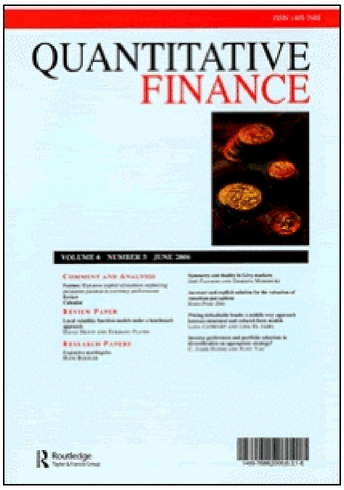

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Stochastic volatility and option pricing

Through a simple Monte Carlo experiment, *Dimitrios Gkamas* documents the effects that stochastic volatility has on the distribution of returns and the inability of the normal distribution utilized by the Black–Scholes model to fit empirical returns. He goes on to investigate the implied volatility patterns that stochastic volatility models can generate and potentially explain.

Introduction

Derivative securities, contracts for which their value is derived from the underlying asset upon which they have been written, are financial instruments which permit both the efficient transfer of risk that emanates from the holding of a financial asset and also provide efficient ways to speculate in the global market place. The pricing of these financial products, as Vasicek (1998) noted, 'is one of the great achievements of modern finance'. The ability to price these financial instruments relies on the law of one price. Put simply, in efficient markets, if a contract pays a certain amount of cash under certain states of the world and another contract (or a combination of contracts, with the same payoff profile) exists, then both contracts must trade at the same price, otherwise arbitrage opportunities exist.

Futures and options, also called contingent claims, are the basic building blocks of all derivative securities. The first category obligates the buyer (seller) to take (make) delivery of the underlying asset, at a future date at a price agreed at the initiation date of the contract. However, derivatives that belong to the second category give the buyer the right to trade the underlying asset at a prespecified date(s) in the future at a price agreed at the initiation date of the contract. Of course the holder of the option will exercise his right only if he is going to gain from the transaction. With no possibility of losses, a premium is demanded by the seller of the option. This is not the case for futures.

This article briefly surveys the standard option pricing model, the so-called Black–Scholes, for pricing commonly traded, also called plain vanilla, option contracts as well as a survey of a competing alternative, the so-called stochastic volatility, that is potentially better suited to explain observed option prices. I will restrict my presentation to option pricing models which assume that asset volatility is driven by a Brownian motion that is different from and possibly correlated to the Brownian motion that drives the asset price. For alternative specifications see Dupire (1994), Derman and Kani (1994), Avellaneda *et al* (1995), Hobson and Rogers (1998), Bensoussan *et al* (1994, 1995), Coleman *et al* (1999) and Dempster and Richards (2000).

The Black–Scholes option-pricing model and its biases

Black and Scholes (1973), by utilizing no-arbitrage arguments, derived a closed-form solution for the price European call (put) option on a non-dividend paying stock. A European call (put) option is the simplest form of a contingent claim which permits, but does not obligate, the buyer of it to demand a trade to take place at the time of the contract's expiration between him and the seller (writer) at a prespecified price (the exercise price) that has been agreed upon the initiation of the contract. If the contract is a call (put), the holder of the option may purchase from (sell to) the writer the underlying asset at the prespecified price on the expiry date of the option.

Black and Scholes (1973) arrived at their formula for a European call option (C) that is a function of two variables—the current price of the underlying asset (S) upon which the option has been written, and the time to expiration of the contract (T)—and three parameters which are assumed to be constant: the exercise price of the option, the level of the continuously compounded risk free interest rate (r) and the annualized standard deviation (σ), or volatility in practitioners' terms, of the underlying asset's returns. The formula is:

$$C = SN(d_1) - Xe^{-rT}N(d_2) \qquad (1)$$

$$d_1 = \frac{\ln(S/X) + \left(r + 0.5\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S/X) + \left(r - 0.5\sigma^2\right)T}{\sigma\sqrt{T}}$$

where $N(.)$ denotes the normal cumulative density function. A similar formula applies to a European put option (P). Note that the above equation depends only on observables (except volatility) and is independent of the expected return of the underlying asset. Moreover, it can be shown that European calls are related to puts with the same time to expiration and exercise price through the following relation, called put–call parity:

$$P = C - S + Xe^{-rT}. \qquad (2)$$

If this is not the case, arbitrage opportunities exist. If we define the moneyness level (M) of the option as

$$M = \frac{S}{Xe^{-rT}} \qquad (3)$$

then we say that the option is at-the-money if $M = 1$, out-of-the-money if $M < 1$ and in-the-money if $M > 1$. Note also that in some option markets it is customary to quote the price of an option in terms of the volatility input. In addition, practitioners routinely utilize option prices to back out a so-called implied or implicit volatility estimate, as volatility is an important parameter in option pricing.

