The Constant Elasticity of Variance Model and Its Implications For Option Pricing

STAN BECKERS*

I. Introduction

BLACK AND SCHOLES [3] derived their seminal option pricing formula under the assumption that underlying stock price returns follow a lognormal diffusion process:

$$\frac{dS}{S} = \mu\, dt + \sigma\, dZ$$

This implies that the percentage price change, $\frac{dS}{S}$, over the interval dt is normally distributed with instantaneous mean μ and instantaneous variance σ^2. While it is well known that the lognormality assumption does not hold exactly, the pricing of European call options has been studied recently for alternative diffusion models. Specifically, Cox [4] and Cox and Ross [5] focused their attention on the constant elasticity of variance diffusion class:

$$dS = \mu S\, dt + \sigma S^{\alpha/2}\, dZ \quad \text{(the elasticity factor } 0 \leq \alpha < 2\text{)}$$

where the instantaneous variance of the percentage price change is equal to $\sigma^2/S^{2-\alpha}$ and hence is a direct inverse function of the stock price. In the traditionally used lognormal model, which corresponds to the limiting case $\alpha = 2$, the variance rate is not a function of the stock price itself. Both casual empiricism and economic rationale tend to support the inverse relationship. If this relationship is borne out by the empirical data, an option pricing formula based on the constant elasticity of variance diffusion could fit the actual market prices better than the Black-Scholes model. In this article we empirically investigate the relationship between the stock price level and its variance of return and perform a comparative statics analysis of the Black-Scholes prices and those based on two special cases of the constant elasticity of variance class ($\alpha = 1$ and $\alpha = 0$).

I. The Relationship Between the Variance of Stock Price Returns and the Level of the Stock Price

It is sometimes argued that a simple economic mechanism might cause an inverse relationship between the level of the stock price and its variance of return. If a

* Vlaamse Ekonomische Hogeschool and European Institute for Advanced Studies in Management, Brussels. The author gratefully acknowledges the helpful suggestions of Mark Rubinstein, Barr Rosenberg and Larry J. Merville. John Cox kindly provided us with a simplified formula for the square root option price. All remaining errors are of course ours.

firm's stock price falls, the market value of its equity tends to fall more rapidly than the market value of its debt, causing the debt-equity ratio to rise; hence the riskiness of the stock increases. A similar effect could be observed even if a firm has almost no debt. Since every firm faces fixed costs, which have to be met irrespective of its income, a decrease in income will decrease the value of the firm and at the same time increase its riskiness. Both operating and financial leverage arguments can be used to explain the inverse relationship between variance and stock price observed in the literature (Black [1], Schmalensee and Trippi [11]). Black [2] states that cause and effect also may be inverted so that a downturn in the general business climate might lead to an increase in the stock price volatility and hence to a drop in stock prices.

The Constant Elasticity of Variance (CEV) class of stock price distributions establishes a theoretical framework within which this inverse relationship can be empirically tested. The instantaneous standard deviation of the percentage price change for this class is given as $\sigma S_t^{(\alpha-2)/2}$ where $0 \leq \alpha < 2$. The standard deviation of the return distribution fluctuates inversely with the level of the stock price. This relationship can be restated as:

$$\ln\left(\text{stdv}\,\frac{S_{t+dt}}{S_t}\right) = \ln\sigma + \frac{(\alpha-2)}{2}\ln S_t \tag{1}$$

If the CEV model would hold exactly, a simple regression

$$\ln\left(\text{stdv}\,\frac{S_{t+1}}{S_t}\right) = a + b\ln S_t + w_t \tag{2}$$

could be used to check the magnitude of the characteristic exponent α. However, several problems arise in implementing this procedure using daily observations.

Although the relationship specified in (1) is instantaneous, equation (2) is expressed over a finite time period. It can nevertheless be shown that a similar relationship also holds over finite time. As α varies from 2 to 0, the coefficient of $\ln S_t$ in (2) will decrease uniformly from 0 to -1. One can easily verify from appendix A, for instance, that if $\alpha = 1$, then

$$\ln\left(\text{stdv}\,\frac{S_{t+1}}{S_t}\right) = \ln k - \frac{1}{2}\ln S_t \quad \text{where} \quad k = \left((e^{\mu\tau}-1)\frac{e^{\mu\tau}}{\mu}\sigma^2\right)^{1/2}.$$

If the CEV model does not hold exactly, the regression specification as given in (2) will be incomplete. Although various economic factors can impact the standard deviation daily, it is practically impossible to quantify them on a daily basis and it is assumed that the stock markets are efficient enough to reflect these factors via stock price changes.

