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JAMES D. MACBETH and LARRY J. MERVILLE*

I. Introduction

IN THIS RESEARCH we use Cox's [7] and Cox and Ross' [8] constant elasticity of variance diffusion processes to model heteroscedasticity in returns to common stocks. The major goal of this paper is to test the Cox call option valuation model for constant elasticity of variance diffusion processes against the Black-Scholes [4] call option valuation model. We find that common stock prices do appear to be generated by constant elasticity of variance diffusion processes; moreover, we find that the Cox valuation model fits market prices of call options significantly better than the Black-Scholes model. Thus, our results have important implications for empirical analysis of call option data and may very well have important implications for empirical analysis of common stock prices and prices of other financial instruments.

At the theoretical level there are several plausible explanations for changes in stock return variances over time. First, firms may internally change their common stock return distribution through technological innovations and/or mergers and acquisitions. Another area of possible explanation for dynamic variances is contained in multiperiod consumption-investment theory (Rubinstein [16] and Fama [9]). If in each period aggregate consumer-investors plan their consumption and investment over multiple future periods, then the variances for securities may change over time as new information arises and new individuals (preferences) bid for risky assets in the capital markets.

There is considerable evidence in the empirical literature that returns to common stocks are heteroscedastic. Blattberg and Gonedes [5] present evidence that a process in which the variance of returns changes randomly through time fits empirical data better than a stationary Stable Paretian process. Rosenberg [15] finds that the variance of monthly returns to the Standard and Poors 500 Index follows an autoregressive process through time. Using daily returns, Black [3] observes that the variance of returns varies inversely with the stock price. This inverse relationship between variance of returns and the stock price can be modeled by a constant elasticity of variance diffusion process.

II. Constant Elasticity of Variance Diffusion Processes

Denote the stock price at an instant of time t as S and let the change in the stock price over the next small increment of time, dt, be denoted dS. Then, for constants

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μ, θ, and $\delta > 0$; the family of constant elasticity of variance diffusion processes can be described by the stochastic differential equation:

$$dS = \mu S\,dt + \delta S^{\theta/2}\,dZ \tag{1}$$

where dZ is a Wiener process. The instantaneous variance of the stock price is $\delta^2 S^\theta$ and the elasticity of this variance with respect to the stock price equals θ. The instantaneous variance of the percentage price change or return, σ^2, is given by the equation:

$$\sigma^2 = \delta^2 S^{(\theta-2)} \tag{2}$$

which is a decreasing function of the stock price for $\theta < 2$. When θ equals two, the instantaneous variance of return is a constant, δ^2, and the stochastic process generating returns is a lognormal diffusion process, the process assumed in the Black-Scholes valuation model.

III. Call Option Valuation with Constant Elasticity of Variance Diffusion Process

Maintaining all of the assumptions of the Black-Scholes option valuation model, except that the stochastic stock price process is generalized to a constant elasticity of variance process, Cox derives the equilibrium price of a call option by invoking an assumption of risk neutrality and discounting the partial expected value of the terminal stock price by the riskless rate of interest.[1] Letting C denote the call price; S, the stock price; τ, the time to expiration; r, the riskless rate of interest; E, the exercise price; and δ and θ the parameters of the stochastic process; the Cox equation for the value of the call option at any instant of time is:

$$C = S\cdot\left[\textstyle\sum_{n=0}^{\infty} g(S'\,|\,n+1)\cdot G\left(E'\,|\,n+1+\frac{1}{2-\theta}\right)\right] - Ee^{-r\tau}\cdot\left[\textstyle\sum_{n=0}^{\infty} g\left(S'\,|\,n+1+\frac{1}{2-\theta}\right)\cdot G(E'\,|\,n+1)\right] \tag{3}$$

where

$$S' = \left[\frac{2re^{r\tau(2-\theta)}}{\delta^2(2-\theta)(e^{r\tau(2-\theta)}-1)}\right)\cdot S^{2-\theta}$$

$$E' = \left[\frac{2r}{\delta^2(2-\theta)(e^{r\tau(2-\theta)}-1)}\right]\cdot E^{2-\theta}$$

$$g(x\,|\,m) = \frac{e^{-x}x^{m-1}}{\Gamma(m)}; \text{ the gamma density function.}$$

$$G(x\,|\,m) = \int_x^\infty g(y\,|\,m)\,dy$$

[1] While this technique is not always appropriate; for example see Garman [10], it is appropriate under the given set of assumptions.

Table 1 provides a simulated comparison of Black-Scholes call option prices, $\theta = 2$, with Cox option prices for a variety of stock prices, exercise prices, times to expiration, values of θ, and values of σ^2. On an annual basis the riskless rate of return is assumed to be six percent while the two values of σ are twenty and forty percent. Given a daily variance rate of return, σ^2, and a value of θ, the parameter δ is chosen to satisfy the equation:

$$\sigma^2 = \delta^2 S^{(\theta-2)} \tag{4}$$

Thus, for a given value of σ^2, Table 1 contains equilibrium prices of call options on stocks with different values of θ, and therefore different constant elasticity of variance stochastic processes but processes with the same variance rate of return, σ^2. Black-Scholes model prices correspond to the case where $\theta = 2$.

If we assume that the process generating the stock price is a constant elasticity of variance process with $\theta = 2$, then, for the range of parameter values considered in Table 1, the Black-Scholes model systematically underprices options when S is above a particular value of S, say S_c, and systematically overprices options when S is less then S_c. This critical stock price, S_c, is equal to or slightly lower than the exercise price, E.

Therefore, at least for the range of parameters used in the simulations reported in Table 1, we can say that if the stochastic process generating the stock price is a constant elasticity of variance process with θ less than two, the Black-Scholes model, given the proper variance rate, underprices in the money options, overprices out of the money options, and gives approximately the proper price for options at the money. Notice, however, that since the variance rate changes with the stock price, the Black-Scholes model will require a different variance rate each time the stock price changes and even then it will only give (approximately) the correct price for at the money options. This fact may explain why practitioners using the Black-Scholes model constantly adjust the variance rate used in the model.[2] At present, this adjustment in the variance rate is done on an *ad hoc* basis. The constant elasticity of variance model indicates exactly how the variance rate changes for a given change in the stock price.

IV. The Data

Our sample consists of daily closing prices of *all* call options traded on the Chicago Board of Trade Options Exchange for American Telephone and Telegraph (ATT), Avon Products (AVON), Eastman Kodak (ETKD), Exxon (EXXN), International Business Machines (IBM), and Xerox (XERX) from December 31, 1975, to December 31, 1976. Option prices and prices of the underlying stocks are taken from the Wall Street Journal. Dividend information comes from Standard and Poors Stock Record. The riskless return is imputed from bid and ask yields reported in the Wall Street Journal for United States

[2] See Black [1], [2], and [3].