The seminal approach of Black and Scholes (1973) has been used extensively in the past for the pricing and risk-management purposes of derivative portfolios and still enjoys considerable support, at least from practitioners, for pricing simple products traded on exchange and over-the-counter option markets. Nevertheless, over the last forty years, a number of empirical findings have been compiled that cast doubt on one of the key assumptions of Black–Scholes, namely that returns are normally distributed, with constant mean and volatility (see Fama 1965, Boness *et al* 1974, Boothe and Glassman 1987, Brorsen and Yang 1994). It has been discovered that empirical returns are far from normally distributed as they exhibit fat tails (kurtosis) and/or non-zero skewness.

As an illustration, in table 1, I present summary statistics for the daily returns of FTSE 100 during the period 1984–1995 as well as four subperiods. Note that values of kurtosis greater than three imply that there are more extreme returns, both positive and negative, at the tails of the FTSE 100 return distribution in comparison with the normal distribution. Non-zero skewness implies that returns are not distributed symmetrically around the mean, a characteristic of the normal distribution. Negative skewness means that, for a given absolute return bracket, there are more negative than positive returns. The opposite is true in the case of positive skewness. Finally, observe the non-stationarity of the mean and volatility of the series.

A number of explanations have been proposed for the non-normal nature of the distribution of returns such as: (i) returns are generated by a non-normal distribution that is characterised by excess kurtosis and non-zero skewness (Blattberg and Gonedes 1974); (ii) empirical returns are generated from a mixture of distributions with constant mean and different variances (Akgiray and Booth 1987); (iii) stock prices are affected not only by normal shocks but also by systematic jump risk that causes return distribution to deviate from normality (Ball and Torous 1983); and finally, (iv) empirical returns are non-normally distributed because stock-return volatility is non-stationary and varies stochastically over time (Engle and Bollerslev 1986). Although, *a priori,* any of the above propositions provides a possible explanation for the empirical return distribution, with the exception of the third hypothesis, there is overwhelming empirical evidence (see Bollerslev *et al* (1992) for a survey), that volatility is not constant.

Table 1. Average daily return (mean) and standard deviation (SD), their equivalent annualized counterparts (mean an) and (SD an) sample skewness (sk) and kurtosis (kurt).

	Whole sample 1984–1995	Subperiod A 1984–1986	Subperiod B 1987–1989	Subperiod C 1990–1992	Subperiod D 1993–1995
Mean	0.000417	0.000663	0.000469	0.00021	0.00033
Mean an (%)	12.2460	17.2380	12.1420	5.4600	8.5800
SD	0.009327	0.008622	0.011996	0.00903	0.00696
SD an (%)	14.9914	13.8583	19.2814	14.514	11.186
Sk	–1.544831	–0.337481	–2.959632	0.4546	–0.12732
Kurt	26.09622	4.250517	34.77622	6.13589	3.32756

These statistics are calculated by multiplying the daily values by the average number of business days in the observation period and the square root of the average number of business days, respectively.

The non-normality of the empirical returns of assets upon which derivative contracts have been written is also manifested by the presence of Black–Scholes implied volatility (BSIV) smiles; away-from-the-money options are traded at higher implied volatilities in comparison to near-the-money ones, or skews, out-of-the-money (in-the-money) puts (calls) are traded at higher implicit volatilities in comparison to in-the-money (out-of-the-money) puts (calls) that are not consistent with the Black–Scholes model; see Duque and Paxson (1994), Heynen (1994) and Rubinstein (1985).

The Black–Scholes model seems to overprice out-of-the-money calls and underprice in-the-money calls.

