 4.1.2 and used in activities 1 and 2 of the funneled instructional unit. To make the clues somewhat parallel to those in other activities, the first clue dealt with the representational problem.

1. How can the figure be represented by $1^3 + 2^3 + 3^3 + \cdots + n^3$?

This clue was substantially more difficult than the first clues in other activities, but at least it should help focus students' attention on the fact that the whole figure could be represented by $S_3(n)$. From the clue, some students should be able to deduce that the number of small triangles in a big tier is the cube of the ordinal number of the tier, which was suggested by the sub-clue.

1.1. How many small triangles are there in a big tier?

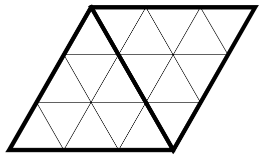
From the researcher's experience, after counting the number of small triangles in the first three big tiers, students could notice that they were the cubes of the first three positive integers. However, some students might not be familiar with these numbers. They might require further sub-clues.

1.1.1. How many big triangles and parallelograms together are there in a big tier?

1.1.2. How many small triangles are there in a big triangle or parallelogram?

The underlying relation for this sub-clue was also used in answering clue 2. From the researcher's experience, students could arrive at the desired relation, usually by counting the number of small triangles in the first few small tiers. However, the strategy employed in activity 1 of the funneled instructional unit and depicted in Figure 4.8 should give students better visualization of the relation. The sub-clue helped guide students along this direction.

1.1.2.1. Put two big triangles together in this manner:



How many △ and ▽ are there in each small tier?

The instructor should find an opportunity to *make the connection* (*Level 3*) between this relation and a similar relation between the side length of a ruled big square and the number of small squares in it, and to extend the connection to similar relations for other regular shapes via *conceptual discourse* (*Level 3*), probably in a debriefing session.

Coupled with this relation, students needed to see that the number of small tiers could be represented by $S_1(n)$ to arrive at the identity between sums of cubes and squares of sums of integers.

2. How can we find the number of all small triangles without using the cubes of integers?

This clue was to reassure students that there was another way of counting the number of small triangles and to help them see that it was key to this activity.

2.1. How many small tiers are there?

2.1.1. How many small tiers are there in each big tier?

From the researcher's experience, even though students could find the answer to this sub-clue and to sub-clue 1.1.2, they had difficulty combining the two answers together. So, the instructor needed to provide them with *probing* or *prompting* questions (*Level 2*). For a group that found that the number of small triangles was $S_3(n)$ without being given clue 1.1.2, the instructor should employ the clue at this point.

3. How can the relation between clue 1 and clue 2 be described by and equation? (The answer to clue 1 should be on the left side of the equation.)

At the end of all the activities, there should be a debriefing session that focused on the ideas behind these derivations and the connections among them (*Level 3: generating conceptual discourse*) in order to help students construct a conceptual framework that held the newly acquired knowledge together.

Table 4.3 Average post-test scores (scaffolded)

	Item	Mean ± SD (%) High-school (42)	Mean ± SD (%) Pre-service (29)	t statistics	p-value
S_1	Restatement	100.0 ± 0.0	98.6 ± 5.16	1.44	0.161
	Derivation	77.4 ± 41.6	27.6 ± 45.5	4.69	0.000
S_2	Restatement	85.7 ± 31.0	53.8 ± 36.7	3.83	0.000
	Derivation	50.7 ± 47.6	2.8 ± 10.3	6.32	0.000
S_3	Restatement	95.2 ± 21.6	79.3 ± 22.3	2.99	0.004
	Derivation	53.8 ± 47.6	17.9 ± 36.8	3.58	0.001
Total	Restatement	93.7± 14.0	77.2 ± 17.2	4.25	0.000
	Derivation	60.6 ± 38.0	16.1 ± 26.3	5.83	0.000