Table 1

A Comparison of Black-Scholes Model Prices and Cox Model Prices

			r = .000164; 6% Annual							
			σ = .010468, 20% annual				σ = .020937, 40% annual			
τ	S	E	$\theta = 2$	$\theta = 0$	$\theta = -2$	θ ; eq −4	$\theta = 2$	$\theta = 0$	$\theta = -2$	$\theta = -4$
	$									
		$ 40.	$10.20	$10.20	$10.20	$10.20	$10.24	$10.28	$10.34	$10.44
30.	50.	50.	1.27	1.27	1.27	1.27	2.41	2.41	2.41	2.42
		60.	.01	.00	.00	.00	.16	.11	.06	.04
		40.	10.60	10.62	10.64	10.69	11.08	11.30	11.58	11.96
90.	50.	50.	2.35	2.36	2.36	2.36	4.30	4.31	4.33	4.37
		60.	.10	.06	.03	.02	1.20	.94	.73	.55
		40.	11.26	11.32	11.42	11.56	12.40	12.81	13.35	14.01
180.	50.	50.	3.55	3.55	3.56	3.57	6.26	6.28	6.34	6.46
		60.	.53	.39	.28	.19	2.78	2.35	1.97	1.64
	50.	40.	11.93	12.05	12.21	12.43	13.60	14.15	14.89	15.65
270.		50.	4.55	4.55	4.57	4.59	7.82	7.85	7.97	8.14
		60.	1.10	.89	.69	.53	4.21	3.65	3.18	2.77
		90.	10.50	10.52	10.54	10.57	11.41	11.56	11.73	11.92
30.	100.	100.	2.54	2.54	2.54	2.54	4.81	4.81	4.82	4.83
		110.	.15	.12	.09	.07	1.47	1.31	1.16	1.03
		90.	11.84	11.94	12.05	12.18	14.44	14.79	15.19	15.67
90.	100.	100.	4.71	4.71	4.72	4.72	8.61	8.62	8.66	8.74
		110.	1.22	1.08	.96	.84	4.72	4.37	4.06	3.78
		90.	13.80	13.98	14.19	14.42	17.95	18.48	19.15	20.00
180.	100.	100.	7.10	7.10	7.12	7.14	12.53	12.56	12.69	12.92
		110.	3.01	2.78	2.56	2.36	8.46	7.98	7.57	7.23
		90.	15.60	15.84	16.11	16.44	20.83	21.51	22.43	23.44
270.	100.	100.	9.10	9.11	9.13	9.18	15.64	15.71	15.95	16.29
		110.	4.73	4.42	4.14	3.89	11.55	10.98	10.54	10.21
		220.	31.13	31.16	31.20	31.25	32.72	33.10	33.53	34.03
30.	250.	250.	6.34	6.34	6.34	6.35	12.03	12.04	12.05	12.09
		280.	.17	.12	.08	.05	2.79	2.38	2.01	1.68
		220.	34.01	34.24	34.52	34.84	39.62	40.59	41.71	43.10
90.	250.	250.	11.77	11.78	11.79	11.81	21.52	21.56	21.65	21.86
		280.	2.20	1.86	1.56	1.29	10.36	9.37	8.48	7.67
		220.	38.55	39.02	39.57	40.21	48.02	49.55	51.46	53.85
180.	250.	250.	17.74	17.75	17.79	17.85	31.31	31.40	31.72	32.31
		280.	6.21	5.55	4.94	4.39	19.50	18.05	16.81	15.74
		220.	42.82	43.47	44.23	45.14	55.03	56.98	59.60	62.41
270.	250.	250.	22.74	22.76	22.83	22.95	39.10	39.27	39.87	40.74
		280.	10.23	9.35	8.55	7.82	27.14	25.39	24.01	22.89

Treasury Bills and is updated weekly. That is, when an option is first traded we select a Treasury Bill that expires just beyond the expiration date of the option and then follow the yield on that Bill through time. For each option expiration date we have a different riskless rate but the riskless rates are generally within ½ of 1% of one another.

V. Estimating θ

From equation (1) it follows that:

$$\frac{(dS - \mu S\, dt)^2}{S^\theta\, dt} \equiv u_t \propto \chi^2(1) \tag{5}$$

where the constant of proportionality is $1/\delta^2$. Our initial technique for estimating θ involved estimating μ from a sample of daily returns and then searching for a value of θ that would yield a sample of u_t values, from this same sample, which when appropriately scaled, would "fit" a Chi Square distribution with one degree of freedom.[3] This approach appears to be very sensitive to the assumption that dZ is a Wiener process since we were unable to find values of θ that would yield a sample of u_t which fit the Chi Square distribution.

Thorpe [18] proposes an alternative technique for estimating θ that exploits the fact that u_t is a decreasing function of θ. He suggests regressing values of u_t, for a particular value of θ, say $\hat{\theta}$, on S_t. For values of $\hat{\theta}$ greater than the true θ the slope coefficient of the regression will be significantly negative and for values of $\hat{\theta}$ less than the true θ the slope coefficient will be significantly positive. There will be a range of values of $\hat{\theta}$ around θ for which the slope coefficient is insignificantly different from zero.

Point estimates of θ can be obtained in a similar manner. For some interval of time dt, it follows from equation (1) that:

$$\ln[(dS - \mu S\, dt)^2] - \ln[dt] = 2\cdot\ln[\delta] + \theta \ln S + \ln[\chi^2_{(1)}] \tag{6}$$

Although the last term does not have an expected value equal to zero, the Chi Square random variables are incorrelated through time, and the above equation can be used as a linear regression equation to obtain consistent estimates of θ.

Using daily returns for 1976 on the six stocks, we calculate the following confidence intervals and point estimates of θ. In obtaining the confidence intervals, we search through integer values of θ only. The numbers in parentheses following the point estimates are standard errors of the estimates. Both of these approaches give credible but imprecise estimates of θ.

Stock	Confidence Region	Point Estimates
ATT	$-2 \le \theta \le 6$	3.84 (1.82)
AVON	$-8 \le \theta \le -2$	−3.63 (1.27)
ETKD	$-1 \le \theta \le 5$	3.04 (1.40)
EXXN	$-1 \le 0 \le 5$	1.62 (1.71)
IBM	$-8 \le \theta \le 2$	−4.16 (2.54)
XERX	$-4 \le \theta \le 2$	−1.69 (1.42)

[3] The values of u_t were scaled so that the sample median value equaled the median value of a Chi Square distribution with one degree of freedom. The goodness-of-fit test was a Chi Square test.

Our final technique for estimating θ relies on the fact that the value of δ is the same for all options written on the same stock. For each stock we arbitrarily select four days during 1976. Then we choose an integer value of θ and use a numerical search routine to calculate an implied value of δ for each observed option price. The time to expiration and the riskless rate of return are measured on a daily basis and the current stock price is reduced by the present value, measured by the riskless rate, of any dividends paid between that day and the expiration date. Starting with our point estimates of θ, we select two or three values of θ until we have approximately the same value of δ for each option price.[4] The values of θ resulting from this numerical search technique are:

Stock	θ
ATT	−2
AVON	0
ETKD	0
EXXN	0
IBM	−4
XERX	1

We have restricted our search to integer values only for simplicity. More detailed search procedures would surely result in better or more exact estimates of θ.