For example, in figure 1, which is a plot of the average BSIV derived from FTSE 100 European calls for a number of moneyness and time-to-expiration categories, the so-called BSIV surface can be found. We observe that average implied volatility are higher for away-from-the-money options in comparison to near-the-money-options. If Black–Scholes is capable of closely fitting observed option prices, we expect the implied volatility surface to be flat, something that is far from true.

The stochastic volatility approach to option pricing

Given the empirical evidence with respect to the distribution of returns and the biases of the Black–Scholes model, a number of researchers proposed option pricing models that explicitly take into account the stochastic nature of asset return volatility. A main assumption is that stock-return variance follows a diffusion process and perfect replication of derivatives is not possible as volatility is not a traded asset. Early studies include those by Johnson and Shanno (1987), Wiggins (1987), Scott (1987) and Hull and White (1987). The first attempts to solve the problem of option pricing under stochastic volatility concentrated mainly on deriving the partial differential equation that a derivative security must satisfy in the case of randomly changing volatility. Subsequently, numerical methods were used to calculate the option price. A notable exception is the approach of Hull and White (1987), who provided an approximate solution to the problem of option pricing under stochastic volatility under the assumption that stock-return volatility is uncorrelated with stock returns.

More recently, a number of researchers provided closed-form solutions for the price of a European call option when volatility is stochastic. They assumed that volatility evolves according to a mean reverting process and permitted the correlation between return and volatility innovations to be different from zero. A pivotal role is played by the characteristic function of the average variance during the life of the option. The characteristic function was used as a means of calculating the risk-neutral probabilities under which we price options using Fourier inversion methods; see Bakshi *et al* (1997), Bates (1996), Ball

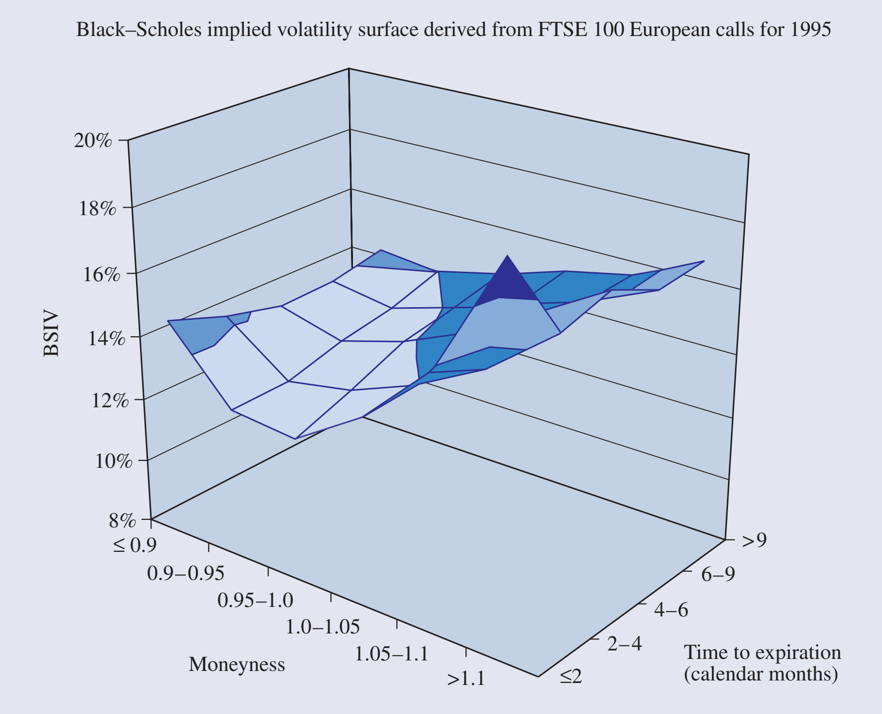

Figure 1. A plot of BSIV with respect to time-to-expiration and moneyness levels.

and Roma (1994) and Heston (1993).

