CHAPTER 2 THEORY

2.1 The Fundamental Physical Equations

The Educational Global Climate Model (EdGCM) is a suite of software that allows users to run a fully functional three dimensions global climate model (GCM) on laptops or desktop computers (Macs and Windows PCs). The heart of a GCM is a model of the Earth's atmosphere. The model numerically solves five fundamental physical equations that are used to describe the evolving state of the atmosphere. These equations are the conservation of mass, conservation of energy, conservation of momentum, conservation of moisture and the ideal gas law in each cell also taking into account the transport of quantities between cells (Chandler et al., 2006). The model equations are as follows.

The conservation of momentum (Newton's second law of motion),

$$\frac{\partial \vec{V}}{\partial t} = -(\vec{V}\cdot\nabla)\vec{V} - \frac{1}{\rho}\nabla p - \vec{g} - 2\vec{\Omega}\times\vec{V} + \nabla\cdot(k_m \nabla \vec{V}) - \vec{F}_d. \tag{2.1}$$

The conservation of mass (continuity equation),

$$\frac{\partial \rho}{\partial t} = -(\vec{V}\cdot\nabla)\rho - \rho(\nabla\cdot\vec{V}). \tag{2.2}$$

The conservation of energy (first law of thermodynamics),

$$\rho c_p \frac{\partial T}{\partial t} = -\rho c_p (\vec{V}\cdot\nabla)T - \nabla\cdot\vec{R} + \nabla\cdot(k_\tau \nabla T) + C + S. \tag{2.3}$$

The conservation of moisture (vapor, liquid, solid),

$$\frac{\partial q}{\partial t} = -(\vec{V}\cdot\nabla)q + \nabla\cdot(k_q \nabla q) + S_q + E. \tag{2.4}$$

The ideal gas law (approximated equation of state),

$$p = \rho R_d T. \tag{2.5}$$

where	$\vec{V}$	velocity	$\vec{R}$	radiation vector
	T	temperature	C	conductive heating
	p	pressure	c_p	heat capacity at const. p
	ρ	density	E	evaporation
	q	specific humidity	S	latent heating
	$\vec{g}$	gravity	S_q	phase-change source
	$\vec{\Omega}$	rotation of the earth	k	diffusion coefficients
	$\vec{F}_d$	drag force of the earth	R_d	dry air gas constant.

2.2 Predictability Measurement

The atmosphere is a dynamical system which is a system that changes over time, it is difficult to provide accurate prediction and determines the predictability. The term predictability measurement may be defined as a useful method for measuring the rate of error growing in a dynamical system. There are many methods to measure the ability of prediction. In this study, the following measurements are utilized.

2.2.1 Lyapunov Exponent (LE)

The notion of Lyapunov exponent (LE) is based on the average rate of exponential separation of two infinitesimally close trajectories in the phase space. The method of Lyapunov characteristic exponents serves as a useful tool to quantify chaos. Especially, LE represents a mean to measure the rate of convergence or divergence of nearby trajectories (McCue, 2005).

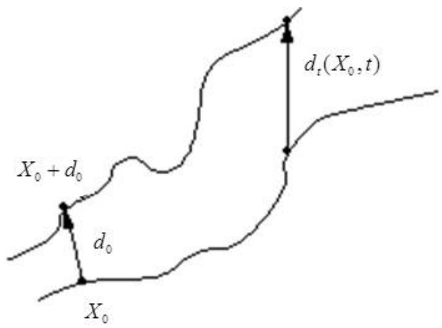

Figure 2.1 The simple measuring chaos in the sense of LE (Elert, 2007).

The growth of the difference d_t between the two trajectories over a time period $\Delta t = t_t - t_0$ can be described by

$$d_t = d_0 e^{\lambda \Delta t} \tag{2.6}$$

where d_0 is called an initial distance (Figure 2.1). Hence the separation rate (λ) is given by (Marc, 1995)

$$\lambda(t) = \frac{1}{\Delta t} \ln\left(\frac{d_t}{d_0}\right). \tag{2.7}$$

For N segments of the nearby trajectories, the average of λ is given by

$$\lambda \equiv \bar{\lambda} = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N+1} \lambda^{(j)}. \tag{2.8}$$

The magnitude of λ is an indicator of the time scale on which chaotic behavior can be predicted for the positive and negative exponent cases, respectively, it works for discrete as well as continuous system. Since λ is proportional to the logarithmic measure of the rate of divergence, the following holds (Leonov, 2007),

1. If $\lambda > 0,$ then the motion is diverge.

2. If $\lambda = 0,$ then the motion is neutral.

3. If $\lambda < 0,$ then the motion is converge.

For a projection of three-dimension phase space, the deformation of a circle along reference trajectories is shown in Figure 2.2.