The inferences to be drawn from our analysis so far are: (1) that θ is different for different securities, and (2) that θ is in general less than two. Therefore, the evidence we have so far suggests that the Cox valuation model has the potential to yield more accurate option prices than the Black-Scholes model.

VI. Estimating δ

In order to obtain model prices we require an estimate of δ. When θ equals two, δ^2 is the, constant, variance rate of return, σ^2, which can be estimated from daily returns to the underlying stock during 1976. However, it appears that θ is less than two and in this case the variance rate changes through time. We overcome this problem by noting that the evidence in Table 1 indicates that even if θ is less than two, the Black-Scholes model, with the correct variance rate of return on day t, σ_t^2, will give approximately the correct price for options at the money. Thus, on a given day, we can take an at the money option and deduce the proper variance rate, σ_t^2, from the Black-Scholes model. Given the variance rate, σ_t^2, and our estimate of θ, we can solve for the value of δ from equation (2). That is, $\delta_t = \sigma_t S_t^{(2-\theta)/2}$. In this way we obtain Black-Scholes and Cox Model prices based on the same instantaneous variance.

We have chosen to compare the Black-Scholes and Cox Models in this manner for two reasons. First, it is now widely recognized that σ changes through time and the Black-Scholes model is typically used with a new value of σ for each time

[4] In principle we could compute implied values of δ for each trading day in 1976 and search for a value of θ that gives implied values of δ that are approximately equal. However, at this juncture we have settled for obtaining a ball park value for θ.

period. Second, for a given instantaneous variance, σ^2, the Black-Scholes and Cox models yield approximately the same price for call options near the money. It is only when the stock price differs from the exercise price that the prices predicted by the two models differ significantly. Our methodology allows us to focus on these differences, leaving issues such as the stationarity of model parameters for future research.

In [12] we assume $\theta = 2$ and compute implied values of σ_t for this data for each day in 1976. On a typical day we observe what the results in Table 1 imply, that is, the implied values of σ for options written on a particular stock decline systematically as the exercise price increases.[5]

Panel *B*, of Table 2 illustrates what we observe for options written on IBM on a typical day in 1976. On the basis of the implied values of σ for options with at least ninety days to expiration, we deduce the value of σ implied by the Black-Scholes model for an at the money option for each trading day in 1976. For example, our estimate of σ for IBM on June 14, 1976 is .012848. With this estimate of σ, θ equal to -4, and the stock price on June 14th, we can infer a value of δ and compute a Cox model price. Panels *C* and *D* of Table 2 give the Black-Scholes and Cox model prices, respectively. Notice that the Cox model prices are consistently closer to the market prices. We observe this kind of improvement on virtually every trading day in 1976 for options written on all six stocks. The difference between the model prices and the market price of a July \$260.00 option illustrates that neither model fits the data perfectly.

VII. The Results

We have computed Black-Scholes and Cox model prices corresponding to our sample of market prices for all options traded in 1976 with at least seven days to expiration. Options which have Black-Scholes model prices less than the lowest quoted market price, \$.0625, have been excluded. We have also excluded options selling for less than \$.50 when the stock price is more than \$5.00 below the exercise price because the CBOE prohibits transactions in these options that will lead to a new open position in that option.

To facilitate reporting our results we define *M*:

$$M \equiv \frac{S - Ee^{-r\tau}}{Ee^{-r\tau}} \tag{7}$$

as a measure of how far the option is in or out of the money. To measure the percent difference between model prices and market prices we use *V*:

$$V \equiv \frac{C_{\text{market}} - C_{\text{model}}}{C_{\text{model}}} \tag{8}$$

[5] We also observe another implication of the constant elasticity of variance diffusion processes. Namely, that the changes through time in the implied values of σ_t are negatively correlated with changes in the stock price.

Table 2

IBM Option Data for June 14, 1976

	Exercise Price	Expiration Dates: January	April	July	October
		Market Prices			
Panel A	$200			$62.00	
	$220			$42.00	$47.50
	$240	$36.50		$23.13	$30.50
	$260	$22.00		$ 7.13	$16.13
	$280	$12.00		$ .94	$ 7.00
		Implied Values of σ			
Panel B	$200			.028412	
	$220			.019312	.018691
	$240	.015404		.014725	.015417
	$260	.012885		.010658	.012824
	$280	.011689		.010158	.011640
		Black-Scholes Model Prices			
Panel C	$200			$61.23	
	$220			$41.38	$43.90
	$240	$33.22		$22.53	$28.13
	$260	$21.95		$ 8.42	$16.15
	$280	$13.73		$ 1.90	$ 8.28
		Cox Model Prices $\theta = -4$			
Panel D	$200			$61.27	
	$220			$41.62	$45.89
	$240	$35.17		$23.05	$29.57
	$260	$21.78		$ 8.45	$16.11
	$280	$11.49		$ 1.39	$ 6.77
	$\tau =$	220.00		33.00	124.00
	r (annual) =	5.9%		5.3%	5.6%
	S^*	$255.85		$260.25	$258.03

S^* is the stock price less the present value of dividends of $2.25 paid September 10 and December 10.

and to measure the dollar difference between market prices and model prices we calculate Y as:

$$Y \equiv C_{\text{market}} - C_{\text{model}} \tag{9}$$

The superiority of the Cox model can be seen in representative Figures 1 through 2. Parts A and B of these figures are scatter diagrams of V versus M for Black-Scholes model prices and Cox model prices, respectively. Parts C and D are similar scatter diagrams of Y versus M. There are a number of inferences to be drawn from these figures.

First, from the preponderance of points in the northeast and southwest quadrants of Parts A and C one can see that, assuming it correctly values at the money options, the Black-Scholes model underprices in the money options and over values out of the money options. When viewing the figures one should remember