Since on any given day only one return observation is available, stdv (S_{t+1}/S_t) cannot be calculated exactly. The absolute value of ln (S_{t+1}/S_t) however, can be used to operationalize the standard deviation if $\ln S_t$ is normally distributed: $|\ln(S_{t+1}/S_t)|$ is a realization of the underlying distribution with expected value approximately proportional to the standard deviation (see appendix C). Since the CEV class of distributions differs from the lognormal in scale only, the ratio of

Table I

Summary of Estimated Regression Statistics for Equation (3) for 47 Stocks

$$\ln \left| \ln \frac{S_{t+1}}{S_t} \right| = a + b \ln S_t + w_t \ (3)$$

Item	Mean Value	Median Value	Extreme Values Min.	Extreme Values Max	Standard Deviation Around Mean.
$\hat{a}$	−2.389	−2.610	−6.607	3.483	1.99
$\lvert T(\hat{a}) \rvert$	8.55	8.975	.120	21.005	6.38
$\hat{b}$	−.552	−.536	−2.231	.438	.52
$\lvert T(\hat{b}) \rvert$	5.68	5.763	.092	11.481	3.04
R^2	.031	.026	.000	.095	.026
DW	1.07	.94	.26	1.64	.38

$E \mid \ln (S_{t+1}/S_t) \mid$ to the standard deviation is approximately constant for the entire class. We can therefore respecify regression (2)[1] as:

$$\ln \left| \ln \frac{S_{t+1}}{S_t} \right| = a + b \ln S_t + w_t \tag{3}$$

Our sample consists of forty-seven stocks (a list is given in Appendix D), each with 1253 daily return observations, covering the period September 18, 1972 through September 7, 1977. Table I and Figure 1 summarize the results of regression (3). In 38 cases the coefficient of $\ln S_t$ was significantly negative, supporting the hypothesis of an inverse relationship. ATT, Syntex and Xerox are the exceptions, having a significantly positive slope coefficient. The low R^2 and Durbin-Watson statistics, however, indicate that the regression specification given in (3) is incomplete. Since the value of the regression coefficients is between 0 and −1 in 33 cases, a CEV model might be better suited to describe the behavior of those stocks than a lognormal model.

To verify the hypothesis that one characteristic exponent holds for all stocks in the sample, a Chow test was used. Deviations from the mean were taken for each of the 47 stocks before pooling the data and running the following regression:

$$ln \left| ln \frac{S_i(t+1)}{S_i(t)} \right| = \begin{matrix} -.376 \ln S_i(t) \\ (-29.369) \end{matrix} \qquad \begin{matrix} i = 1, 47 \\ t = 1, 1253 \end{matrix} \qquad R^2 = .014$$

While the pooled regression involves 46 linearly independent restrictions, the extremely high value of the F statistic[2] $F(46, 58792) = 42.22$ indicates that we can

[1] It was not always possible to take the log of the dependent variable since in a number of cases the logarithmic return is zero. Eliminating these observations will artificially inflate the R^2 of the regression since the omitted variables would have had an extremely large negative value. We therefore substituted (S_t + .0625) for S_t in those cases where in reality no price change occurred, thus replacing the log of zero with a value from the low end of the spectrum.

[2] Whereas theoretically we should have had 1253 × 47 = 58,891 observations, some stocks did not trade on particular days, thereby reducing the number of observations to 58,886.