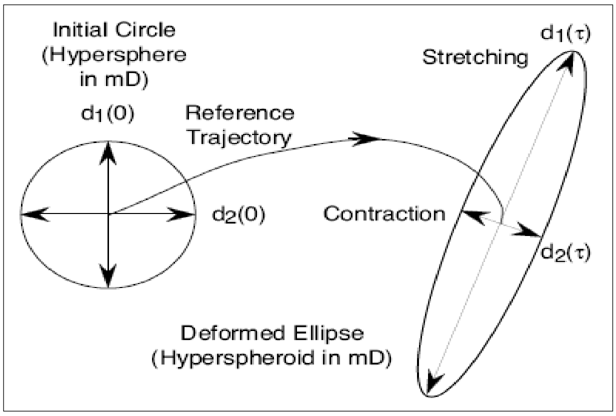

Figure 2.2 Stretching and contraction of trajectories projection of a three-dimension flow (Saiuparad and Sukawat, 2012).

For the stretching along the first direction d_1, the LE λ_1 is positive and the system cannot be stable. For the contraction along the second direction d_2, the LE λ_2 is negative, and then the system is stable. If the third direction is no deformation, the LE λ_3 is zero. These three LEs can be written their signs in the notation (+, −, 0), which specifies chaos for flows. The signs of LEs are sufficient to determine the stability of a dynamical system. It is convenient to arrange the exponents starting from the largest positive on the left λ_1 to the largest negative on the right λ_n

$$\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \quad \forall \lambda_n < \ldots < \lambda_2 < \lambda_1$$

For three-dimension phase space flow, the following properties hold (Kinsner, 2003).

1. If the system has only negative Lyapunov exponents, then the system is called stable.

2. If the system has only zero and negative Lyapunov exponents, then the system is called stable periodic.

3. If the system has only one positive Lyapunov exponent, then the system is called chaos.

2.2.2 Maximum Lyapunov Exponent (MLE)

Physically, LE is a measure of how rapidly nearby trajectories converge or diverge. Often times only the maximal Lyaponov exponent (MLE) is discussed since the maximal exponent is the simplest to calculate from a numerical time series and yields the greatest insight into the dynamics of the system (McCue, 2005). MLE is a measure of the rate of exponential separation of two infinitesimally close trajectories in the phase space and is given by (Boffetta, 1998),

$$\lambda_{\max} = \lim_{t \to \infty} \lim_{d_0 \to 0} \frac{1}{t} \ln \frac{d_t}{d_0} \tag{2.9}$$

where d_t is the distance between the trajectories at time *t*. Consider a reference trajectory and a test trajectory which are separated at time t_0 by a small phase space distance d_0. For a chaotic system, the distance between reference and test trajectories

$d(t)$ will be separated at an exponential rate. Hence, it must be renormalized. Define a rescaling parameter α_i as (Marc, 1995),

$$\alpha_i \equiv \frac{d(t_i)}{d(t_0)} \tag{2.10}$$

where t_i is the time at which $d(t_i) \geq D$. The rescaling of the test trajectory is performed when the distance $d(t_i)$ becomes greater than or equal to the threshold *D*. Then,

$$\lambda_1 = \frac{1}{t_1} \ln \frac{d_1}{d_0} = \frac{1}{t_1} \ln \alpha_1 \tag{2.11}$$

where $\lambda_i \equiv \lambda(t_i)$ and $d_i \equiv d(t_i)$. For successive threshold crossing and subsequent rescalings,

$$\begin{aligned} \lambda_2 &= \frac{1}{t_2} \ln \frac{d_2 \cdot \alpha_1}{d_0} = \frac{1}{t_2} \ln(\alpha_1 \cdot \alpha_2) \\ \lambda_3 &= \frac{1}{t_3} \ln \frac{d_3 \cdot \alpha_2 \cdot \alpha_1}{d_0} = \frac{1}{t_3} \ln(\alpha_1 \cdot \alpha_2 \cdot \alpha_3) \\ &\vdots \end{aligned} \tag{2.12}$$

and so on. Therefore, the instantaneous LE is written by

$$\lambda_n = \frac{1}{t_n - t_0} \sum_{i=1}^{n} \ln \alpha_i \tag{2.13}$$

where
$$\alpha_i = \frac{d(t_i)}{d(t_0)} \tag{2.14}$$

Hence, MLE can be defined as

$$\lambda \equiv \bar{\lambda} = \frac{1}{N} \sum_{i=1}^{N} \lambda_n \tag{2.15}$$

where N represents the total step number of evolution.