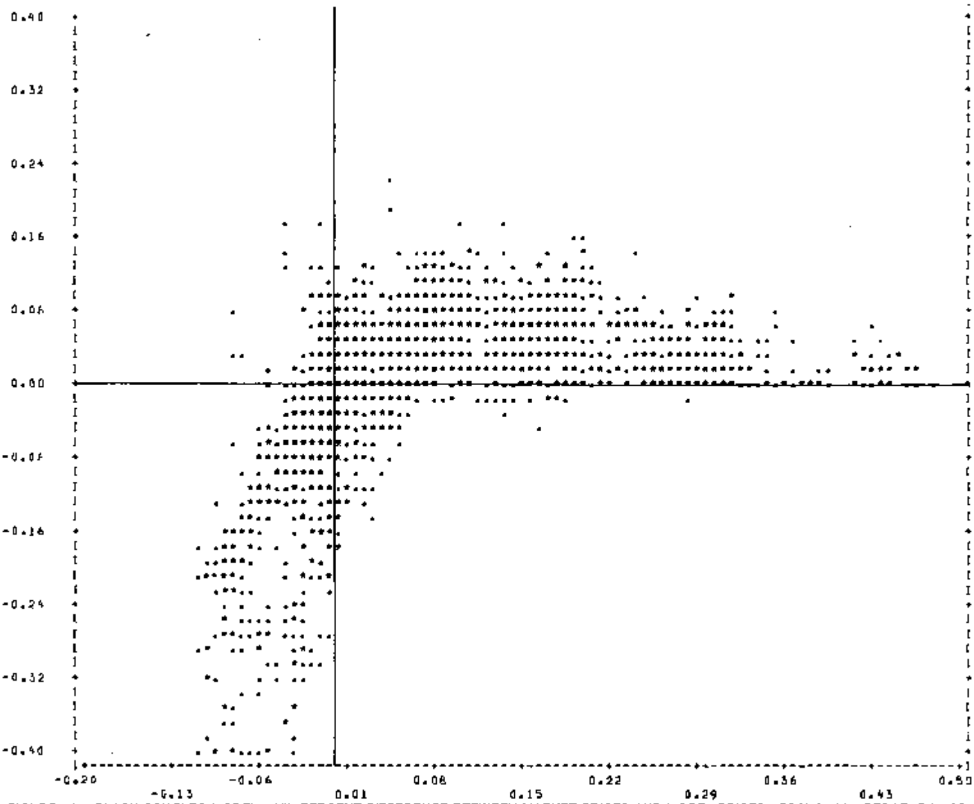

FIGURE 1A BLACK-SCHOLES MODEL, "V", PERCENT DIFFERENCE BETWEEN MARKET PRICES AND MODEL PRICES VERSUS "M", PERCENT IN OR OUT OF THE MONEY.

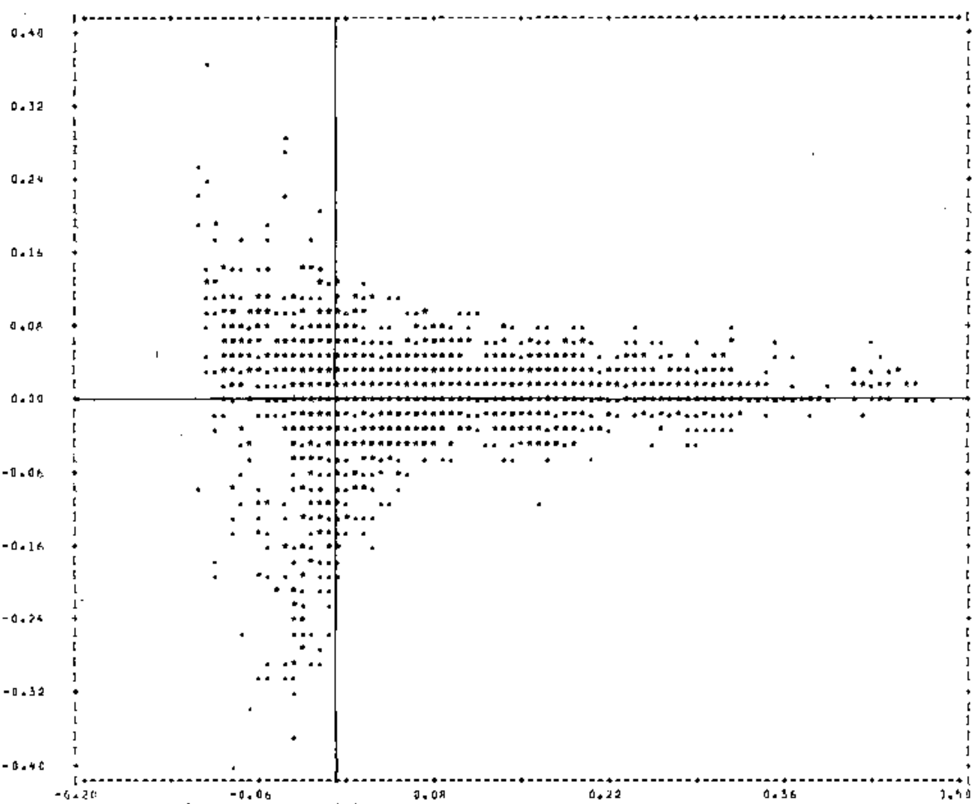

FIGURE 1B COX MODEL, $\theta = -4$. "V", PERCENT DIFFERENCE BETWEEN MARKET PRICES AND MODEL PRICES VERSUS "M", PERCENT IN OR OUT OF THE MONEY.

Figure 1. Options Written on IBM Stock 1976, 2390 Observations

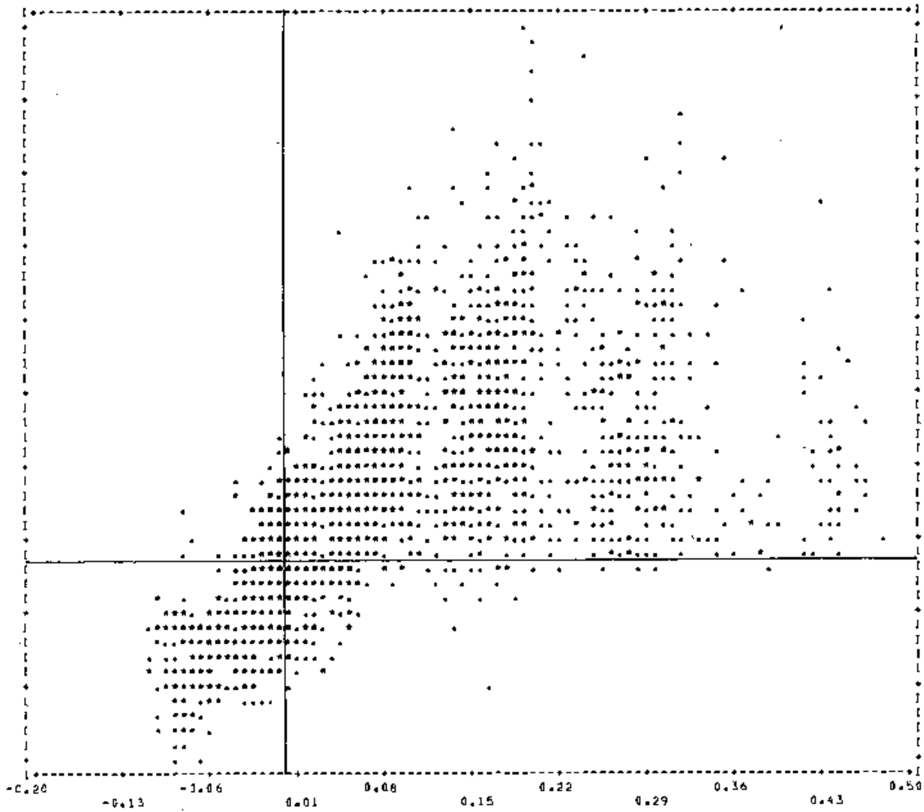

FIGURE 1C BLACK-SCHOLES MODEL, "Y", DOLLAR DIFFERENCE BETWEEN MARKET PRICES AND MODEL PRICES VERSUS "M", PERCENT IN OR OUT OF THE MONEY.

