ESTIMATION IN THE CONSTANT ELASTICITY OF VARIANCE MODEL

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ABSTRACT

The constant elasticity of variance (CEV) diffusion process can be used to model heteroscedasticity in returns to common stocks. In this diffusion process, the volatility is a function of the stock price and involves two parameters. Similar to the Black-Scholes analysis, the equilibrium price of a call option can be obtained for the CEV model. The purpose of this paper is to propose a new estimation procedure for the CEV model. A merit of our method is that no constraints on the elasticity parameter of the model are imposed. In addition, frequent adjustments of the parameter estimates are not required. Simulation studies indicate that the proposed method is suitable for practical use. As an illustration, real examples on the Hong Kong stock option market are carried out. Various aspects of the method are also discussed.

KEYWORDS

Constant Elasticity of Variance; Diffusion Process; Least Squares; Option Pricing; Volatility.

1. INTRODUCTION

The valuation of options has been one of the main issues in the areas of modern finance, and is of great interest to actuaries. In particular, stock options are often used in the study of option pricing. Suppose that the stock price process S_t is defined as the solution of a stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \tag{1}$$

where μ is known as the expected rate of return; σ is the standard deviation of the percentage price return and often called the stock price volatility; and B_t is a Wiener process. In option pricing theory, the risk-neutrality assumption allows us to replace the expected rate of return by the risk-free rate of interest; and hence the only unobservable value in (1) is the volatility. The parameter σ can be estimated from the history of stock prices, that is, using the sample standard deviation of the return rate. Given model (1) and a certain set of assumptions, Black and Scholes (1973) obtained exact formulas for pricing European options. Through the Black-Scholes (BS) formulas, an observed option price in the market can be used to find an implied value of σ. This alternative estimate is termed an implied volatility.

Much attention has been paid to the constant volatility assumption in (1) which seems not very suitable in real cases. There is considerable evidence in the literature, indicating that stock returns are heteroscedastic. For example, see Black (1976), Blattberg and Gonedes (1974), and MacBeth and Merville (1979). In view of this property, Cox (1975) and Cox and Ross (1976) studied the constant elasticity of variance (CEV) diffusion process which takes the form

$$dS_t = \mu S_t dt + \delta S_t^{\frac{\theta}{2}} dB_t, \tag{2}$$

where δ and θ are constants. The volatility of model (2) is $\delta S_t^{\frac{\theta}{2}-1}$. The variance rate of S_t is $\delta^2 S^\theta$ and the elasticity of this variance with respect to S_t is θ. It is easily seen that model (1) is equivalent to model (2) when $\theta = 2$, and that the volatility is a increasing (decreasing) function of S_t when $\theta > 2$ ($\theta < 2$). Under model (2) and the set of assumptions in the BS framework, Cox (1975) derived the equilibrium price of a call option for $\theta < 2$. Emanuel and MacBeth (1982) extended the pricing formula to the case of $\theta > 2$.

Models (1) and (2) are sometimes referred to as the BS and CEV models respectively. In this paper, the number and abbreviation of the models are used interchangeably. Both the BS and CEV option pricing formulas are given in the Appendix. The option pricing formula for the CEV model certainly has a more complex form than that for the BS model.

MacBeth and Merville (1980) investigated the problem of estimating δ and θ and proposed a three-stage procedure. The first stage is to find a reasonable point estimate of θ. It relies on the fact that, given the true value of θ, the value of δ is the same for all options written on the same stock. For an integer value of θ and an observed option price, an application of a numerical search routine to the CEV option pricing formula yields an implied value of δ. Then an arbitrary set of observed option prices from the same stock generates a set of implied values of δ for the same θ. The same steps are repeated for different integer values of θ. The final estimate of θ is the one for which the implied values of δ are most nearly constant. The second stage is based on their simulation study indicating that the BS model with the correct variance rate of return will give approximately the correct price for at-the-money options even if the underlying stock price process follows the CEV model with $\theta < 2$. Thus the BS implied volatility calculated using an at-the-money option price is treated as a good estimate of the volatility of the CEV model. The final stage is to obtain an etimate of δ using the results in the first two stages. As the market option price changes, the value of δ needs to be re-estimated. Hence daily adjustment of δ is possibly required.