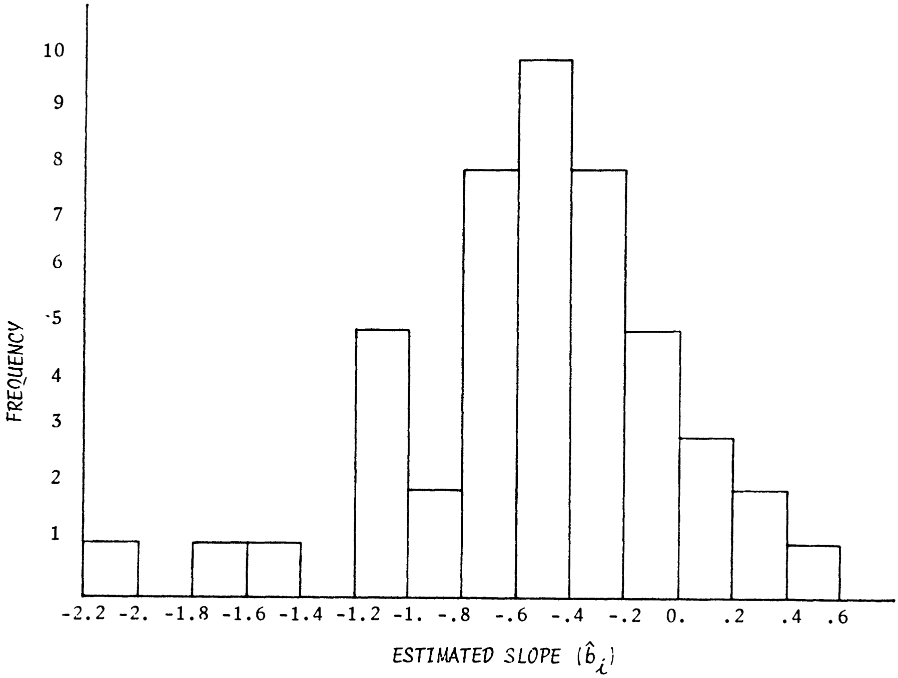

Figure 1. Frequency distribution of estimated slope coefficients ($\hat{b}_i$) for equation (3) for 47 stocks.

soundly reject the hypothesis that the slope coefficient is the same at −.376 for all stocks. It can therefore be concluded that, while the CEV class may be supported by the data for an individual stock, it is highly unlikely that a single model can be applied uniformly across all stocks.

Since the inverse relationship between stock price and volatility is supported by the data, a quick check was performed to verify the underlying rationale. Fluctuations in the debt-equity ratio due to changes in the stock price are thought to cause the volatility to move in the opposite direction of the stock price. This causality was tested by relating leverage to the variance of the stock price return. Leverage information was collected for the four full calendar years (1973–1976) of our observation interval. The following measure was used as a descriptor of market leverage[3]:

$$\frac{\text{(long term debt + preferred stock) at book value} \\ + \text{common stock at market value}}{\text{common stock at market value}}$$

This measure of market leverage (ML) was then related to the actual variance of the stock price return over the fiscal year using the simple regression:

[3] While we would have preferred to value LT debt and preferred stock at market prices, this information is less readily available. Since their prices vary less proportionately than the stock prices, this inaccuracy is considered acceptable. The Compustat tapes were used as a source.

$$(\text{Var}_i(t) - \overline{\text{Var}_i}) = .143{*}10^{-4}\, (\text{ML}_i(t) - \overline{\text{ML}_i}) \qquad R^2 = .045$$
$$(2.967) \qquad \text{DW} = 2.17$$

To eliminate any differences in the level of response between stocks, all variables were defined as deviations from their intertemporal mean. Although an increase in market leverage does significantly affect the risk to the stockholders, a number of other factors apparently also impact this relationship. A more detailed analysis of the intertemporal behavior of the variance would certainly be warranted, but is beyond the scope of this paper.

Having empirically established the validity of using the CEV class to describe stock price behavior, it would be interesting to investigate the effect of this alternative model specification on option pricing. Specifically, since the CEV class might capture the actual stock price behavior better than the lognormal model, the corresponding option prices could give a better fit to the actual option market prices than the Black-Scholes model prices. The following section attempts to clarify this question using a comparative statics analysis.

II. A Comparative Statics Analysis

Cox [4] derived an option pricing formula which holds if the stock price follows a CEV diffusion. The derivation is based on an argument first presented by Cox and Ross [5]: if the return stream on a European call option can be spanned by the underlying stock and the risk-free asset, then the resulting differential equation which governs the call will hold for any set of investor preferences. In particular, a solution obtained under any specific assumption about investor preferences will have complete generality. Assuming risk neutrality, the value of an option is merely the expected future value of the call at expiration discounted to the present at the risk-free rate. The solution to the option pricing problem then depends on finding the distribution of the stock price at expiration. Cox [4] found that if the stock price follows the CEV diffusion, the continuous part of the density of S_T, conditional on S_t $(t < T)$ is

$$f(S_T, T; S_t, t) = (2 - \alpha)k^{1/2-\alpha}(xy^{1-2\alpha})^{1/2(1/2-\alpha)}e^{-x-y}I_{1/2-\alpha}(2(xy)^{1/2}) \tag{4}$$

where $\tau = T - t$

$$k = \frac{2\mu}{\sigma^2(2-\alpha)(e^{\mu(2-\alpha)\tau}-1)}$$

$$x = kS_t^{2-\alpha}e^{\mu(2-\alpha)\tau}$$

$$y = kS_T^{2-\alpha}$$

I_q = modified Bessel function of the first kind of order q.