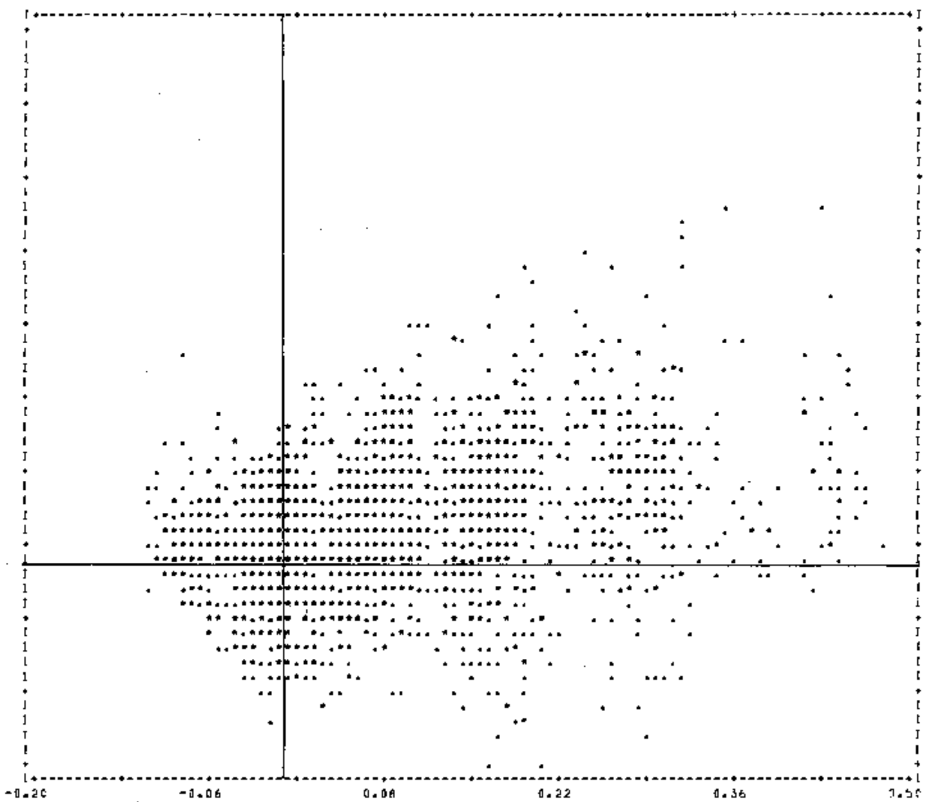

FIGURE 1D COX MODEL, $\theta = -4$. "Y", DOLLAR DIFFERENCE BETWEEN MARKET PRICES AND MODEL PRICES VERSUS "M", PERCENT IN OR OUT OF THE MONEY.

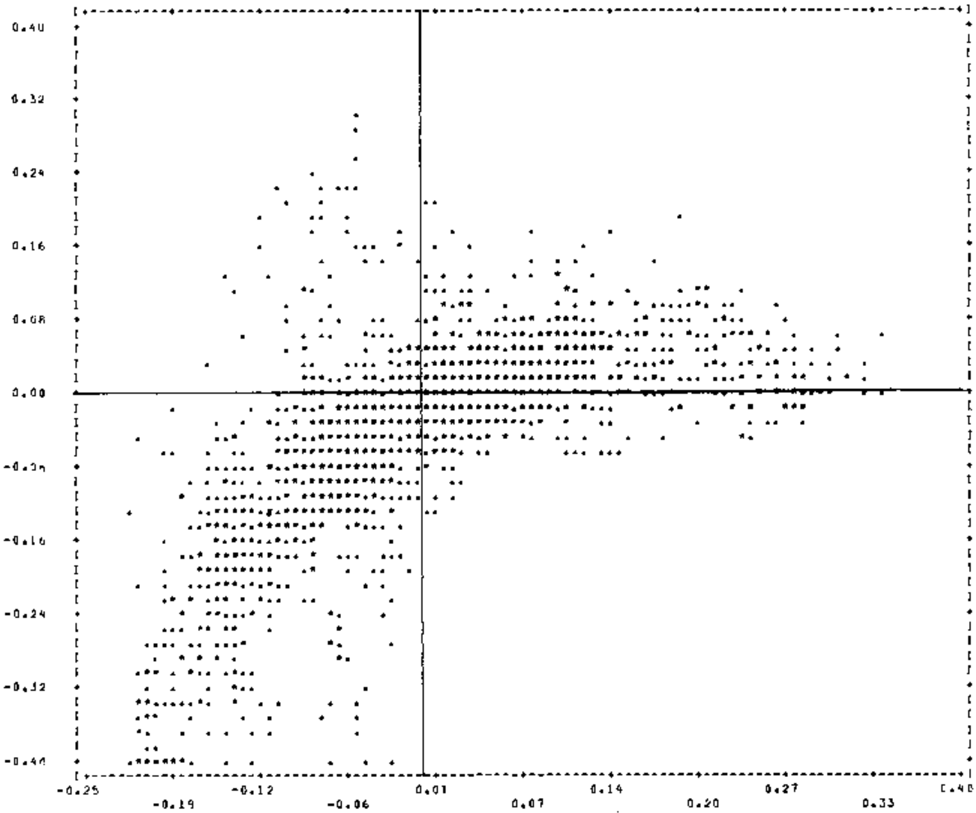

FIGURE 2A BLACK-SCHOLES MODEL, "V", PERCENT DIFFERENCE BETWEEN MARKET PRICES AND MODEL PRICES VERSUS "M", PERCENT IN OR OUT OF THE MONEY.