In the method of MacBeth and Merville (MM), the idea used in the first stage is intuitively clear and actually implied by model (2) while the third stage is straightforward. As pointed out by Manaster (1980), the validity of the second stage is in doubt. It is simply because the BS implied volatility is not the true volatility when the value of θ is other than 2. Their estimates of δ may therefore differ systematically from the true value. Furthermore, even though, their second stage is supported by simulation results, it holds only for $\theta < 2$. The two parameters of the CEV model, in principle, should not require adjustments as frequently as the BS parameter. Because of the use of the BS implied volatility, this valuable feature of the CEV model cannot come into play in their method.

In this paper, we propose a new estimation procedure for the CEV model in which neither constraints on θ nor frequent adjustments of the parameter estimates are needed. Section 2 uses the log-linear property of the variance of the percentage price return and the results of Chesney, Elliott, Madan, and Yang (1993) to estimate the parameters through the least-squares method. Although this simple idea allows us to jointly estimate δ and θ, the linearization of the variance causes certain numerical problems. Section 3 introduces a two-stage approach as a remedy and demonstrates its practicality through simulations. Real examples on the Hong Kong stock option market are carried out in Section 4. Finally some remarks are given in Section 5.

2. Least-Squares Estimation

We now introduce a least-squares procedure through which the two parameters of model (2) can be estimated jointly. To obtain point estimates of δ and θ denoted by $\hat{\delta}$ and $\hat{\theta}$, we first find an estimate of $\sigma_t^2 = \delta^2 S_t^{\theta-2}$ which is the square of the volatility at time t in model (2). Using the results of Chesney, Elliott, Madan, and Yang (1993), we have the following estimate of σ_t^2

$$V_t = \frac{2}{\alpha \Delta t}\left[\frac{S_{t+\Delta t}^{1+\alpha} - S_t^{1+\alpha}}{(1+\alpha)S_t^{1+\alpha}} - \frac{S_{t+\Delta t} - S_t}{S_t}\right]\frac{1}{\Delta t}, \tag{3}$$

where α is a constant and Δt is the length of a small time interval. It was shown that the conditional expectation $E(V_t|S_t)$ converges to σ_t^2 as $\Delta t \to 0$, and that the conditional variance $Var(V_t|S_t)$ is minimized when

$$\alpha = -\frac{13}{11} - \frac{12}{11}\frac{\mu}{\sigma_t^2}. \tag{4}$$

Table 1. Simulation results using (5)

True value		$\hat{\theta}$		$\hat{\delta}$	
θ	δ	Mean	S.D.	Mean	S.D.
-4	7000	-3.9963	0.7222	9881.2	31003
-3	1300	-3.0117	0.6993	1701.9	4911.4
-2	250	-1.9890	0.6483	283.98	722.34
-1	45	-1.0054	0.6221	47.639	118.93
0	8	0.0110	0.6205	8.3722	22.125
1	1.50	1.0044	0.5741	1.4051	3.7028
2	0.25	1.9981	0.6319	0.2939	0.7888
3	0.05	3.0058	0.5838	0.0584	0.1079
4	0.01	3.9988	0.6453	0.0092	0.0311

Computationally, we start with an initial guess of α, and then obtain a value of V_t from (3). The parameter μ can be easily estimated using the sample return denoted by $\hat{\mu}$. Replacing σ_t^2 by V_t and μ by $\hat{\mu}$ in (4), we have a revised value of α. We iterate the two steps until the last two values of α are within a certain tolerance.