To be more specific, the majority of the studies assumed that stock-return variance (V) evolves according to the following mean reverting process:

$$dV = \beta(\mu_V - V)\,dt + \sigma_V\sqrt{V}\,dW_V \qquad (4)$$

where μ_V is the mean reversion level of the variance process, β is the speed at which the variance reverts to μ_V, σ_V denotes the volatility of the volatility process and dW_V denotes Brownian motion, possibly correlated with the Brownian motion that drives stock returns with correlation coefficient ρ_{SV}. The European call option price in the case that volatility is stochastic is given by

The formulae for cf_1, cf_2 can be found in

$$C = SP_1 - Xe^{-rT}P_2 \qquad (5)$$

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln(X)}cf_j}{i\phi}\right]d\phi \qquad j = 1, 2 \qquad (6)$$

Bakshi *et al* (1997). Note that the above integral cannot be computed analytically but can be calculated relatively easily by numerical techniques.

Effect of stochastic volatility on the distribution of returns

The effect that stochastic volatility has on the distribution of returns can be examined by performing a simple Monte Carlo experiment. According to Black–Scholes stock prices follow the process:

$$\frac{dS}{S} = \mu dt + \sigma dW_S \qquad (7)$$

whereas in the case that volatility is stochastic the above equation becomes

$$\frac{dS}{S} = \mu dt + \sqrt{V}dW_V \qquad (8)$$

with V evolving according to equation (4) and $\mathrm{Cov}(dW_S, dW_V) = \rho_{SV}$. We generate 1000 random prices utilizing the two stochastic processes above and examine the empirical distribution of the continuously compounded returns for the case that volatility is stochastic and for the case that returns follow geometric Brownian motion with constant volatility.

In table 2, I provide sample kurtosis and skewness statistics for the distribution of returns that results for various levels of σ_V and ρ_{SV} as these are the two key parameters of the volatility process. Table 2 shows, first, with respect to the effect that the correlation coefficient between returns and volatility innovations has on the distribution of returns, that negative values are associated with negative skewness and positive values are associated with positive skewness. In other words, a left-skewed distribution results when the correlation is negative and a right-skewed one occurs when the correlation is positive. The consequence of this finding is that European options will be mispriced by the Black–Scholes model, as it does not take into account the skewed nature of the return distribution that results when volatility is stochastic. In the case that the correlation between returns and volatility innovations is negative, the Black–Scholes model underprices out-of-the-money puts in comparison to out-of-the-money calls, as it does not take into account that extreme negative returns are more probable than extreme positive returns. The opposite is true if ρ_{SV} is positive. Second, when the correlation between returns and volatility innovations is zero, table 2 shows that extreme positive and negative returns are equally more probable in comparison to the normal distribution, as implied by the coefficient of excess kurtosis. As a result, Black–Scholes underprices both out-of-the-money puts and calls.

Table 2. Skewness and kurtosis of simulated sample returns for a number of values for the volatility of volatility (σ_V) and correlation between return and volatility innovation (ρ_{SV}) parameters.

Skewness				Kurtosis			
	ρ_{SV}				ρ_{SV}		
σ_V	–0.5	0	0.5	σ_V	–0.5	0	0.5
0.55	–0.09	–0.01	0.06	3.55	3.26	3.23	3.22
0.60	–0.09	0.00	0.08	3.60	3.51	3.48	3.45
0.65	–0.10	0.01	0.09	3.65	4.19	4.15	4.12

The observation interval is 0.001 years, the drift of the diffusion process (μ) is 20% and the volatility (σ) is 22.36% for the case that is assumed constant. In the case that asset prices are characterized by stochastic volatility, β = 4.6 and μ_V = 23.36% whereas instantaneous volatility at time zero is assumed to be at its mean reversion level (μ_V). For the case that the asset price follows geometric Brownian motion with constant volatility the sample skewness is –0.01 and the sample kurtosis is 2.98.

The biases of the Black–Scholes that potentially can be explained by the stochastic volatility model

Given the ability of stochastic volatility to generate return distributions that are similar to those observed empirically, it will be interesting to examine the capability of the stochastic volatility option-pricing model to explain observed option prices. In other words, it will be interesting to investigate what BSIV surfaces the stochastic volatility model is capable of generating and consequently explaining.

To be more precise, I calculated option prices for a number of time-to-expiration and moneyness categories by utilizing equation (5) and used these option prices as input to the Black–Scholes model in order to back out implied volatilities. The implicit volatilities that result by utilizing those inputs to the Black–Scholes model stochastic volatility option prices are plotted with respect to a number of ρ_{SV} and σ_V levels in figures 2 and 3 respectively.