The probability that $S_T = 0$ is given by $G\left(\dfrac{1}{2-\alpha}, r\right)$ where $G(m, v)$ is the complimentary gamma distribution and r is the risk-free rate. This probability approaches zero as α approaches 2 (the lognormal case). Given that the probability distribution of S_T is known, Cox obtains the following option price:

$$C(S, \tau) = S_t \sum_{n=0}^{\infty} g(n + 1, x) G\left(n + 1 + \frac{1}{2 - \alpha}, kK^{2-\alpha}\right)$$

$$- Ke^{-r\tau} \sum_{n=0}^{\infty} g\left(n + 1 + \frac{1}{2 - \alpha}, x\right) G(n + 1, kK^{2-\alpha})$$

where $g(m, v) = \dfrac{e^{-v} v^{m-1}}{\Gamma(m)}$ is the gamma density function

$$k = \frac{2r}{\sigma^2 (2 - \alpha)(e^{r(2-\alpha)\tau} - 1)}$$

$$x = kS_t^{2-\alpha} e^{r(2-\alpha)\tau}.$$

K = striking price.

In our comparative statics analysis we concentrate on two special cases of this general formula: the square root model ($\alpha = 1$) and the absolute model ($\alpha = 0$). Cox and Ross [5] discuss both models as limiting cases of pure Markov jump processes. In order to compare the Black-Scholes model and either of these two models, we need to ensure that equivalent inputs are used in all models. Each of the three models under consideration depends upon only five data inputs: the stock price (S), time to maturity (τ), the exercise price (K), the risk free rate (r) and an estimate of the stock price volatility (σ^2). We assume that $\mathrm{Var}(S_T/S_t)$ is the same for all the models compared in order to ensure that consistent volatility estimates are used in the comparative statics analysis. Appendices A and B derive the relationships between σ^2 square root and σ_{BS}^2 and between σ^2 absolute and σ_{BS}^2 respectively.

Using these results we can proceed with a comparative statics analysis of the three models. Note, however, that the option pricing formula for the CEV class contains an infinite summation which makes evaluation difficult in those cases where convergence is slow. This problem is easily solved for the absolute model because Cox and Ross [5] introduce an alternative formulation for the option price:

$$C(S, \tau) = (S - Ke^{-r\tau}) N(y_1) + (S + Ke^{-r\tau}) N(y_2) + v(n(y_1) - n(y_2))$$

where $N(\cdot)$ = cumulative unit normal distribution function
$n(\cdot)$ = unit normal density function

$$v = \sigma \left(\frac{1 - e^{-2r\tau}}{2r}\right)^{1/2}$$

$$y_1 = \frac{S - Ke^{-r\tau}}{v}$$

$$y_2 = \frac{-S - Ke^{-r\tau}}{v}.$$

Similarly, Cox[4] developed an approximative formula for the value of the option for the square root model. Let

$$y = \frac{4rS}{\sigma^2(1 - e^{-r\tau})} \quad \text{and} \quad z = \frac{4rK}{\sigma^2(e^{-r\tau} - 1)}.$$

Let w be a parameter which takes on the values 0 or 4.

Let $h(w) = 1 - \frac{2}{3}(w + y)(w + 3y)(w + 2y)^{-2}$

$q(w) =$

$$\frac{1 + h(h-1)\left(\frac{w + 2y}{(w + y)^2}\right) - h(h-1)(2-h)(1-3h)\left(\frac{(w+2y)^2}{2(w+y)^4}\right) - \left(\frac{z}{(w+y)}\right)^h}{\left\{2h^2\left(\frac{w+2y}{(w+y)^2}\right)(1 - (1-h)(1-3h)\left(\frac{w+2y}{(w+y)^2}\right)\right\}^{1/2}}$$

Then $C(S, \tau) = SN(q(4)) - Ke^{-r\tau}N(q(0))$ where $N(\cdot)$ is the cumulative unit normal distribution function.