Given V_t, the method of least squares can be employed. The least-squares estimates $\hat{\delta}$ and $\hat{\theta}$ minimize the sum of squares of deviations between $\ell n\, V_t$ and $\ell n\, \sigma_t^2$

$$\sum_{t=1}^{n-1} (\ell n\, V_t - \ell n\, \sigma_t^2)^2, \tag{5}$$

where n is the number of data points. Taking logarithm of σ_t^2 produces a linear function of $\ell n\, S_t$, and hence simplifies the minimization problem.

From model (2), we generate 1000 stock prices with an initial stock price of \$30, a risk-free interest rate of 5%, and $\Delta t = 0.0025$. We assume the initial volatility to be around 0.25. We summarize the simulation results in Table 1. For each pair of $\hat{\delta}$ and $\hat{\theta}$, the means and standard deviations (S.D.) are calculated from 1000 simulations. Table 1 shows that the simple linear least-squares method produces reasonably good estimates for θ but not for δ. The means of $\hat{\delta}$ are close to the true values except for values of δ over 250, and all the standard errors appear very large relative to the means. The large standard deviations of $\hat{\delta}$ can be explained by the fact that a small deviation between $\hat{\theta}$ and θ yields a relatively large difference between $\hat{\delta}$ and δ. In fact, the variation of $\hat{\theta}$ has an exponential effect on that of $\hat{\delta}$. There is another problem of empirical nature. Theoretically, the values of S_t and $S_{t+\Delta t}$ are different with probability 1. However, in real cases, two consecutive stock prices in a short period of time often have the same value. When applying (3) to real data, it is very likely that we have a large number of undefined values of $\ell n\, V_t$. These numerical difficulties motivate us to consider a two-stage approach to estimate the parameters in the next section.

3. Two-Stage Estimation

Following the first stage of the MM's method, we make use of the fact that all options written on the same stock have the same values of δ and θ. We arbitrarily select m call option prices $(C_1, ...C_m)$ with the same underlying stock in the observation period. For each C_j and a given value of θ, we use a numerical search routine to calculate the implied value of δ, denoted by $\delta_j(\theta)$, from the CEV option pricing formula. To measure the degree of dispersion among δ_j's , we consider the absolute relative error defined as

$$U(\theta) = \sum_{j=1}^{m} \frac{|\delta_j(\theta) - \bar{\delta}_j(\theta)|}{\bar{\delta}_j(\theta)}, \tag{6}$$

where $\bar{\delta}_j$ is the mean of δ_j's. Then our final point estimate $\hat{\theta}$ is the one that minimizes U. After obtaining $\hat{\theta}$ in the first stage, we estimate δ by minimizing the sum

$$\sum_{t=1}^{n-1} (V_t - \delta^2 S_t^{\hat{\theta}-2})^2, \tag{7}$$

where V_t's are defined in (3). Hence we have

$$\hat{\delta} = \sqrt{\frac{\sum V_t S_t^{\hat{\theta}-2}}{\sum S_t^{2\hat{\theta}-4}}}. \tag{8}$$

We now perform a simulation study to assess the performance of the proposed method. For a given pair values of δ and θ, we simulate 5 call option prices from model (2) with exercise prices \$26, \$28, \$30, \$32, \$34. We set the initial stock price, risk-free interest rate, time to maturity as \$30, 5%, and 0.25 respectively. In this simulation study, we arbitrarily consider six cases characterized by integer values of θ ranging from -2 to 3. The corresponding δ's are set in the way that the initial volatility is around 30%. The first step is to select the best θ according to (6). For simplicity, we just try integer values of θ from -3 to 4. Figure 1 displays the plots of U of (6) versus θ. We see that the measure U always attains its minimum at the true value of θ for all the six cases. If the CEV model is correct, this measure does give a clear indication where the true θ is located. The second step is to calculate the estimates of δ using (8). Table 2 shows that V_t of (3) together with (7) indeed produce a very good estimate of δ given that $\hat{\theta}$ is close to θ.