4.2.2.1 *Students' performance on recall and derivation*

Table 4.3 shows the average post-test scores of high-school students and pre-service mathematics teachers. The high-school students performed much better after the intervention. The mean scores were 100, 85.7 and 95.2 per cent in restating the formulas for sums of integers, sums of squares, and sums of cubes respectively. The majority of them also knew how to derive the formulas, although not quite up to the satisfactory levels. The pre-service teachers, on the other hand, showed less improvement after the intervention when compared to the high-school students. In restating the formulas, the mean scores were 98.6, 53.8, and 79.3 per cent. The pre-service teachers still showed poor ability to derive the formulas with the averages scores of 27.6, 2.8, and 17.9 per cent for the three formulas. The post-test scores of the high-school students were significantly higher than those of the pre-service teachers in all items tested, expect in restating the formula for sums of integers where almost all of the participants had no trouble. The results indicated that good understanding of basic mathematics was a prerequisite for more advanced mathematics.

Table 4.4 compares the average pre- and post-test scores of high-school students while Table 4.5 compares those of pre-service teachers.

Table 4.4 Average pre- and post-test scores of high-school students (scaffolded)

	Item	Mean ± SD (%)		t statistics	p-value
		Pre-test	Post-test		
S_1	Restatement	51.2 ± 50.0	100.0 ± 0.0	6.33	0.000
	Derivation	14.3 ± 35.4	77.4 ± 41.6	7.71	0.000
S_2	Restatement	4.8 ± 21.6	85.7 ± 31.0	13.63	0.000
	Derivation	0.0 ± 0.0	50.7 ± 47.6	6.91	0.000
S_3	Restatement	4.8 ± 21.6	95.2 ± 21.6	19.72	0.000
	Derivation	0.0 ± 0.0	53.8 ± 47.6	7.32	0.000
Total	Restatement	20.2 ± 24.3	93.7± 14.0	16.33	0.000
	Derivation	4.8 ± 11.8	60.6 ± 38.0	8.82	0.000

Table 4.5 Average pre- and post-test scores of pre-service teachers (scaffolded)

	Item	Mean ± SD (%)		t statistics	p-value
		Pre-test	Post-test		
S_1	Restatement	6.9 ± 25.8	98.6 ± 5.16	19.06	0.000
	Derivation	0.0 ± 0.0	27.6 ± 45.5	3.27	0.003
S_2	Restatement	0.0 ± 0.0	53.8 ± 36.7	7.90	0.000
	Derivation	0.0 ± 0.0	2.8 ± 10.3	1.44	0.161
S_3	Restatement	0.0 ± 0.0	79.3 ± 22.3	19.11	0.000
	Derivation	0.0 ± 0.0	17.9 ± 36.8	2.63	0.014
Total	Restatement	2.3 ± 8.6	77.2 ± 17.2	21.83	0.000
	Derivation	0.0 ± 0.0	16.1 ± 26.3	3.29	0.003

The post-test scores were significantly higher than the pre-test scores in all items tested but to a different extent. For sums of integers, the increase in restating the formula was higher among the pre-service teachers, when compared to the high-school students. However, the increase in deriving the formula was higher among the high-school students. For the other two summations, the increases were higher among the high-school students for all items tested. The results suggested that the newly developed instructional unit although could promote participants' understanding, it still could not bring the lower achievers with poorer prior knowledge (the pre-service teachers) up to the same level as the higher achievers (the high-school students), especially regarding the derivations of the formulas.

Table 4.6 Average worksheet scores (scaffolded)

	Mean ± SD (%)			
Item	**High-school (42)**	**Pre-service (29)**	***t* statistics**	***p*-value**
S_1	98.0 ± 6.2	91.4 ± 21.8	1.10	0.290
S_2	82.0 ± 19.9	88.6 ± 15.7	–0.76	0.460
S_3	86.0 ± 23.2	85.7 ± 9.8	0.03	0.973
Total	91.0 ± 7.7	89.3 ± 15.1	0.26	0.790

Table 4.6 shows the average worksheet scores of both groups of participants. The participants were asked to give answers and record their findings in the given worksheets during the implementation. The results indicated that there was no significant difference between the average worksheet score of high-school students and that of pre-service teachers. The results suggested that the work done in the worksheets did not reflect the capabilities of individual participants. One or two persons in each group might understand the topic and filled out the worksheet while other might not quite understand. Another possible reason was that the worksheets could not be used to measure students' achievement.