Starting with figure 2, in which I investigate the effect that the correlation coefficient between return and volatility innovations has on the implicit volatility surface that the stochastic volatility model is capable of generating. In panel A, the BSIV surface that results when the correlation between return and volatility innovations is assumed to be zero is plotted. As we can see, away-from-the-money options exhibit higher BSIVs compared to their near-the-money counterparts. In other words, the BSIV surface exhibits a 'smile' with respect to the moneyness level when the underlying asset's volatility is stochastic and uncorrelated with it. This observation can be explained by the fat-tailed nature of the distribution of returns that results in this case, as we demonstrated in the previous section. In addition, we observe that stochastic volatility is important mainly for short-term options given the fact that, as the time to expiration of the option contract increases, the implicit volatility surface flattens; in other words the implicit volatility 'smile' is more significant for short-term options than long-term ones.

The effect of positive correlation between return and volatility innovations is examined in panel B. Here a number of comments may be made. First, out-of-the-money calls are traded at higher BSIVs, whereas for in-the-money calls the oppo-

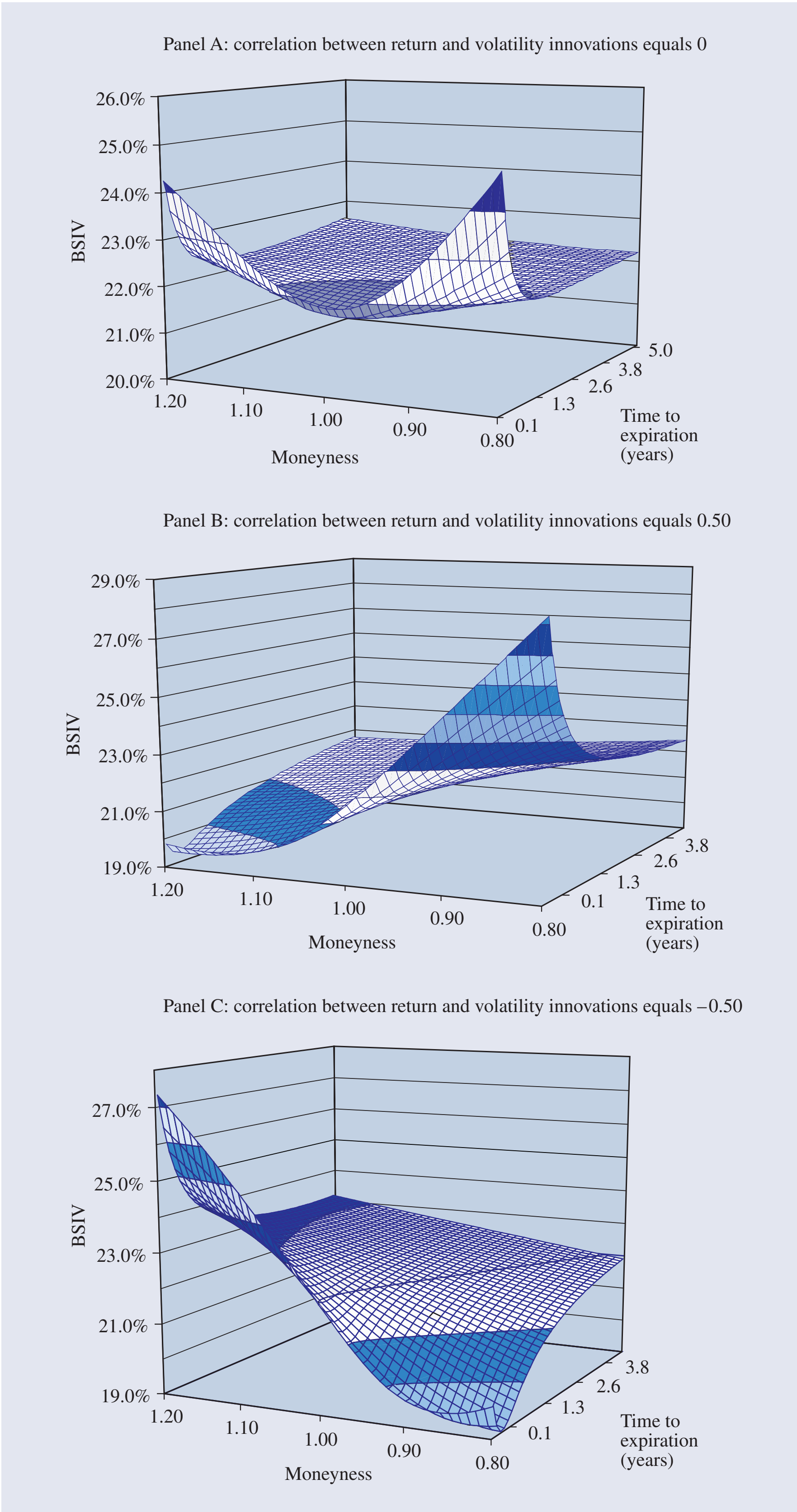

Figure 2. A plot of BSIV for different moneyness and time-to-expiration levels that result by utilizing European call option prices generated by the stochastic volatility option pricing model as inputs to the Black–Scholes model. Three different correlations are shown. The volatility of volatility coefficient $\sigma_V = 0.6$ and the values of the remaining structural parameters of the volatility process are the same as in table 2.