Both the exact formula (using 995 as the upper bound to the summation)[5] and the approximative formula were used in order to evaluate the square root option price. Table II summarizes the comparative statics results under various assumptions about the stock price, time to maturity and volatility. In those cases where the approximate formula results diverged from the exact formula, the approximate value is reported in brackets under the "exact" option price.

As a general rule, it can be inferred that for in-the-money and at-the-money options the model price increases as the characteristic exponent decreases, whereas exactly the opposite is true for out-of-the money options. In other words, the model prices for the square root and absolute options are higher than the Black-Scholes model prices for at-the-money and in-the-money options. This behavior is consistent across stock price levels and the differences between the three models become more apparent as the time to maturity and volatility increase. Contrastingly, the differences between prices for the out-of-the money options seem to decrease as one moves further away from the lognormal diffusion case. However, some deviant behavior is observed for the high volatility, long maturity slightly out-of-the-money options; the model prices for the square root model and absolute model tend to be higher than the corresponding Black-Scholes prices. Higher volatility (because of relatively low stock price) relative to the Black-Scholes model and the long time to maturity may explain this phenomenon.

On the basis of his perception of the option market, Black [1] claims that there are systematic differences between the Black-Scholes model and actual market prices. According to the model, the market overprices options that are way out-

[4] Personal communication. The derivation of this approximative formula is based on Sankaran [9] and [10].

[5] It was verified that the upper bound was sufficiently high to ensure convergence for all cases under consideration.

Table II

Call Values for Alternative Diffusion Models

		$S = 20$ $\alpha = 2$ (BS Model)			$r = 1.05$ $\alpha = 1$ (Square Root)			$\alpha = 0$ (Absolute Model)		
		Months to Maturity			Months to Maturity			Months to Maturity		
σ	K	1	4	7	1	4	7	1	4	7
.2	10	10.041	10.161	10.281	10.041	10.161	10.281	10.041	10.161	10.281
	15	5.061	5.245	5.439	5.061	5.248	5.454	5.061	5.252	5.472
	20	.501	1.084	1.502	.503	1.097	1.532	.504	1.101	1.541
	25	.0	.037	.161	.0	.029	.144 (.143)	.0	.021	.118
	30	.0	.0	.008	.0	.0	.004	.0	.0	.001
.3	10	10.041	10.161	10.281	10.041	10.162	10.288 (10.287)	10.041	10.163	10.307
	15	5.061	5.289	5.570	5.061	5.320	5.654	5.062	5.351	5.725
	20	.731	1.536	2.093	.735	1.574	2.179 (2.180)	.736	1.581	2.196
	25	.004	.217	.554	.002	.196	.541 (.540)	.001	.162	.478
	30	.0	.018	.115	.0	.010	.086	.0	.004	.050
.4	10	10.041	10.162	10.293	10.041	10.170	10.349 (10.348)	10.041	10.191	10.454
	15	5.065	5.416	5.824	5.068	5.505 (5.506)	6.033 (6.037)	5.073	5.579	6.194
	20	.960	1.990	2.685	.970	2.072 (2.073)	2.878 (2.882)	.972	2.084	2.936
	25	.029	.516	1.062	.022	.501 (.500)	1.111 (1.110)	.015	.435	1.034
	30	.0	.106	.383	.0	.078 (.077)	.350 (.348)	.0	.043	.254
.2	30	10.122	10.489	10.878	10.122	10.495	10.908	10.122	10.504	10.944
	35	5.148	5.760	6.399	5.150	5.798	6.478	5.153	5.830	6.539
	40	1.003	2.167	3.004	1.006	2.193	3.063	1.007	2.202	3.082
	45	.022	.506	1.103	.019	.487 (.486)	1.093	.016	.455	1.047
	50	.0	.075	.323	.0	.059	.287	.0	.043	.237
.3	30	10.122	10.579	11.140	10.123	10.639	11.307 (11.309)	10.124	10.702	11.450
	35	5.219	6.251	7.171	5.235	6.363	7.393 (7.396)	5.250	6.439	7.520
	40	1.461	3.073	4.186	1.471	3.147 (3.148)	4.357 (4.360)	1.472	3.161	4.392
	45	.162	1.255	2.235	.149	1.248	2.298	.133	1.189	2.224
	50	.007	.435	1.107	.004	.393 (.392)	1.082 (1.081)	.003	.324	.955
.4	30	10.129	10.831	11.649	10.136	11.010 (11.011)	12.067 (12.073)	10.147	11.157	12.389
	35	5.388	6.894	8.095	5.427	7.121 (7.124)	8.557 (8.567)	5.461	7.244	8.813
	40	1.920	3.979	5.370	1.941 (1.940)	4.145 (4.147)	5.755 (5.764)	1.943	4.168	5.873
	45	.419	2.103	3.428	.397	2.156	3.670 (3.675)	.365	2.073	3.633
	50	.057	1.033	2.124	.044 (.043)	1.001 (1.000)	2.221 (2.220)	.030	.871	2.068