Table 2. Simulation results using the two-stage method

θ	δ	$\hat{\theta}$	$\hat{\delta}$
-2	270	-2	267.38
-1	50	-1	51.199
0	9	0	9.1553
1	1.65	1	1.6559
2	0.30	2	0.2999
3	0.06	3	0.0607

Table 3. Simulation results for $\hat{\delta}$ of (8)

True value			$\hat{\delta}$	
θ	δ	$\hat{\theta}$	Mean	S.D.
-4	7000	-3.9963	6995.1	1137.7
-3	1300	-3.0117	1318.2	169.68
-2	250	-1.9890	251.45	27.362
-1	45	-1.0054	45.216	3.5961
0	8	0.0110	8.0173	0.2265
1	1.50	1.0044	1.4975	0.0349
2	0.25	1.9981	0.2483	0.0053
3	0.05	3.0058	0.0496	0.0011
4	0.01	3.9988	0.0098	0.0002

To investigate the variability of $\hat{\delta}$ of (8), we perform another simulation similar to the one in the previous section. Here we use the means of $\hat{\theta}$ in Table 1 as our final estimates of θ in the first stage, and the results are given in Table 3. Not surprisingly, the means and standard deviations of (8) look much better than those shown in Table 1. Although the first stage of our method is more or less the same as that of the MM's method, our second stage has a stronger theoretical basis and is computationally easier.

4. Application to Hong Kong Stock Options

The Hong Kong stock option market began in late 1995 and most of the option contracts are on blue chips. Call option prices for three popular stocks, Cheung Kong Holdings Limited (CKH), Hong Kong Telecommunications Limited (HKT), and Swire Pacific Limited 'A' (SPA), from July 1, 1996 to June 30, 1997, are used

Table 4. Parameter estimates

Stock	CEV		BS
	$\hat{\theta}$	$\hat{\delta}$	$\hat{\sigma}$
CKH	4	0.004	0.268
HKT	4	0.234	0.299
SPA	0	16.16	0.243

in the analysis. The data are obtained from Research and Planning Division, the Stock Exchange of Hong Kong. The time series plots of S_t and V_t for the three stocks are shown in Figure 2. The plots of V_t suggest that the volatility for each stock does vary through time. Roughly speaking, for CKH and HKT, the volatility increases with the stock price; and the reverse is true for SPA.

The objectives of this section are to apply our two-stage method to the Hong Kong data and compare the performance of the CEV and BS models. In this application, we not only calculate the estimates of δ and θ but also the estimate of the volatility of model (1) denoted by $\hat{\sigma}$. Instead of using the implied volatility or the sample standard deviation, we use the method of least squares together with V_t of (3) to obtain $\hat{\sigma}$. The results are summarized in Table 4. As expected, the value of $\hat{\theta}$ for CHK and HKT is greater than 2 and that for SPA is less than 2.

We have mentioned in Section 1 that the BS model works quite well for pricing at-the-money calls. On the other hand, empirical evidence suggests that the BS model systematically overprices in-the-money call options and underprices out-of-the-money call options if the underlying stock process follows the CEV model with $\theta < 2$. It is interesting to see if similar systematical pricing biases still hold for $\theta > 2$.

We compute the BS and CEV model prices for CKH and HKT using the estimates shown in Table 4. For having more reasonably quoted market prices, we only include the options with high trading volume. We use a risk-free interest rate of 5% in our calculations. We now define $M = (S - K)/K$ as a measure of how far the option is in or out of the money where K represents the exercise price of the option. Furthermore, we use $V = (C_{market} - C_{model})/C_{model}$ to measure the percent difference between model price C_{model} and market price C_{market}. Figures 3 and 4 plots V versus M for CKH and HKT respectively. Figure 3 clearly reveals similar pricing biases of the BS model and shows that the measure V for the CEV model looks more randomly scattered around the horizontal line. As for HKT, the underpricing problem can also be seen to a certain degree. Although Figure 4 does not exhibit much overpricing of in-the-money options for the BS model, we see that the points for the BS model are distributed farther away from zero than those for the CEV model.