4.2.2.2 *Students' perception*

Both high-school students and pre-service teachers reflected on what they had learned and what they had gained upon participating in the instructional unit. The results of students' reflection are shown in Table 4.7 as percentage of students covering each type of reflection. As shown in the table, both high-school students and pre-service teachers had similar thought on the instructional unit, albeit with lower achievement in the latter group. Regarding their comparison between the newly developed instructional unit and traditional instruction, the students enjoyed and preferred the new instructional unit. The main reason for their preference was that the scaffolding activities in the new learning unit helped them to learn with comprehension. The cooperative work also helped them to think together, to share idea, and to discuss with their group mates. Another important reflection point was that the illustrations made them understand more. They enjoyed this type of learning activity. They had a chance to review their prior knowledge on summations. Most importantly they reflected that the teacher helped facilitate them in conducting the given tasks which were very helpful for them.

Table 4.7 Students' reflection: HS—high-school, PS—pre-service (scaffolded)

Topic	Students' reflection	% covering	
		HS	PS
The newly developed instructional unit versus traditional instruction	Illustrations were good for understanding	76.2	75.9
	Given a chance to participate in activities	45.2	48.3
	Liked being in a group	42.9	44.8
	Liked to think for one self	40.5	44.8
	Teacher provided guidance	21.4	24.1
	It was fun	19.0	17.2
	Given a chance to review about summations	–	17.2
Advantages of the learning unit	Illustrations and 3-D objects supplement learning	71.4	75.9
	Learning by concepts not by rote	35.7	37.9
	Cooperative learning	35.7	34.5
	New ways of learning about summations	30.9	24.1
	There was rhyme and reason for doing things	19.0	24.1
Disadvantages of the learning unit	Difficult and complicated process of learning	23.8	34.5
	Could not figure out the illustrations	11.9	34.5
	Too time consuming	7.1	6.9
Comments and suggestions	Should teach summation basics before instruction	23.8	–
	Should review before starting lessons	–	20.7

Concerning the advantages and disadvantages of the instructional unit, the students voiced that the newly developed instructional unit encouraged them to think critically so that they could construct knowledge by themselves. They did not have to memorize the formulas like in traditional instruction. They really appreciated the learning activities that helped them and led them step-by-step in their higher-order thinking, resulting in better understanding. They were so proud that they were able to derive the formulas from the given illustrations. They said that they never had experience like this in their previous learning. The derivations of the formulas were more authentic to them than ready-made formulas. However, about twenty per cent of the high-school students and thirty per cent of the pre-service teachers voiced that the mathematical content was rather difficult for them: they were not able to follow and catch up with the learning activities. Some of them had already forgotten the formulas and it was difficult for them to recover their knowledge. About ten per cent of the high-school students and thirty per cent of the pre-service teachers stated that they could not visualize from the given illustrations and thus resulting in their failure to understand. Some students seemed to think that too much time was wasted. This was in contrast to other students who actively participated in the activities that voiced that more time should be given. However, most students agreed that the learning activities encouraged them to work cooperatively which benefited all the members of the group, and to learn more actively resulting in more participation in all the activities and better construction of knowledge. Nevertheless, the students commented that the instructor should review necessary basic knowledge on summations before the actual implementation of the unit.

CHAPTER V
DISCUSSION

Overview

This chapter interprets and discusses the research findings. The purely mathematical part needs no discussion here. However, the researcher has employed the novel geometric derivations of basic summation formulas in the two novel instructional units in order to facilitate students' derivation of these formulas. The effects of both novel instructional units on students' learning and attitude are discussed.