2.2.3 Finite Size Lyapunov Exponent (FSLE)

The finite size Lyapunov exponent (FSLE) is the average exponential separation of two trajectories at finite errors δ in the phase space. It is a generalization of the LE's concept to finite separations which is defined as

$$\lambda(\delta) = \left\langle \frac{1}{t} \ln \left(\frac{\|\delta \mathbf{x}(t)\|}{\|\delta \mathbf{x}(0)\|} \right) \right\rangle \tag{2.16}$$

In order to compute the average of growth rate after a given time interval at every time step Δt, FSLE is given by,

$$\lambda(\delta) = \frac{1}{\Delta t}\left\langle \ln\left(\frac{\|\delta\mathbf{x}(t+\Delta t)\|}{\|\delta\mathbf{x}(t)\|}\right)\right\rangle \tag{2.17}$$

where $\delta\mathbf{x}(t)$ is the distance between the trajectories with a suitable norm for an initial time.

$\delta\mathbf{x}(t+\Delta t)$ is the distance between the trajectories at the time $t + \Delta t$, where Δt is the forecast period.

$\langle \bullet \rangle$ is the averaged over many trajectories.

Moreover, FSLE tends to MLE in the limit of infinitesimal separation between trajectories (Aurell, 1997),

$$\lim_{\delta \to 0} \lambda(\delta) = \lambda_{\max} \tag{2.18}$$

2.4.4 Finite Time Lyapunov Exponent (FTLE)

The finite time Lyapunov exponent (FTLE) is applied to fully developed turbulence model and to atmospheric predictability. FTLE has been defined for a prescribed finite time interval to study the local dynamics on the attractor. The sensitivity of trajectories over finite time intervals t to perturbations of the initial conditions can be associated with FTLE. For a system of equation written in state space from $\dot{x} = u(x)$, a trajectory $x(t)$ and the equations for small deviations δx from this trajectory can be expressed by the equation (Eckhardt and Yao, 1993),

$$\delta\dot{x}_i = (\partial u_i / \partial x_j)\delta x_j \tag{2.19}$$

where δx is the deviation from the trajectory. The equation for FTLE is given by

$$\lambda_{\Delta t}(x(t), \delta x(0)) = \frac{1}{\Delta t}\log\frac{\|\delta\mathbf{x}(t+\Delta t)\|}{\|\delta\mathbf{x}(t)\|} \tag{2.20}$$

FTLE measures convergence or divergence of nearby trajectories and thus providing a quantitative measure of a system's sensitivity to initial condition. Positive FTLE indicates exponential divergence of a nearby trajectory and conversely, negative FTLE indicates exponential convergence.

2.4.5 Local Lyapunov Exponent (LLE)

By definition, the local Lyapunov exponent (LLE) of a dynamical system characterizes the rate of separation of infinitesimally closed points of the trajectory (Guégan and Leroux, 2008). This local rate has been defined as a finite time version of global LE, which provides information on how a perturbation to a system's trajectory will exponentially increase or decrease in finite time. LLE depends on the orientation of initial conditions in dynamical systems and also depends on the magnitude of the time interval. Thus, LLE is a short time LE in the limit where the time interval approaches zero which can be expressed as (McCue and Troesch, 2004)

$$\lambda_{local}(\mathbf{x}(t)) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \log \frac{\|\delta \mathbf{x}(t+\Delta t)\|}{\|\delta \mathbf{x}(t)\|} \quad (2.21)$$

where $\mathbf{x}(t)$ is the trajectory with the time t

$\delta \mathbf{x}(t)$ is the distance between the trajectories at the time t

$\delta \mathbf{x}(t+\Delta t)$ is the distance between the trajectories at the time $t+\Delta t$,

Δt is the time interval or forecast period.

LLE exhibits the property as the practical quantity that controls the limit on predictability which is related to the inherent dynamical instability in a chaotic system. In a real sense, one may predict forward only for a finite time in a chaotic system because one has knowledge of local phase space position only to a finite resolution. This uncertainty in phase space location is increase or decrease exponentially rapidly as time evolves along the trajectory. Therefore, the finite time LLE is the critical quantities that govern predictability (Abarbanel et al., 1992).

2.4.6 Supremum Lyapunov Exponent (SLE)

The supremum Lyapunov exponent (SLE) provides a measure of the average rate of convergence or divergence of nearby trajectories (Saiuparad and Sukawat, 2012). The system with more positive exponent indicates sensitive dependence on the initial conditions, that is chaotic dynamics and unpredictability. The definition of SLE for predictability measurement is described as follows.