Table II—*Continued*

		$S = 20$ $\alpha = 2$ (BS Model)			$r = 1.05$ $\alpha = 1$ (Square Root)			$\alpha = 0$ (Absolute Model)		
		Months to Maturity			Months to Maturity			Months to Maturity		
σ	K	1	4	7	1	4	7	1	4	7
.2	50	10.203	10.916	11.718	10.204	10.954	11.816	10.205	10.992	11.902
							(11.817)			
	55	5.304	6.554	7.665	5.314	6.619	7.789	5.324	6.669	7.872
							(7.790)			
	60	1.504	3.251	4.506	1.509	3.290	4.595	1.511	3.302	4.623
	65	.158	1.302	2.373	.149	1.287	2.390	.138	1.250	2.347
	70	.005	.421	1.125	.004	.387	1.087	.003	.341	1.005
						(.386)	(1.086)			
.3	50	10.229	11.394	12.596	10.241	11.545	12.918	10.253	11.664	13.131
						(11.546)	(12.922)			
	55	5.597	7.559	9.086	5.631	7.724	9.411	5.659	7.818	9.565
						(7.725)	(9.415)			
	60	2.192	4.609	6.279	2.206	4.721	6.536	2.209	4.742	6.588
						(4.722)	(6.540)			
	65	.565	2.587	4.170	.545	2.615	4.320	.519	2.554	4.259
							(4.322)			
	70	.094	1.345	2.673	.079	1.307	2.715	.064	1.202	2.566
						(1.306)				
.4	50	10.345	12.157	13.785	10.387	12.490	14.477	10.428	12.702	14.910
						(12.494)	(14.491)			
	55	6.034	8.702	10.641	6.099	9.028	11.313	6.147	9.172	11.636
						(9.032)	(11.328)			
	60	2.880	5.969	8.055	2.911	6.217	8.633	2.915	6.252	8.809
						(6.220)	(8.646)			
	65	1.102	3.936	5.995	1.082	4.073	6.434	1.040	3.994	6.452
						(4.074)	(6.443)			
	70	.340	2.507	4.397	.306	2.536	4.684	.263	2.375	4.560
						(2.535)	(4.687)			

of-the-money and underprices options that are way into-the-money. Options with less than three months to maturity also tend to be underpriced. Under these circumstances, the square root and absolute models hold little promise since they tend to worsen the fit to the market prices rather than improve it.

However, a recent study by MacBeth and Merville [7] argues that the deviations between market prices and Black-Scholes model prices counterpose those observed by Black. For a limited sample (6 options) studied over the 1976 calendar year, they find that the model understates (overstates) the market prices for deep into (out of) the money options. Although their conclusions can only be considered tentative because of the limited sample, it appears that the CEV class might have been better suited to describe the behavior of those options in 1976.

It is obvious that further extensive study is needed to conclusively establish the deviation between Black-Scholes and market prices. If the MacBeth-Merville findings are confirmed, the CEV class of option pricing should be considered as a prime alternative formulation. In that case it will also be comforting to know that, as can be inferred from Table 2, the simplified formula suggested by Cox for the square root model can readily be used as a substitute for the general formulation.

III. Conclusion

On the basis of the empirical study performed, it appears that the constant elasticity of variance class could be a better descriptor of the actual stock price behavior than the traditionally used lognormal model. While no general model apparently applies for all stocks, most of the stocks analyzed in this paper exhibited a significant negative relationship between the level of the stock price and its volatility. Some evidence also exists that part of this relationship can be attributed to changes in the debt-equity ratio caused by fluctuations in the stock price. On the basis of this limited evidence, a model based on the simple variance nonstationarity inherent in the constant elasticity of variance class might be preferable to the traditionally used lognormal model.