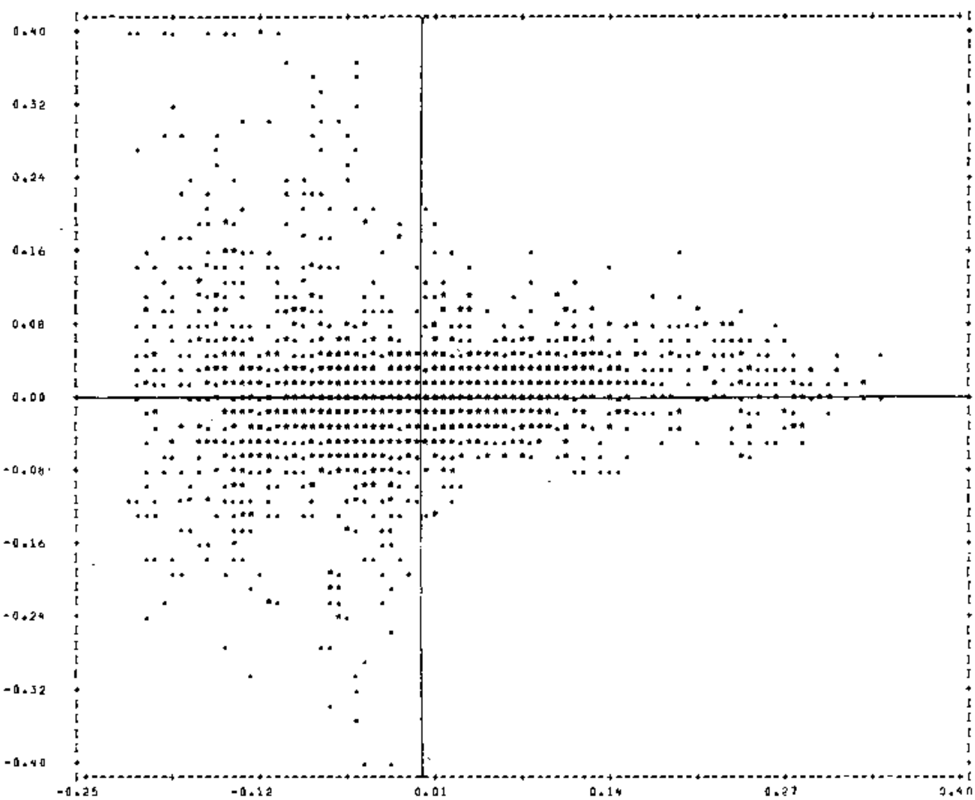

FIGURE 2B COX MODEL, θ = 0. "V", PERCENT DIFFERENCE BETWEEN MARKET PRICES AND MODEL PRICES VERSUS "M", PERCENT IN OR OUT OF THE MONEY.

Figure 2. Options Written on ETKD Stock 1976, 2097 Observations

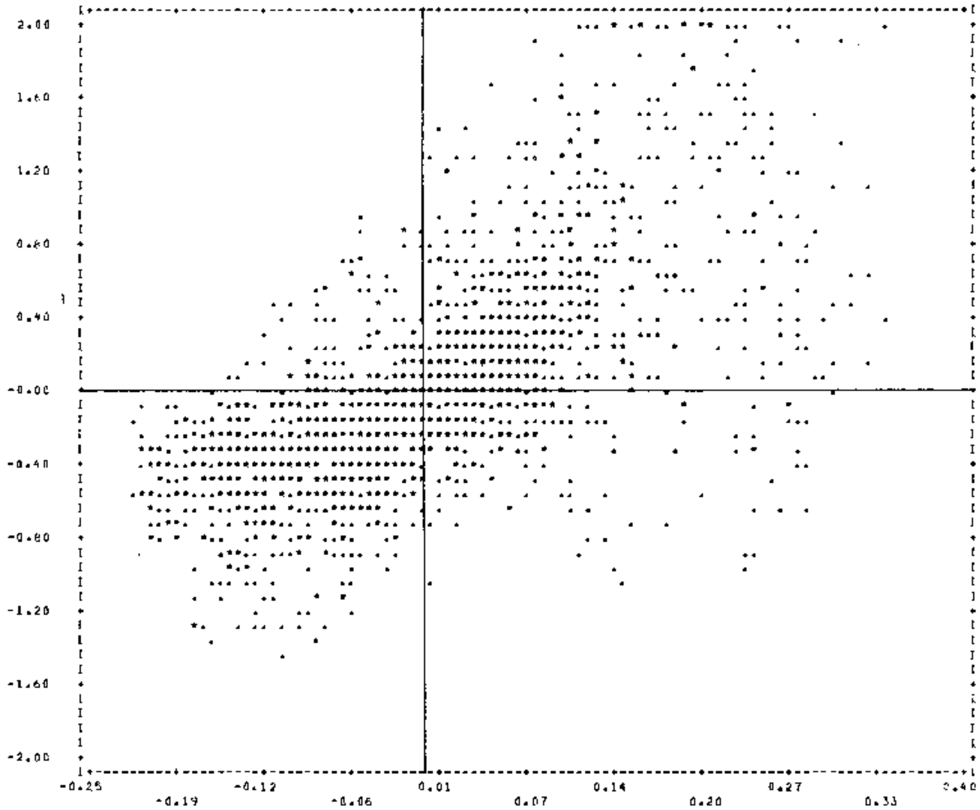

FIGURE 2C BLACK-SCHOLES MODEL, "Y", DOLLAR DIFFERENCE BETWEEN MARKET PRICES AND MODEL PRICES VERSUS "M", PERCENT IN OR OUT OF THE MONEY.

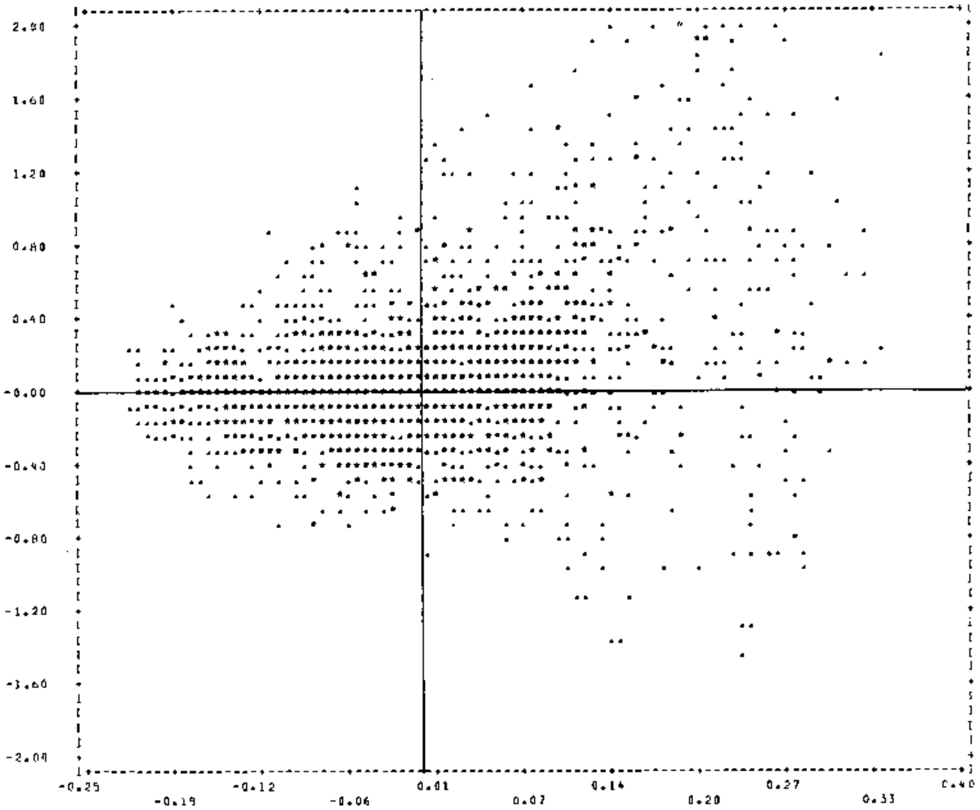

FIGURE 2D COX MODEL, $\theta = 0$. "Y", DOLLAR DIFFERENCE BETWEEN MARKET PRICES AND MODEL PRICES VERSUS "M", PERCENT IN OR OUT OF THE MONEY.

that when deviations of model prices from market prices are measured as a fraction of model prices, the deviations appear greater for out of the money options because the model prices are typically small for out of the money options, but when the deviations are measured in terms of dollars, they are larger for in the money options.