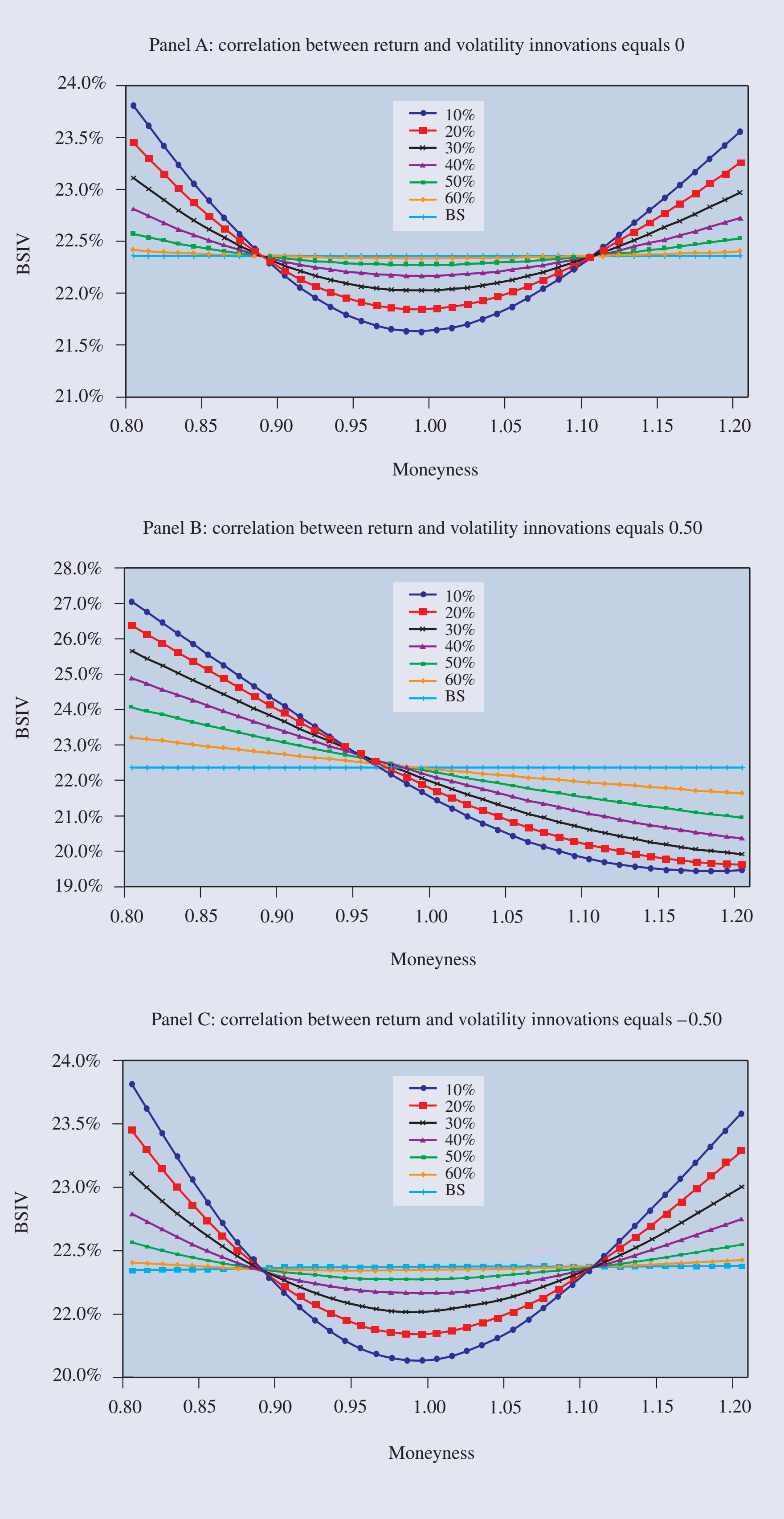

Figure 3. A plot of the BSIVs for different moneyness and vvl that result by utilizing European call option prices as described in figure 2. Three different correlations are shown. The time to expiration of the option contracts is three calendar months and the values of the remaining structural parameters of the volatility process are as in table 2.

site is true compared to at-the-money options. In other words, the volatility 'smile' when the correlation between return and volatility innovations is positive becomes a wry grin or skew.

This finding can be explained by the fact that positive correlation results in a return distribution for the underlying asset that is skewed to the right as high positive returns are associated with high volatility and high negative returns with low. Consequently, out-of-the-money call options have a higher probability of finishing in-the-money and out-of-the-money puts have a lower probability of being exercised than implied by the normal distribution. This asymmetry of the distribution of the underlying asset is translated to higher implicit volatilities for out-of-the-money calls and lower volatilities for in-the-money calls (as these contracts are linked to out-of-the-money puts through the put–call parity) when compared to at-the-money options. Again this effect is mitigated as the time to expiration increases. Finally, for similar reasons, the opposite pattern is observed in the case that the correlation between return and volatility innovations is negative, see panel C.

In figure 3, the effect of volatility of volatility on the BSIV surface, generated by utilizing stochastic volatility option prices as input to the Black–Scholes model, is examined. When the correlation between return and volatility innovations is zero (panel A), the resulting implicit volatility smile intensifies as the volatility of volatility level (vvl) increases. As we have demonstrated, this feature is a consequence of the increasingly fatter tails of the distribution of returns of the underlying