5.1 Funneled Instructional Unit

The results indicated that the developed instructional unit could improve both students' performance on basic summation formulas and students' retention of the formulas. Students' comments suggested that this was because the instruction was understandable and enabled them to enjoy the learning experience. This could not have happened if it was not simple enough, yet a simpler instruction might not better elicit understanding from students. According to Ma and Xu (2004), the relationship between attitude toward mathematics (ATM) and achievement in mathematics (AIM) was an imbalanced and reciprocal one, with AIM demonstrated causal predominance (priority) over ATM. This relationship was most significant during the later junior and early senior high-school grades. The implication of their findings for the developed instruction in particular, and mathematics instruction in general, was that the beginning of the instruction had to be understandable in order to promote students' attitude toward the instruction. Once their attitude became positive, better outcomes could be expected. This AIM-ATM-AIM cycle should be maintained throughout the instruction.

There was one result that might be a cause for concern, i.e., the result that there was no significant difference in the average post-test scores between the control group and the treatment group. One could only conclude that the developed instruction helped average students to perform at the level of higher-achieving ones. It would be ideal if one could attain two equally achieving groups or could regroup them randomly. The result from the retention test suggested that, had one been able to do so, the result from the posttest would have been clearer. A larger sample size would also help a great deal.

In designing the instruction, the researcher followed Albert Einstein's advice to *make everything simple but not simpler*. Good instruction should enable students to learn the intended knowledge with understanding, without unnecessary mental loads on the learners. One has to be careful that an innovative derivation of a basic summation formula, while elegantly designed, may inadvertently invoke irrelevant knowledge that makes the learning more complicated than what is intended. Not only should one engage students' preconceptions (the first principle in the findings by the Committee on *How People Learn* (Fuson, Kalchman, & Bransford, 2005)), but those preconceptions should be the simplest one that still allows students to make progress toward the learning objectives of that particular instruction. In addition, different students come to class with different preconceptions. So, one may need to prepare different ways of engaging students so that none would be left out. In addition, one has to find ways to assess students' prior knowledge in order to select the instruction properly.

The developed instruction is a novel one that enables instructors to broaden their repertoire. The researcher would like to encourage experts in different fields to adapt this approach. Bear in mind that the developed instruction has to conform to the three principles in Fuson et al. (2005), together with the suggestion in the previous paragraph, as much as possible. Another way to contribute to better teaching and learning is to develop novel tests that can elicit students' prior knowledge on a particular topic of interest. A good assessment goes hand in hand with good instruction. Instructors are also encouraged to keep an eye on research that could provide them with more and better ways of assessing and teaching.

5.2 Scaffolded Instructional Unit

The pre-service mathematics teachers had extremely low scores in the pre-test, reflecting their unsatisfactory achievement in spite of their claimed interest in mathematics. The researcher suspects that mathematics teachers who are not currently teaching summations cannot do any better. This has been found out by another study. The high-school students, although performing better in the pre-test, may not really be better than the pre-service teachers because the high-school students had only recently received the instruction and thus had still fresh memories of the summations taught, be it in terms of formulas or derivations. Conceptually, one cannot be sure that the groups were different in their mastery. Nonetheless, it is more important for pre-service and in-service mathematics teachers to have thorough understanding of basic mathematics in general, and of the topic they are teaching in particular. Such understanding cannot be developed through traditional instruction. Thus, in pre-service programs, it is imperative to adopt the kind of instruction that emphasizes understanding instead of having pre-service teachers enroll in more advanced mathematics courses. In addition, there should be more professional development programs that employ the same strategy in this country.

Working together as a group may lead to better interactions and development, but the success depends on group members who are competent and have desire to try and compete. There were some groups that progressed with enthusiasm, while others were stuck and had to be prompted more.

In addition to cooperative learning of students in each group and the final whole class discussion of students' accomplishment or effort, during the instructional period, the instructor guided the students by giving purposeful hints and suggestions. This was done for individual groups thus addressing their specific needs and stages of learning. Some groups were quick to grasp the concept of using the physical aids as extra components for association of things learned. Although others were slower in grasping the concepts involved, they eventually came to grip with the problems because the physical aids together with the scaffolding by the instructor facilitated the learning. The physical aids might have helped lessen the mental load in terms of imagination and might have added new dimensions to tackling complicated problems.