Second, a comparison of Parts *B* and *D* with Parts *A* and *C* clearly indicates that the Cox model, with some θ less than two, fits the data better than the Black-Scholes model. With more precise estimates of θ and δ the Cox model should yield even better results.

Finally, Part *A* of Figure 2, corresponding to options written on Eastman Kodak, and representative of options written on Avon, Exxon, and Xerox, indicates that the Black-Scholes model does not always overprice out of the money options. We have examined the options corresponding to points in the northwest quadrants of this Figure and find that these are generally deep out of the money options with a short time to expiration and market prices between \$.50 and \$2.00. Since the trading volume in these options is often low, we suspect that at least some of these observations are the result of non-synchronous stock and option prices.[6] Since Cox model prices of these options are even lower than Black-Scholes model prices, the Cox model underprices these options more than the Black-Scholes model.

To indicate the consistency of our results we have grouped option prices according to whether the options are in or out of the money and whether the expiration date is near, less than ninety days in the future, or far greater than ninety days in the future. The sample means $\bar{M}$, $\bar{V}$, and $\bar{Y}$, sample standard deviations $s(M)$, $s(V)$, and $S(Y)$ of M, V, and Y, and the root mean squared dollar difference between model prices and market prices, RMSFE, of option prices in each category are reported for the Black-Scholes and Cox models in Table 3. N is the number of option prices in each category.

Whether one uses the mean of V, the mean of Y or the root mean squared dollar forecast error as a criterion, the Cox model fits the data better for virtually every group of option prices for every stock. If one *a priori* assumes that the two models fit the data equally well and considers each category as an independent observation, then the probability that the Cox model RMSFE will be smaller in twenty-two out of twenty-four categories is much less than .0001.

One source of potential error in our analysis arises from the fact that American call options on the CBOE can be exercised at anytime prior to the stated expiration date. When the underlying stock pays dividends, it may be rational to exercise an option just prior to an ex-dividend day. Assuming $\theta = 2$, Roll [14] derives an explicit valuation equation for a call option with known dividends to be paid prior to the expiration date. Our adjustment for dividends assumes the probability of early exercise is essentially zero. A non-zero probability of early exercise will be reflected in a higher market price than the Black-Scholes model would predict and consequently an implied value of σ that is greater than the

[6] Another possible explanation is that the process generating stock prices may have a jump component. See Merton [13].

Table 3

Summary of Percent and Dollar Differences Between Market Prices and Model Prices

						Black Scholes		Cox		Black Scholes		Cox		Black Scholes	Cox
Money	Stock	Expiration	N	$\bar{M}$	$s(M)$	$\bar{V}$	$s(V)$	$\bar{V}$	$s(V)$	$\bar{Y}$	$s(Y)$	$\bar{Y}$	$s(Y)$	RMSFE	RMSFE
	ATT														
Out		Near	97	−.03	.02	−.15	.14	−.10	.14	−$ .12	$ .12	−$ .08	$ .12	$ .17	$ .15
Out		Far	404	−.04	.03	−.09	.09	.02	.09	− .13	.13	.03	.12	.18	.13
In		Far	875	.09	.06	.06	.08	.05	.07	.34	.34	.28	.33	.51	.43
In		Near	373	.11	.08	.08	.01	.00	.08	.19	.31	.17	.31	.36	.35
	AVON														
Out		Near	238	−.07	.05	−.07	.16	.01	.23	− .08	.19	.00	.23	.20	.23
Out		Far	515	−.06	.05	−.06	.07	.00	.06	− .17	.17	.01	.15	.24	.15
In		Far	928	.13	.10	.04	.04	.02	.03	.28	.29	.14	.24	.40	.28
In		Near	456	.16	.13	.01	.05	.00	.05	.11	.29	.04	.27	.31	.27
	ETKD														
Out		Near	256	−.06	.04	−.08	.15	.00	.17	− .10	.32	.03	.34	.34	.33
Out		Far	848	−.09	.06	−.10	.10	.01	.08	− .35	.31	.02	.23	.47	.23
In		Far	632	.09	.07	.03	.04	.01	.03	.39	.63	.20	.55	.74	.58
In		Near	361	.10	.08	.04	.05	.03	.05	.50	.60	.40	.56	.79	.69
	EXXN														
Out		Near	168	−.04	.03	−.10	.15	−.04	.19	− .12	.16	−.06	.16	.20	.18
Out		Far	544	−.07	.04	−.09	.10	.00	.12	− .18	.21	−.01	.18	.28	.18
In		Far	684	.08	.06	.02	.05	.01	.04	.16	.35	.11	.31	.38	.33
In		Near	330	.09	.07	−.01	.06	−.02	.06	.06	.31	.03	.29	.31	.29
	XERX														
Out		Near	199	−.08	.05	−.01	.16	.05	.19	.01	.27	.07	.29	.27	.30
Out		Far	557	−.10	.07	−.07	.08	.00	.09	− .17	.19	− .01	.17	.25	.17
In		Far	798	.16	.12	.02	.03	.01	.03	.23	.33	.12	.30	.40	.32
In		Near	367	.16	.12	.02	.05	.01	.05	.20	.34	.16	.33	.39	.36
	IBM														
Out		Near	196	−.03	.03	−.18	.15	−.07	.14	− .86	.74	− .36	.74	1.13	.83
Out		Far	376	−.03	.03	−.07	.08	.05	.06	− .80	.83	.56	.61	1.15	.83
In		Far	1096	.11	.08	.06	.04	.03	.03	2.09	1.55	.86	1.12	2.61	1.41
In		Near	722	.17	.12	.03	.04	.01	.04	1.12	1.17	.66	1.06	1.62	1.26

true value of σ. Since early exercise is more likely the deeper the option is in the money, it is possible that the observed pattern in implied values of σ is the result of a potential early exercise effect.

To investigate this possibility we have computed Roll model prices of call options with exactly one dividend remaining for all our data using our original implied values of σ. If the probability of early exercise is non-zero, then the Roll model price will exceed the market price because we are using an implied value of σ that is too big. We find no evidence of an early exercise effect in the prices of options written on International Business Machines, Xerox, and Eastman Kodak. We do find instances where the prices of options written on Avon and Exxon exhibit an early exercise effect, but the price differences are typically only a few cents, and we conclude that our analysis of the Avon and Exxon data would be virtually unchanged had we used Rolls model to obtain implied values of σ. However, prices of American Telephone and Telegraph options do exhibit a significant early exercise effect which suggests that our estimate of θ is too low.

VIII. Summary and Conclusions

In [12] we document the systematic deviations between Black-Scholes model prices and market prices of call options exemplified by Parts *A* and *C* of Figure 1 and 2. We were surprised by our findings as they are opposite to statements by Black [1], which are widely accepted.