asset that result when the vvl increases. The effect of increasing the vvl in the positive correlation case is shown in panel B. As we have noted previously, positive correlation results in a right-skewed return distribution that increases the value of out-of-the-money calls relative to out-of-the-money puts and consequently, through the put–call parity, in the money calls. This wry grin or skew pattern is attenuated as the vvl increases. The higher the vvl the more right-skewed the distribution of returns for the underlying asset becomes and as a result the more valuable out-of-the-money calls relative to out-of-the-money puts become. This is translated into an underpricing by the Black–Scholes model of out-of-the-money calls and an over-pricing of in-the-money calls as it assumes normally distributed returns.

The opposite pattern is observed in panel C where the effect of increasing volatility for a negative correlation is investigated. Increasing vvls result in increasing values of BSIVs generated by the stochastic volatility model for in-the-money calls relative to out-of-the-money counterparts. In addition, the Black–Scholes model seems to overprice out-of-the-money calls and underprice in-the-money calls. This is the case because, as we have noted before, negative correlation between return and volatility innovations results in a left-skewed distribution of returns for the underlying asset. This left skewness is exacerbated as the vvl increases. Consequently, the higher probability of out-of-the-money puts being exercised compared to the normal distribution utilized in the Black–Scholes models results in higher BSIV for these options. The opposite is true for out-of-the-money calls. For these contracts the probability of being exercised becomes lower and lower as the vvl increases for a given degree of negative correlation. Consequently, the BSIV curve using option prices, generated by the stochastic volatility model, as inputs to the Black–Scholes model, exhibits a J shape as we move from out-of-the-money to in-the-money European call contracts.