Assume that the evolution of the atmosphere is governed by a nonlinear dynamical system of n-dimension continuous-time, defined by the differential equation,

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)) \quad (2.22)$$

$$\dot{\mathbf{x}}(t) = \frac{d\mathbf{x}(t)}{dt} \quad (2.23)$$

where $\mathbf{x}(t)$ is the trajectories of the finite-dimensional state space $\mathbb{R}^n$, $\mathbf{x}(t) \in \mathbb{R}^n$ is the state space at time t, that is $\mathbf{x} = [x_1 \; x_2 x_3 \ldots \; x_n]^T$ and $\mathbf{F}$ is an n-dimensional vector field. Assume the vector field $\mathbf{F}$ generates the vector in space $\mathbf{x}(t) = \mathbf{f}(\mathbf{x}, t)$, such that

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(\mathbf{f}(\mathbf{x}, t)) \quad (2.24)$$

The solution of (2.24) under the initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is written as

$$\mathbf{x}(t) = \mathbf{f}(\mathbf{x}_0, t) \quad (2.25)$$

where $\mathbf{f}(\mathbf{x}_0, t) : \mathbb{R}^n \to \mathbb{R}^n$ is the map which describes time evolution of all phase points such that $\mathbf{f}(\mathbf{x}_0, 0) = \mathbf{x}_0$.

Let the set $\{\mathbf{f}(\mathbf{x}_0,t): t \in \mathbb{N}\}$ is the trajectory of the system through $\mathbf{x}_0$. Consider an *n*-dimensional discrete-time smooth dynamical system of a nonlinear model solution that depends only on the initial condition

$$\mathbf{f}(\mathbf{x},t+\Delta t) = M\left[\mathbf{f}(\mathbf{x}_0,t)\right],\ t \in \mathbb{N} \tag{2.26}$$

where $\mathbf{f}(\mathbf{x}_0,t) \in \mathbb{R}^n$ is the vector in a state space of the system at the time *t*, Δt is the time interval, *M* is the time integration of the numerical scheme from the initial condition $\mathbf{f}(\mathbf{x}_0,t)$ to time evolution of the next state $\mathbf{f}(\mathbf{x},t+\Delta t)$.

Define the surface air temperature $\mathbf{x}_i(t)$ as the state space of the dynamical system at time *t* at the point $\mathbf{x}_i$, i=1,2,3,…,N (N is the the total number of experiment data points). The Lyapunov exponent (LE) for $\mathbf{x}_i(t)$ can be written as

$$\lambda = \frac{1}{\Delta t}\ln\frac{\left\|\delta(\mathbf{x}_i(t+\Delta t))\right\|}{\left\|\delta(\mathbf{x}_i(t))\right\|},\ i=1,2,\ldots,N \tag{2.27}$$

Then the characteristic exponents of $\mathbf{x}_i(t)$ are defined by

$$\lambda_{\inf} = \lim_{\Delta t\to\infty}\inf\frac{1}{\Delta t}\ln\frac{\left\|\delta(\mathbf{x}_i(t+\Delta t))\right\|}{\left\|\delta(\mathbf{x}_i(t))\right\|},\ i=1,2,\ldots,N \tag{2.28}$$

and

$$\lambda_{\sup} = \lim_{\Delta t\to\infty}\sup\frac{1}{\Delta t}\ln\frac{\left\|\delta(\mathbf{x}_i(t+\Delta t))\right\|}{\left\|\delta(\mathbf{x}_i(t))\right\|},\ i=1,2,\ldots,N \tag{2.29}$$

If in (2.28) and (2.29) the limits exist and

$$\lim_{\Delta t\to\infty}\inf\frac{1}{\Delta t}\ln\frac{\left\|\delta(\mathbf{x}_i(t+\Delta t))\right\|}{\left\|\delta(\mathbf{x}_i(t))\right\|} = \lim_{\Delta t\to\infty}\sup\frac{1}{\Delta t}\ln\frac{\left\|\delta(\mathbf{x}_i(t+\Delta t))\right\|}{\left\|\delta(\mathbf{x}_i(t))\right\|},\ i=1,2,\ldots,N \tag{2.30}$$

then the characteristic exponent of $\mathbf{x}_i(t)$ exists and defined by (Saiuparad and Sukawat, 2012),

$$\Lambda = \lim_{\Delta t\to\infty}\frac{1}{\Delta t}\ln\frac{\left\|\delta(\mathbf{x}_i(t+\Delta t))\right\|}{\left\|\delta(\mathbf{x}_i(t))\right\|},\ i=1,2,\ldots,N \tag{2.31}$$

Therefore, the characteristic exponential of $\mathbf{x}_i(t)$ is the rate of divergence or convergence of two nearby trajectories. The supremum of (2.31) over all of trajectories $\mathbf{x}_i(t)$ in the phase space is the point of interest. The supremum Lyapunov exponent (SLE) is defined as (Saiuparad and Sukawat, 2012)**,**

$$\Lambda_{\sup} = sup\lim_{\Delta t\to\infty}\frac{1}{\Delta t}\ln\frac{\left\|\delta(\mathbf{x}_i(t+\Delta t))\right\|}{\left\|\delta(\mathbf{x}_i(t))\right\|},\ i=1,2,\ldots,N. \tag{2.32}$$