We now feel that the evidence we have presented warrants the conclusion that the stochastic process generating stock prices can best be considered, at least for the pricing of call options, as a constant elasticity of variance process with parameter θ generally less than two.[7] Moreover, we have demonstrated that under these circumstances, the Cox call option valuation model prices are closer to market prices than Black-Scholes model prices. In addition, viewing the stochastic process as a constant elasticity of variance process with $\theta < 2$ explains why Black observes a negative relationship between the sample variance of daily returns and the stock price, and why practitioners who use the Black-Scholes model to value call options must constantly adjust the variance rate they input to the model.

Our findings have obvious implications for practitioners who currently use the Black-Scholes model. With estimates of θ and δ, the calculations involved in computing Cox model prices are not difficult. The infinite sums typically converge quickly, especially for values of θ less than 1. We are currently investigating better techniques for estimating θ and δ as well as considering the time series stationarity behavior of these parameters.

Our analysis also has implications for empirical research in option pricing. A number of researchers have used implied values of σ from the Black-Scholes model to analyze the stochastic process generating stock prices and/or to form

[7] Some other preliminary analysis we have done suggests that the parameter θ does change through time. It is possible that it can be greater than two.

option trading strategies.[6] Some of the findings of these and other empirical studies of option pricing may warrant new interpretation in view of our results.

We did not allow for the possibility of early exercise, and with the exception of options written on ATT, our results would not be different had we used Roll's exact model to incorporate the possibility of early exercise. However, we did discover that a careful analysis of call option prices on dividend paying stocks should incorporate Roll's exact model rather than the two commonly used approximations based upon the original Black-Scholes model. In our expanded analysis of ATT option prices, we find that the approximation we used, which assumes early exercise will not occur, and the approximation which assumes that early exercise will occur for certain, can both yield prices significantly below the exact Roll model price.

Finally, constant elasticity of variance diffusion processes may provide a framework to analyze non-stationarity in returns to common stocks and other financial instruments and thereby provide improved methodological techniques for empirical tests of financial market efficiency.

REFERENCES

1. Fischer Black, "Fact and Fantasy in the Use of Options," *Financial Analysts Journal*, (July-August 1975), pp. 36-72.
2. ———, "Forecasting Variance of Stock Prices for Option Trading and Other Purposes" Proceedings of the Center for Research in Security Prices Seminar, University of Chicago, November, 1975.
3. ———, "Studies of Stock Price Volatility Changes," Proceedings of the meetings of the American Statistical Association, Business and Economics Statistics, Section, Chicago, 1976.
4. F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, Volume 81, (1973), pp. 637-654.
5. R. C. Blattberg and N. J. Gonedes, "A Comparison of the Stable and Student Distributions as Stochastic Models for Stock Prices," *Journal of Business*, Volume 47, (1974), pp. 244-280.
6. D. Chiras and S. Manaster, "The Informational Content of Option Prices and a Test of Market Efficiency," *Journal of Financial Economics*, Volume 6, (June/September 1978).
7. John Cox, "Notes on Option Pricing I: Constant Elasticity of Diffusions," Unpublished draft, Stanford University, September, 1975.
8. John Cox and Stephen Ross, "The Valuation of Options for Alternative Stochastic Processes," *Journal of Financial Economics*, Jan/March 1976.
9. Eugene Fama, "Multiperiod Consumption-Investment Decisions," *American Economic Review*, LX, March 1970.
10. Mark Garman, "A General Theory of Asset Valuation Under Diffusion State Processes," Working Paper No. 50, Research Program in Finance, University of California, Berkeley, 1976.
11. H. A. Latané and R. J. Rendleman, "Standard Deviations of Stock Price Ratios Implied in Option Prices," *Journal of Finance*, Vol. 31, No. 2, (1976), pp. 369-381.
12. J. D. MacBeth and L. J. Merville, "An Empirical Examination of the Black-Scholes Call Option Pricing Model," *The Journal of Finance*, (December 1979).
13. R. C. Merton, "Option Pricing When Underlying Stock Returns are Discontinuous," *Journal of Financial Economics*, Vol 1, No. 1 & 2 (January/March 1976), pp. 125-144.
14. R. Roll, "An Analytic Valuation Formula for Unprotected American Call Options on Stocks with Known Dividends," *Journal of Financial Economics*, Vol 5, No. 2, (November 1977).

[6] For example, Latané and Rendleman [11], Schmalensee and Trippi [17] and Chiras and Manaster [6].

15. Barr Rosenberg, "The Behavior of Random Variables with Nonstationary Variance and the Distribution of Security Prices," Manuscript, University of California, Berkeley, 1973.
16. Mark Rubinstein, "A Discrete-Time Synthesis of Financial Theory: Part I Optimal Decision and Sharing Rules; Part II Valuation and Efficiency, Part III Extensions and Prospective," Working Papers #20, #21, and #26, Research Program in Finance, University of California, Berkeley, 1974.
17. R. Schmalensee and R. Trippi, "Common Stock Volatility Expectations Implied by Option Premia," *The Journal of Finance*, Vol. 33, No. 1, (March 1978), pp. 129–147.
18. Edward O. Thorpe, "Common Stock Volatilities in Option Formulas," Proceedings of the Center for Research in Security Prices Seminar, University of Chicago, May 1976.

DISCUSSION

STEVEN MANASTER*: MacBeth and Merville (hereafter MM) test the Cox call option model for constant elasticity of variance diffusion process against the constant variance option model of Black and Scholes (hereafter BS). Given that the Cox model includes the BS model as a special case, it is clear that the Cox model must explain observed option prices at least as well as the BS model. The empirical observation that stock return variances move inversely with stock price almost guarantees that the Cox model, for reasonable parameter values, will perform better than the BS model. Therefore, it is not surprising that, on the basis of their tests, MM conclude that the Cox model fits the market data better than the BS model. Rather than discussing MM's conclusions, in these comments I should like to address two related questions; first, what is the most useful method to estimate the Cox model parameters δ and θ; and second, is the extra effort required to estimate these parameters justified by the improved results relative to the BS model.

The principle benefit of the Cox model relative to the BS model is noted by MM. They correctly explain that if the variance rate (σ^2) changes with stock price changes, the BS model will require constant ad hoc adjustments in the value of σ^2 employed in the BS formula. The Cox model, on the other hand, automatically adjusts for changes in the variance rate. The principle cost associated with the Cox model is that two parameters, δ and θ, must be estimated rather than σ alone.

Under either the Cox or the BS assumptions the parameters, δ, θ and σ define characteristics of the underlying stock. In principle their values can be estimated from stock price and return data alone; no reliance on option prices nor on a specific valuation formula is required. In practice however, researchers have assumed a particular option valuation formula and used observed option prices to calculate implied parameter values. The implied parameters are constructed so that prices calculated from the formula closely fit the observed option prices. In using implied parameter values, care must be taken not to employ the same model and data twice, once to estimate the parameters and again to make inferences regarding the validity of the model.

MM estimate the parameters δ and θ using a three step procedure. First, they

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