5. REMARKS

At the first glance, it is natural to consider the sum of the squared deviations between σ_t^2 and V_t and use the non-linear least-squares method to estimate the two parameters jointly. This can be done easily using some statistical software package like SAS. However, the non-linear method does not result in acceptable estimates of the parameters. The resulting correlation matrix indicates that $\hat{\delta}$ and $\hat{\theta}$ are highly correlated.

In our two-stage method, it may take a long time to search for the best estimate of θ. The simulation results in Section 2 suggest that we can make use of (5) to obtain a good initial estimate of θ. To do this, we simply ignore those points with zero V_t in the estimation. The true θ is likely to be somewhere near the initial estimate. It is expected that the best estimate can be obtained within a few tries. Furthermore, in the previous examples, we have restricted our search to integer values of θ only for simplicity. A more detailed search would certainly produce a more accurate estimate.

For checking model adequacy, we may examine the distribution of the following random variables

$$X_t = \frac{dS_t - \hat{\mu} S_t dt}{\hat{\delta} S_t^{\frac{\hat{\theta}}{2}}}. \tag{9}$$

If the CEV model is correct, the histogram of X_t's should roughly look like a normal curve with zero mean. This simple graphical method can also informally check whether the estimates are plausible.

In this paper, we use V_t of (3) and the least-squares method to develop an estimation procedure for the CEV model. It is worthy to mention that this idea can be extended to any general diffusion processes for stock prices. As shown in the real examples, the simplest case is the BS model; and the constant volatility is estimated by the square root of the mean of V_t's.

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References

BLACK, F. & SCHOLES, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81, 637-654.

BLACK, F. (1976). Sudies of stock price volatility changes. *Proceedings of the meetings of the American Statistical Association, Business and Economics Statistics*, Section, Chicago.

BLATTBERG, R.C. & GONEDES, N.J. (1974). A comparison of the stable and student distributions as stochastic models for stock prices. *Journal of Business*, **47**, 244-280.

COX, J. (1975). Notes on option pricing I: Constant elasticity of diffusions. *Unpublished draft*, Stanford University.

COX, J. & ROSS, S. (1976). The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, **3**, 145-166.

CHESNEY, M., ELLIOTT, R.J., MADAN, D. & YANG, H. (1993). Diffusion coefficient estimation and asset pricing when risk premia and sensitivities are time varying. *Mathematical Finance*, **3**, 85-99.

EMANUEL, D.C. & MACBETH, J.D. (1982). Further results on the constant elasticity of variance call option pricing model. *Journal of Financial and Quantitative Analysis*, **42**, 533-554.

MACBETH, J.D. & MERVILLE, L.J. (1979). An empirical examination of the Black-Scholes call option pricing model. *Journal of Finance*, **34**, 1173-1186.

MACBETH, J.D. & MERVILLE, L.J. (1980). Tests of the Black-Scholes and Cox Call Option Valuation Models. *Journal of Finance*, **35**, 285-301.

MANASTER, S. (1980). Discussion of MacBeth, J.D. and Merville, L.J. "Tests of the Black-Scholes and Cox Call Option Valuation Models". *Journal of Finance*, **35**, 301-303.

Appendix

Consider a European call option with an exercise price of K, maturing at time T. Let the risk-free interest rate be r. Then, from the risk-neutral valuation argument, the call option price is the discounted expected payoff, that is,

$$C = e^{-r(T-t)}E[\max(S_T - K, 0)].$$

BS option pricing formulas:

$$C = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

where

$$\begin{aligned}
d_1 &= \frac{\ell n\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \\
d_2 &= d_1 - \sigma\sqrt{T-t}, \\
N &= \text{standard normal cdf}, \\
\sigma &= \text{constant volatility in model (1)}.
\end{aligned}$$