While the simplicity should be emphasized in designing the funneled instructional unit, the scaffolded instructional unit always contains a certain degree of difficulty. The derivation of the formula for sums of cubes is a good example. It was employed in both instructional units, but while the participants in the former felt that it was easy, those in the latter felt that it was difficult. The time spent on this derivation in the scaffolded unit was about the same as the time spent on the first three activities combined. Since this unit is time-consuming, especially when implemented on relatively low achievers, its implementation may require some curriculum adjustment. A regular instructor, who knows about students' capabilities and curriculum requirements, should be able to select a proper instructional unit. One may opt for one or two scaffolded activities and conduct the remaining derivations in a funneled fashion.

CHAPTER VI
CONCLUSIONS

Overview

This chapter summarizes the research findings. The novel geometric derivations of sums of powers of integers are described. The newly developed instructional unit has been shown to enhance students' performance in the derivation of the formulas. The implications, limitations and recommendations of this research study are also discussed.

6.1 Summary of the Research Findings

In purely mathematical term, the researcher has come up with a family of novel geometric derivations of the formulas for sums of powers of integers whose dimensional numbers are one more than the powers. However, since one can only reach the third dimension pictorially, the derivation of the formula for sums of cubes can only be drawn by a composite picture. The formulas for sums of higher powers can be deduced and be mathematically generalized.

For the sums of integers, squares, and cubes, the newly devised geometric derivations (physical aids and illustrations) have been used to enhance students' understanding and facilitate formula derivation in an instructional unit, wherein the pre-service mathematics teachers and high-school students were instructed to derive the formulas for sums of squares and cubes by first being exposed to sums of integers pictorially. Most students, who originally could not write nor derive these formulas, ended up doing so quite well.

It is recommended that the pictorial and physical representations plus other existing ones should be used to help students learn conceptually with long term retention.

6.2 Implications of the Research Study

Instructors should use the newly developed physical aids and/or those of others in helping students understand concepts involved in summation formulas. As shown in the review and the preambles, there have been several ways of illustrating the derivations. The physical aids may be more helpful in promoting students' understanding.

Pre-service teachers in general should have the experience of working with students using the physical aids under guidance. The researcher believes that the conceptualization ability acquired through this type of exposure, i.e., scaffolding with physical aids, will be retained for long duration. This should result in their better future instruction in their in-service years.

6.3 Limitations

Although the physical aids may be helpful to some students, other alternative physical aids may be more useful to the others. Moreover, there are also counter arguments to physical aids in the sense that those who are good at imagination may be deprived of the opportunity to exploit it. The polemic is always on whether one should focus on the abstract or on the physical in developing the mathematical mind.

This study's results came from limited numbers of students in only two schools and one university. More extended studies with a larger population of students and teachers should yield results that can be generalized, leading to possible change in practice and policy.

6.4 Recommendations

There are recommendations for further study and those for further development.

6.4.1 Recommendations for further study

In this research study, the newly developed instructional units were implemented by doctoral students whose expertise was not in teaching summations to high-school students or pre-service mathematics teachers. Thus, the instructional units should be implemented by an instructor experienced in teaching summations, or at least experienced in teaching high-school mathematics, instead of a doctoral student or a university instructor whose expertise lay elsewhere.

At present, the majority of our students are female, which, according to TIMSS 2007 (Martin, Mullis, & Foy, 2008), might stem from the fact that, Thai female students outperformed Thai male students on average (p.59), and thus more of the former progressed to the high-school and university level. In fact, among the participating non-Middle-Eastern countries, Thailand had the largest difference between female student's average mathematics score in grade eight and that of male students. This should not be taken a sign to celebrate the success of a feminist movement, but rather as a sign of the failure of mathematics education in this country. From a relatively recent study with extremely large sample size (more than two hundred thousand participants), males tended to be better at visuospatial abilities that utilized Euclidean reasoning while females tended to be better at object-location memorization (Collaer, Reimers, & Manning, 2007; Silverman, Choi, & Peters, 2007). According to Herlitz, Nilsson, and Bäckman (1997), better memorization in females applied to a variety of tasks, both recall and recognition in nature. In addition, both visual and verbal working memories affect mathematical performance (Miller & Bichsel, 2004). So, instruction that promotes rote learning suits female students better while instruction that employs visual aids should benefit male students more than their female counterparts. Thus, one may hypothesize that the main reason for the difference in average mathematics score between the two genders is the prevalence of traditional mathematics instruction in this country, with the detrimental effect that students of both genders have very little chance of understanding. In fact TIMSS 2007 (Martin, Mullis, & Foy, 2008) also found that eighth graders from Thailand had the second lowest average mathematics score among students from the participating non-Islamic countries in Asia, Australia, Europe, and North America (p.35). So, it would be interesting to see whether the newly developed instructional units could both enhance students' performance and close the gap between the two genders. To be able to answer the latter, the number of participants has to be large enough and be representative of the population, and the assessment has to be more thorough and discriminating.