Summary

Many empirical studies have documented a relatively poor fit of the Black–Scholes model to market data, as is evident from the non-flat BSIV surface that many option markets exhibit. I have presented an alternative model that explicitly takes into account the stochastic nature of return volatility for pricing derivative securities. By generating option prices with the stochastic volatility model and using these prices as inputs to the Black–Scholes in order to back out BSIVs, I have demonstrated the various implicit volatility patterns that the stochastic volatility option-pricing model can generate and potentially explain.

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Further reading

Akgiray V and Booth G 1987 Compound distribution models of stock returns an empirical comparison *J. Financial Res.* **10** 269–97

Avellaneda M, Levy A and Paras A 1995 Pricing and hedging derivative securities in markets with uncertain volatilities *Appl. Math. Finance* **2** 73–88

Bakshi G, Cao C and Che Z 1997 Empirical performance of alternative option pricing models *J. Finance* **52** 2003–49

Ball A C and Roma A 1994 Stochastic volatility option pricing *J. Financial Quant. Anal.* **29** 589–605

Ball A C and Torous W 1983 A simplified jump process for common stock returns *J. Financial Quant. Anal.* **18** 53–65

Bates D 1996 Jumps and stochastic volatility: exchange rate processes implicit in Deutsche Mark options *Rev. Financial Stud.* **9** 69–108

Bensoussan A, Crouhy M and Galai D 1994 Stochastic equity volatility related to the leverage effect: I. Equity volatility behaviour *Appl. Math. Finance* **1** 63–85 1995 Stochastic equity volatility related to the leverage effect: II. Valuation of european equity options and earrants *Appl. Math. Finance* **2** 43–59

Black F and Scholes M 1973 The pricing of options and corporate liabilities *J. Political Economy* **81** 637–54

Blattberg R C and Gonedes N J 1974 A comparison of the stable and student distributions as statistical models for stock prices *J. Business* **47** 244–80

Bollerslev T, Chou C and Kroner K 1992 ARCH modelling in finance: a review of the theory and empirical evidence *J. Econometrics* **52** 5–59

Boness A, Chen A and Jatusipitak S 1974 Investigations of nonstationarity in prices *J. Business* **47** 518–37

Boothe P and Glassman D 1987 The statistical distribution of exchange rates: empirical evidence and economic implications *J. Int. Economics* **22** 297–319

Brorsen W and Yang S 1994 Nonlinear dynamics and the distribution of daily stock index returns *J. Financial Res.* **17** 187–203

Coleman T F, Li Y and Verma A 1999 Reconstructing the unknown local volatility function *J. Computational Finance* **2** 77–102

Dempster M A H and Richards D G 2000 Pricing American options fitting the smile *Math. Finance* **10** 157–77

Derman E and Kani I 1994 Riding on a smile *Risk* **7** 32–9

Dupire B 1994 Pricing with a smile *Risk* **7** 18–20

Duque J and Paxson D 1994 Implied volatility and dynamic hedging *Rev. Futures Markets* **13** 381–428

Engle R and Bollerslev T 1986 Modelling the persistence of conditional variances *Econometric Rev.* **5** 1–50

Fama E 1965 The behavior of stock market prices *J. Business* **38** 34–105

Heston S L 1993 A closed form solution for options with stochastic volatility with applications to bond and currency options *Rev. Financial Stud.* **6** 327–43

Heynen R 1994 An empirical investigation of observed smile patterns *Rev. Futures Markets* **13** 317–53

Hobson D and Rogers L 1998 Complete models with stochastic volatility *Math. Finance* **8** 27–48

Hull C J and White A 1987 The pricing of options on sssets with stochastic volatilities *J. Finance* **42** 281–300

Johnson H and Shanno D 1987 Option pricing when the variance is changing *J. Financial Quant. Anal.* **22** 143–51

Rubinstein M 1985 Nonparametric tests of alternative option pricing models using all reported trades and quotes on the 30 most active CBOE options classes from August 23 1976 through August 31 1978 *J. Finance* **30** 455–80

Scott L O 1987 Option pricing when the variance changes randomly: theory estimation and an application *J. Financial Quant. Anal.* **22** 419–38

Vasicek O F 1998 Prologue in Briys C E, Bellalah M and Minh M H 1998 *Option Futures and Exotic Derivatives* (New York: Wiley)

Wiggins J B 1987 Option values under stochastic volatility: theory and empirical estimates *J. Financial Economics* **19** 351–72

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