CEV option pricing formulas:

$$C = SM_1 - Ke^{-r(T-t)}M_2,$$

where

$$\begin{aligned}
M_1 &= \sum_{n=0}^{\infty} g(S'|n+1)G(K'|n+p), && \theta < 2 \\
&= 1 - \sum_{n=0}^{\infty} g(S'|n+p)G(K'|n+1), && \theta > 2 \\
M_2 &= \sum_{n=0}^{\infty} g(S'|n+p)G(K'|n+1), && \theta < 2 \\
&= 1 - \sum_{n=0}^{\infty} g(S'|n+1)G(K'|n+p), && \theta > 2 \\
S' &= \frac{2re^{r(T-t)(2-\theta)}S^{2-\theta}}{\delta^2(2-\theta)\left(e^{r(T-t)(2-\theta)}-1\right)}, \\
K' &= \frac{2rK^{2-\theta}}{\delta^2(2-\theta)\left(e^{r(T-t)(2-\theta)}-1\right)},
\end{aligned}$$

$$
\begin{aligned}
g(x|m) &= \tfrac{e^{-x}x^{m-1}}{\Gamma(m)}, \\
G(x|m) &= \textstyle\int_x^\infty g(y|m)dy, \\
p &= 1 + \tfrac{1}{|2-\theta|}, \\
\theta &= \text{the elasticity of variance in model (2).}
\end{aligned}
$$