The scaffolded instructional unit emphasizes students' thought process in the derivation of basic summation formulas. The key to this unit is the third activity, which requires students to extend the derivation in the second activity to three dimensions. However, it seems that the post-test fails to measure students' thought process in general, and their understanding of extension in particular. As mentioned in section 2.2.2.2, extension is one of the advanced core concepts in mathematics. Thus, it is important to know about students' understanding of extension. A carefully designed in-depth interview should be able to elicit this kind of information.

6.4.2 Recommendations for further development

At the beginning of the scaffolded instructional unit, students had to be induced into thinking pictorially rather slowly and more explanations given to the easiest one, i.e., sums of integers. Once they grasped the concept, they could go on to sums of squares and sums of cubes more easily. The learning process should be modified to suit the learners of different levels of achievement. For example, the teacher might need to use more scaffolding questions or more clues. Additionally, different students may have different way of learning. The illustration used in the study may be appropriate for only some students due to their multiple intelligences. Alternative physical models such as Siu's (1989) and Kalman's (1991) should be constructed and tried out.

The relation between the current activity for sums of cubes in the scaffolded instructional unit and other activities is rather weak. In retrospect, Schrage's (1992) derivation of the formula for sums of cubes depicted in Figure 2.21 is much more closely related to other activities. So, to facilitate students' construction of a conceptual framework that holds the newly acquired knowledge together, it should replace the last activity. The clues for the new activity will also be more similar to those in other activities and less confusing to students. Thus, students should take less amount of time to complete the activity, enhancing the attractiveness of the instructional unit as a whole.

With the last activity in the scaffolded instructional unit replaced, the two instructional units would have almost nothing in common, which would increase the instructor's workload if one wants to keep the option of selecting or mixing the two instructional units open. Thus, the activities in the funneled instructional unit should all be replaced to match those in the scaffolded one, both in their contents and order. This way, the two units represent two ends of a spectrum, enabling the instructor to conveniently mix the two units to suit the level of students and the time constraint.

REFERENCES

Anghileri, J. (2006). Scaffolding practices that enhance mathematics learning. *Journal of Mathematics Teacher Education, 9*, 33–52.

Beery, J. (2009). Sums of powers of positive integers. *Loci: Convergence*. Retrieved from MathDL website: http://mathdl.maa.org/mathDL/46/?pa=content&sa=viewDocument&nodeId=3284 doi:10.4169/loci003284

Beery, J., & Stedall, J. (2009). *Thomas Harriot's Doctrine of Triangular Numbers: The "Magisteria Magna"*. Zurich: European Mathematical Society.

Benjamin, A. T., & Orrison, M. E. (2002). Two quick combinatorial proofs of $\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2$. *The College Mathematics Journal, 33*(5), 406–408.

Bernoulli, J. (1959). Jacques (I) Bernoulli on the "Bernoulli numbers" (J. Ginsburg, Trans.). In D. E. Smith (Ed.), *A Source Book in Mathematics* (pp. 85–90). Mineola, NY: Dover Publications.

Bliss, J., Askew, M., & Macrae, S. (1996). Effective teaching and learning: scaffolding revisited. *Oxford Review of Education, 22*(1), 37–61.