The comparative statics analysis of the Black-Scholes, square root and absolute models indicates that their prices differ systematically. The constant elasticity of variance class yields prices which are higher than the Black-Scholes prices for in-the-money and at-the-money options, while the reverse is true for out-of-the-money options. Some recent empirical evidence suggests that these are exactly the deviations being observed in practice between the Black-Scholes price predictions and market prices. If these results hold true for an extensive sample, the constant elasticity of variance class could become a prime alternative model specification to the canonical Black-Scholes formula.

Appendix A

For the square root model the distribution of S_T can be written as

$$f(S_T, T) = k(x/y)^{1/2}e^{-x-y}I_1(2(xy)^{1/2}) \quad \text{from (4) where } \alpha = 1$$

or

$$f(S_T, T) = kx \sum_{n=0}^{\infty} \frac{e^{-x}x^n}{(n+1)!} f_{1,n+1}(y) \quad \text{(by Feller [6], p. 58)}$$

where

$$f_{a,v}(x) = \frac{1}{G(v)} a^v x^{v-1} e^{-ax}$$

is the gamma density function.

As $y = kS_T$,

$$f(y) = x \sum_{n=0}^{\infty} \frac{e^{-x}x^n}{(n+1)!} f_{1,n+1}(y).$$

The moments of $f(y)$ are equal to

$$E(y^j) = x \sum_{n=0}^{\infty} \frac{e^{-x}x^n}{(n+1)!} (n+j)(n+j-1)\cdots(n+1).$$

Hence

$$\text{Var}\,\frac{S_T}{S_t} = \frac{e^{\mu\tau}}{\mu S_t}\,\sigma^2(e^{\mu\tau} - 1)$$

or

$$\sigma^2_{\text{square root}} = \frac{\mu S_t \text{Var}}{e^{\mu\tau}(e^{\mu\tau} - 1)}.$$

Assuming that the lognormal diffusion holds, then

$$\text{Var}\left(\frac{S_T}{S_t}\right) = e^{2\mu\tau + \sigma^2_{BS}\tau}(e^{\sigma^2_{BS}\tau} - 1)$$

(by the properties of the lognormal distribution).
Hence the σ^2 estimate for the square root model which corresponds to a given σ^2_{BS} is

$$\sigma^2 = \frac{\mu S_t e^{2\mu\tau + \sigma^2_{BS}\tau}(e^{\sigma^2_{BS}\tau} - 1)}{e^{\mu\tau}(e^{\mu\tau} - 1)} = \frac{\mu S_t e^{\mu\tau} e^{\sigma^2_{BS}\tau}(e^{\sigma^2_{BS}\tau} - 1)}{(e^{\mu\tau} - 1)}.$$

Appendix B

Relationship Between $\sigma^2_{\text{absolute}}$ and σ^2_{BS}

For the absolute model the distribution of the stock price at time T can be written as (Cox and Ross, [5]):

$$f(S_T, T; S_t, t) = (2\pi Z)^{-1/2}\left\{\exp\left(-\frac{(S_T - S_t e^{\mu\tau})^2}{2Z}\right) - \exp\left(-\frac{(S_T + S_t e^{\mu\tau})^2}{2Z}\right)\right\}$$

where $Z = \frac{\sigma^2}{2}(e^{\mu\tau} - 1)$.

Hence $E(S_T) = S_t\, e^{\mu\tau}$ and $\text{Var}\,(S_T) = E(S_T - S_t e^{\mu\tau})^2$

Letting $S_t e^{\mu\tau} = M$

$$E(S_T - M)^2 = 2(2\pi Z)^{-1/2} M Z e^{-M^2/2Z} + \left\{Z(2\pi Z)^{-1/2}\int_{-M}^{M} e^{-x^2/2Z}dx - 2M^2 + 4M^2(2\pi Z)^{-1/2}\int_0^M e^{-x^2/2Z}dx\right\}.$$

where $x = S - M$

$y = S + M$

The integrals are hard to evaluate because they both depend on the unknown Z which in turn is a function of σ^2. A numerical solution technique can be used to find the variance input to the absolute model which corresponds to the Black-Scholes variance, by solving the following equation:

$$e^{2\mu\tau+\sigma_{BS}^2\tau}(e^{\sigma_{BS}^2\tau}-1) = (1/S_t^2)\left\{(2\pi Z)^{-1/2}2MZe^{-M^2/2Z} + Z(2\pi Z)^{-1/2}\int_{-M}^{M} e^{-x^2/2Z}dx - 2M^2 + (2\pi Z)^{-1/2}4M^2\int_0^M e^{-x^2/2Z}dx\right\}$$

where $Z = \sigma_{abs}^2(e^{\mu\tau}-1)/2\mu$ and $M = S_te^{\mu\tau}$.