Figure 1: Plots of U

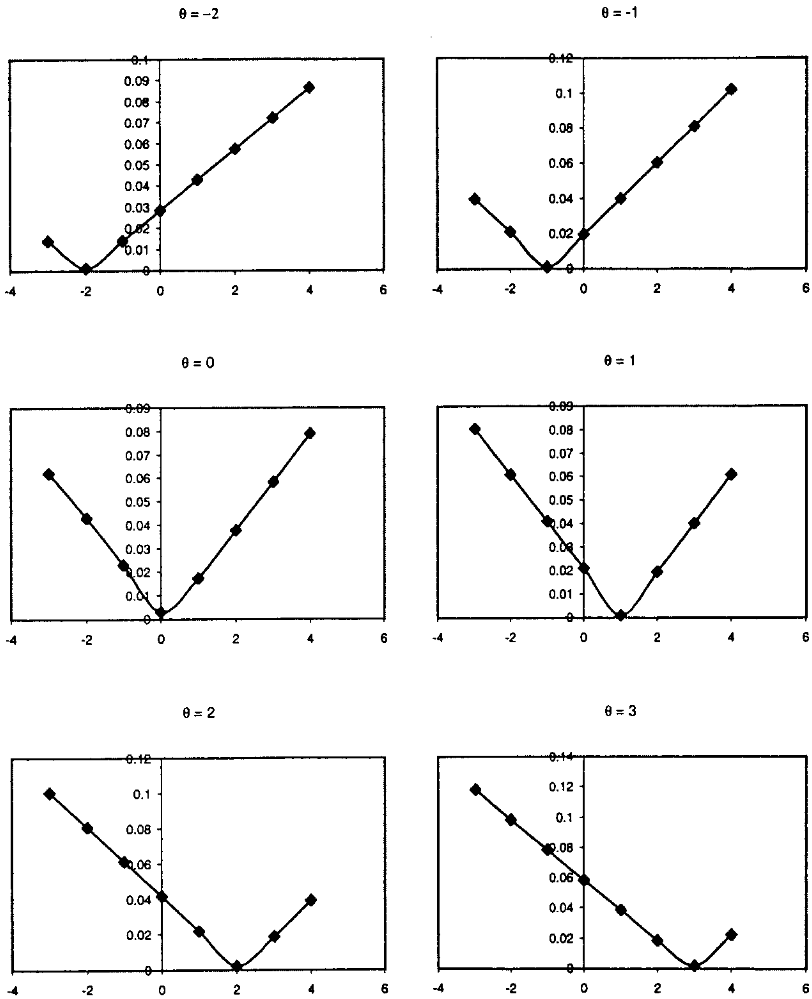


Figure 2: Plots of S_t and V_t

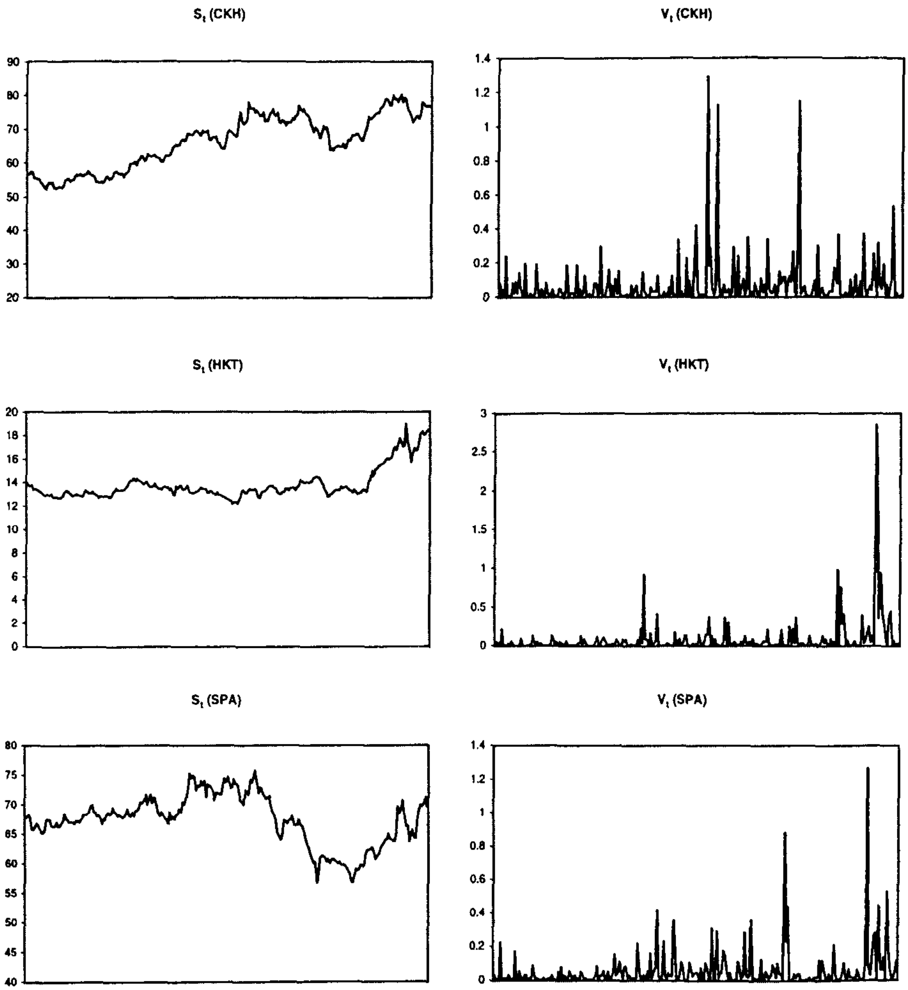
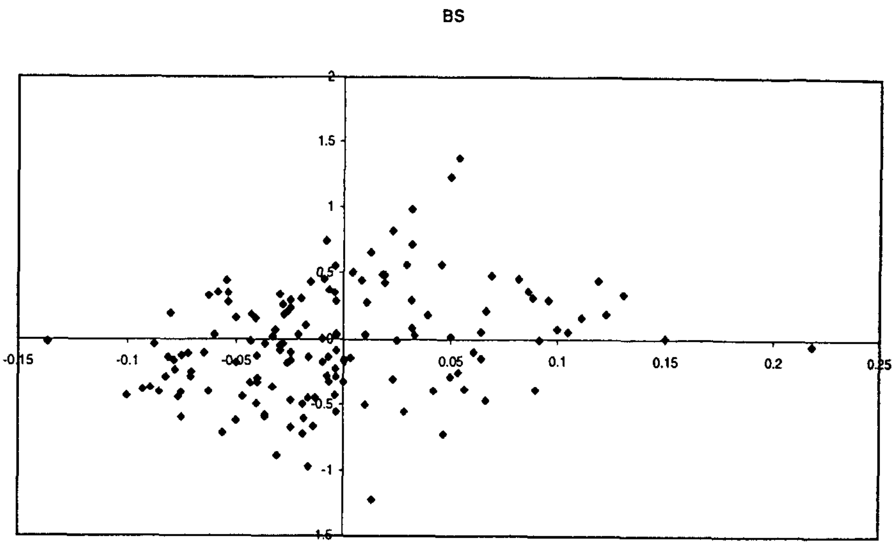