Burrows, B. L., & Talbot, R. F. (1984). Sums of powers of integers. *The American Mathematical Monthly, 91*(7), 394–403.

Chilaka, J. O. (1983). Proof without words: sum of squares. *Mathematics Magazine, 56*(2), 90.

Conway, J. H., & Guy, R. K. (2006). *The Book of Numbers*. New York: Copernicus.

Chuang, P.-C. (1989). $\sum_{i=1}^{n} \sum_{j=i}^{n} j = \sum_{i=1}^{n} i^2$. *The College Mathematics Journal, 20*(2), 123.

Collaer, M. L., Reimers, S., & Manning, J. T. (2007). Visuospatial performance on an internet line judgment task and potential hormonal markers: sex, sexual orientation, and 2D:4D. *Archives of Sexual Behavior, 36*(2), 177–192.

Cupillari, A. (1989). Proof without words: $1^3 + 2^3 + 3^3 + 4^4 + \cdots + n^3 = \frac{\{n(n+1)\}^2}{4}$. *Mathematics Magazine, 62*(4), 259.

DeLoache, J. S. (1995). Early symbol understanding and use. In D. Medin (Ed.), *The Psychology of Learning and Motivation* (Vol. 33, pp. 65–114). New York: Academic Press.

Dienes, Z. P. (1995). *Some Thoughts on the Dynamics of Learning Mathematics*. Wolfville, Nova Scotia.

Edwards, A. W. F. (1986). A quick route to sums of powers. *The American Mathematical Monthly, 93*(6), 451–455.

Edwards, A. W. F. (1987). *Pascal's Arithmetical Triangle: The Story of a Mathematical Idea*. London: Charles Griffin.

Farlow, S. J. (1995). Count the dots: $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$. *The College Mathematics Journal, 26*(3), 190.

Flores, A. (1998). Sum of cubes. *The College Mathematics Journal, 29*(1), 61.

Fry, A. L. (1985). Proof without words: sum of cubes. *Mathematics Magazine, 58*(1), 11.

Fuson, K. C., Kalchman, M., & Bransford, J. D. (2005). Mathematical understanding: an introduction. In M. S. Donovan & J. D. Bransford (Eds.), *How Students Learn: Mathematics in the Classroom* (pp. 217–256). Washington, DC: National Academies Press.

Golomb, S. W. (1965). A geometric proof of a famous identity. *The Mathematical Gazette, 49*(368), 198–200.

Herlitz, A., Nilsson, L.-G., & Bäckman, L. (1997). Gender differences in episodic memory. *Memory & Cognition, 25*(6), 801–811.

Kalman, D. (1991). $(1 + 2 + \cdots + n)(2n + 1) = 3(1^2 + 2^2 + \cdots + n^2)$. *The College Mathematics Journal, 22*(2), 124.

Knuth, D. E. (1993). Johann Faulhaber and sums of powers. *Mathematics of Computation, 61*(203), 277–294.

Kung, S. H. (1989). Sum of squares. *The College Mathematics Journal, 20*(3), 205.

Leibniz, G. W. (2008). *The Early Mathematical Manuscripts of Leibniz* (J. M. Child, Trans.). New York: Cosimo.

Love, J. B. (1977). Proofs without words: cubes and squares. *Mathematics Magazine, 50*(2), 74.

Ma, X., & Kishor, N. (1997). Assessing the relationship between attitude toward mathematics and achievement in mathematics: a meta-analysis. *Journal for Research in Mathematics Education, 28*(1), 26–47.

Ma, X., & Xu, J. (2004). Determining the causal ordering between attitude toward mathematics and achievement in mathematics. *American Journal of Education, 110*(3), 256–280.

Martin, M. O., Mullis, I. V. S., & Foy, P. (2008). *TIMSS 2007 International Mathematics Report: Findings from IEA's Trends in International Mathematics and Science Study at the Fourth and Eighth Grades*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Lynch School of Education, Boston College.

Miller, H., & Bichsel, J. (2004). Anxiety, working memory, gender, and math performance. *Personality and Individual Differences, 37*(3), 591–606.