A numerical solution technique based on Muller's method [8] was used in order to solve for the root of this equation. Specifically, letting $S_t = 40$ and $r = 1.05$ annually, the following table gives the corresponding σ_{abs} for different σ_{BS} and times to maturity:

	TMT (in months)		
σ_{BS}	1	4	7
.2	.1769	.1818	.1868
.3	.4006	.4194	.4394
.4	.7185	.7727	.8504

Appendix C

Relationship between $E(|x|)$ and σ

The relationship between $E(|x|)$ and the standard deviation will be derived under the assumption that x is normally distributed $N(\mu, \sigma^2)$.

$$E(|x|) = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} |x|\, e^{-(x-\mu)^2/2\sigma^2}$$

Letting $y = \dfrac{x-\mu}{\sigma}$

$$E(|x|) = (2\pi\sigma^2)^{-1/2}\int_{-\infty}^{\infty} |\sigma y + \mu|\, e^{-y^2/2}\sigma dy$$

$$= \sigma(2/\pi)^{1/2}\int_{\mu/\sigma}^{\infty} ye^{-y^2/2}dy + \mu - \mu(2/\pi)^{1/2}\int_{-\infty}^{-\mu/\sigma} e^{-y^2/2}dy$$

Since μ is negligibly small in comparison to σ and close to zero for daily data.

$$E(|x|) \cong \sigma(2/\pi)^{1/2}.$$

Appendix D:

The 47 Stocks in the Sample

ASA, ATT, Atlantic Richfield, Avon, Boeing, Burroughs, Chase, Citicorp, Coca Cola, Control Data, Deere, Delta Air, Digital Equipment, Du Pont, Ford, General Electric, General Motors, General Telephone, Homestake, Honeywell, IBM, International Minerals, ITT, Kennecott, Loews, McDonalds, Merrill Lynch, Monsanto, National Semiconductor, Northwest Air, Occidental, Pfizer, Polaroid, RCA, Schlumberger, Searle, Sears, Skyline, Sperry, Syntex, Texas Instruments, Tiger, U.S. Steel, Upjohn, Western Union, Westinghouse, and Xerox.

REFERENCES

1. F. Black. "Fact and Fantasy in the Use of Options." *Financial Analysts Journal* 31 (July–August 1975).
2. F. Black. "Studies of Stock Price Volatility Changes." *Proceedings* of the 1976 Meetings of the American Statistical Association, Business and Economic Statistics Division.
3. F. Black and M. Scholes. "The Pricing of Options and Other Corporate Liabilities." *Journal of Political Economy* 81 (May–June 1973).
4. J. Cox. "Notes on Option Pricing I: Constant Elasticity of Variance Diffusions." (Working Paper Stanford University, September 1975).
5. J. Cox and S. Ross. "The Valuation of Options for Alternative Stochastic Processes." *Journal of Financial Economics* 3 (Jan–March 1976).
6. W. Feller. *An Introduction to Probability Theory and its Applications.* Vol. 2, 2d ed. (New York: J. Wiley and Sons, 1971).
7. J. MacBeth and L. Merville. "An Empirical Examination of the Black-Scholes Call Option Pricing Model" *Journal of Finance* 34 (December 1979).
8. D. E. Muller. "A Method for Solving Algebraic Equations Using an Automatic Computer." *Mathematical Tables and Other Aids to Computation* 10 (Spring 1956).
9. M. Sankaran. "On the Non-Central Chi-Square Distribution." *Biometrika* 46 (July 1959).
10. ———. "Approximations to the Non-Central Chi-Square Distribution." *Biometrika* 50 (August 1963).
11. R. Schmalensee and R. R. Trippi. "Common Stock Volatility Expectations Implied by Option Premia." *Journal of Finance* 33 (March 1978).