Figure 3: Plots of V vs M for CKH



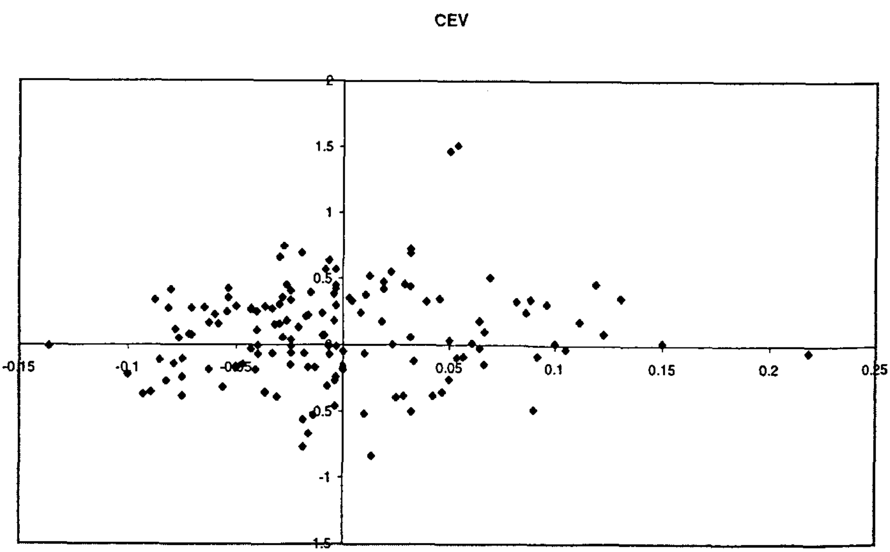
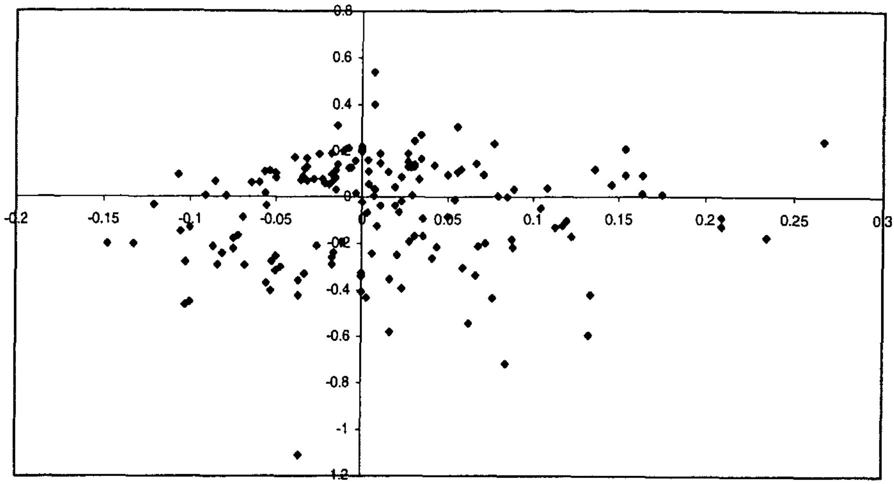


Figure 4: Plots of V vs M for HKT

BS



CEV