Nelsen, R. B. (1990). Proof without words: squares of triangular numbers. *Mathematics Magazine, 63*(3), 178.

Nelsen, R. B. (2000). *Proofs Without Words II: More Exercises in Visual Thinking*. Washington, DC: The Mathematical Association of America.

Paul, J. L. (1971). On the sum of the *k*th powers of the first *n* integers. *The American Mathematical Monthly, 78*(3), 271–272.

Richards, I. (1984). Proof without words: sum of integers. *Mathematics Magazine, 57*(2), 104.

Rogoff, B., & Wertsch, J. V. (1984). *Children's Learning in the "Zone of Proximal Development"*. San Francisco: Jossey-Bass.

Sakmar, I. A. (1997). Proof without words: sums of squares. *Mathematics Magazine, 70*(3), 212.

Schoenfeld, A. H. (1988). When good teaching leads to bad results: the disasters of "well taught" mathematics courses. *Educational Psychologist, 23*(2), 145–166.

Schrage, G. (1992). Proof without words. *Mathematics Magazine, 65*(3), 185.

Shult, D. L., & Tenenbaum, J. (1988). A three-dimensional multiplication table inspires mathematical discovery. *Mathematics Teacher, 81*, 92–95.

Silverman, I., Choi, J., & Peters, M. (2007). The hunter-gatherer theory of sex differences in spatial abilities: data from 40 dountries. *Archives of Sexual Behavior, 36*(2), 261–268.

Siu, M.-K. (1984). Proof without words: sum of squares. *Mathematics Magazine, 57*(2), 92.

Somchaipeng, T., Kruatong, T., & Panijpan, B. (in press). Discs as models for physical and mathematical proofs of series: square of sum and sum of squares. *Mathematics Teacher*.

Stein, R. G. (1971). A combinatorial proof that $\sum k^3 = (\sum k)^2$. *Mathematics Magazine, 44*(3), 161–162.

Turner, B. (1977). Completing the cube. *Mathematics Teacher, 70*(1), 67–70.

Turner, B. (1980). Sums of powers of integers via the binomial theorem. *Mathematics Magazine, 53*(2), 92–96.

Turner, B. (1981). Point-and-line proof for the sum of the cubes. *The Two-Year College Mathematics Journal, 12*(4), 270–271.

Uttal, D. H., Scudder, K. V., & DeLoache, J. S. (1997). Manipulatives as symbols: a new perspective on the use of concrete objects to teach mathematics. *Journal of Applied Developmental Psychology, 18*, 37–54.

Vygotsky, L. S. (1978). *Mind in Society: The Development of Higher Psychological Processes*. Cambridge, MA: Harvard University Press.

Williams, M. (1989). Vygotsky's social theory of mind. *Harvard Educational Review, 59*(1), 108–126.

Witmer, E. E. (1935). The sums of powers of integers. *The American Mathematical Monthly, 42*(9), 540–548.

Wood, D., Bruner, J. S., & Ross, G. (1976). The role of tutoring in problem solving. *Journal of Child Psychology & Psychiatry & Allied Disciplines, 17*(2), 89–100.

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BIOGRAPHY (cont.)

PUBLICATIONS

Busadee, N., Laosinchai, P., & Panijpan, B. (2011). Possibility and probability lessons from popular sports. *Mathematics Teacher, in press*.

Busadee, N., Panijpan, B., Laosinchai, P., & Ruenwongsa, P. (2010). Enhancing highschool students' achievement in permutation and combination through nontraditional word problems, sport problems and probabilistic games. *The International Journal of Learning, 17*(7), 413–428.

Laosinchai, P., & Panijpan, B. (2011). A geometric interpretation of Pascal's formula for sums of powers of integers. *The American Mathematical Monthly, in press*.

Piyayodilokchai, H., Ruenwongsa, P., Ketpichainarong, W., Laosinchai, P., & Panjaburee, P. (2011). Promoting students' understanding of SQL in a database management course. *The International Journal of Learning, 17*(11), 325–337.