THE CAUSAL SOLUTIONS AND FOURIER TRANSFORM OF SOME OPERATOR RELATED TO THE BESSEL DIAMOND OPERATOR

SUDPRATHAI BUPASIRI

THIS RESEARCH WAS SUPPORTED FOR PERSONNEL IN
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สุดประไทย บุพศิริ. 2558. **ผลเฉลยคอแซลและการแปลงฟูริเยร์ของตัวดำเนินการบางตัวที่สัมพันธ์กับตัวดำเนินการเบสเซิลไดมอนด์**

อาจารย์ที่ปรึกษางานวิจัย: ผศ.ดร. คำสิงห์ นนเลาพล

บทคัดย่อ

งานวิจัยครั้งนี้ เป็นการศึกษาสมการ $\oplus_B^k u(x) = \delta$, $\oplus_B^k(P \pm i0) = \delta$ และ $\oplus_{B,8}^k Y(t) = \delta$ โดยที่ $\oplus_B^k$ และ $\oplus_{B,8}^k$ เป็นตัวดำเนินการที่สัมพันธ์กับตัวดำเนินการเบสเซิลไดมอนด์กระทำซ้ำกัน k ครั้ง กำหนดดังนี้

$$\oplus_B^k = \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^4 - \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^4\right]^k$$

และ

$$\oplus_{B,8}^k = \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^8 - \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^8\right]^k$$

โดยที่ $p+q=n, B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}, v_i = 2\alpha_i + 1, \alpha_i > -\frac{1}{2}$ [4], $x_i > 0, i = 1,2,\ldots,n, P = P(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, k$ เป็นจำนวนที่ไม่เป็นลบ และ n เป็นมิติของ $\mathbb{R}_n^+$ จุดประสงค์ของงานนี้จะศึกษาผลเฉลยมูลฐานของตัวดำเนินการ $\oplus_B^k$,$\oplus_{B,8}^k$ และผลเฉลยคอแซลและแอนติคอแซลของตัวดำเนินการ $\oplus_B^k$

Sudprathai Bupasiri. 2015. **The Causal Solutions and Fourier Transform of Some Operator Related to the Bessel Diamond Operator**.

Faculty of Education, Sakon Nakhon Rajabhat University.

Research Advisor: Asst. Prof. Dr. Kamsing Nonlaopon

ABSTRACT

In this research, we consider the solution of the equations $\oplus_B^k u(x) = \delta$, $\oplus_B^k (P \pm i0) = \delta$, and $\oplus_{B,8}^k Y(t) = \delta$ where $\oplus_B^k$ and $\oplus_{B,8}^k$ are the operator related to the Bessel diamond operator iterated k-time and are defind by

$$\oplus_B^k = \left[\left(\sum_{i=1}^{p} B_{x_i} \right)^4 - \left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^4 \right]^k$$

and

$$\oplus_{B,8}^k = \left[\left(\sum_{i=1}^{p} B_{x_i} \right)^8 - \left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^8 \right]^k$$

where $p + q = n, B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}, v_i = 2\alpha_i + 1, \alpha_i > -\frac{1}{2}$ [4], $x_i > 0, i = 1, 2, \ldots, n, P = P(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, k$ is a nonnegative integer and n is the dimension of $\mathbb{R}_n^+$. In this work we study the elementary solution of the operator $\oplus_B^k$,$\oplus_{B,8}^k$ and causal and anticausal solution of the operator $\oplus_B^k$.

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CHAPTER I

INTRODUCTION

In this chapter, we studied basic concept of the causal and anticausal solution of the ultra-hyperbolic Bessel and Bessel diamond operator which will be used in later chapters.

H. Yildirim, M.Z. Sarikaya and S. Ozturk [7] have first introduced the elementary solution of the n-dimensional Bessel diamond operator and the Fourier-Bessel transform of their convolution and showed that the solution of the convolution form $(-1)^k S_{2k}(x) * R_{2k}(x)$ is a unique elementary solution of the $\diamondsuit_B^k u(x) = \delta$.

Consider the Bessel ultra-hyperbolic operator iterated k-times,

$$\Box_B^k = \left[\sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j}\right]^k$$

Yildirim, Sarikaya and Ozturk [7] has shown that the generalized function $R_{2k}(x)$ define by (4.1.1) is the unique elementary solution of the operator $\Box_B^k$, that is $\Box_B^k R_{2k}(x) = \delta$ where $x \in \mathbb{R}_n^+ = \{x : x = (x_1, \dots x_n), x_1 > 0, x_2 > 0, \dots, x_n > 0\}$. Yildirim, Sarikaya and Ozturk [7] studied the Bessel diamond operator, iterated k-times,

$$\begin{aligned} \diamondsuit_B^k &= \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2\right] \\ &= \left[\sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j}\right]^k \left[\sum_{i=1}^{p} B_{x_i} + \sum_{j=p+1}^{p+q} B_{x_j}\right]^k, \end{aligned} \qquad (1.1.1)$$

Yildirim, Sarikaya and Ozturk [7] showed that the function $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is the unique elementary solution for the operator $\diamondsuit_B^k$, where $*$ indicates convolution, and $S_{2k}(x), R_{2k}(x)$ are defined by (4.1.1) and (4.1.4) respectively, that is,

$$\diamondsuit_B^k \left((-1)^k S_{2k}(x) * R_{2k}(x)\right) = \delta(x). \qquad (1.1.2)$$

Furthermore, the operator $\oplus^k$ was first studied by Kananthai, Suantai and Longani [3] . The $\oplus^k$ operator can be expressed in the form

$$\oplus^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k$$
$$\cdot\left[\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} + i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k\left[\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k. \quad (1.1.3)$$

Satsanit [22] has studied the Green function and Fourier transform for o-plus operators, iterated k-times, defined by

$$\begin{aligned}\oplus^k &= \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k\\ &= \Diamond^k\left[\left(\frac{\triangle+\square}{2}\right)^2 + \left(\frac{\triangle-\square}{2}\right)^2\right]^k\\ &= \Diamond^k\left(\frac{\triangle^2+\square^2}{2}\right)^k\\ &= \Diamond^k \circledcirc^k. \end{aligned} \quad (1.1.4)$$

where

$$\circledcirc^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k.$$

The purpose of this work is to study the operator

$$\begin{aligned}\oplus_B^k &= \left[\left(\sum_{i=1}^{p}B_{x_i}\right)^4 - \left(\sum_{j=p+1}^{p+q}B_{x_j}\right)^4\right]^k\\ &= \left[\left(\sum_{i=1}^{p}B_{x_i}\right)^2 - \left(\sum_{j=p+1}^{p+q}B_{x_j}\right)^2\right]^k\left[\left(\sum_{i=1}^{p}B_{x_i}\right)^2 + \left(\sum_{j=p+1}^{p+q}B_{x_j}\right)^2\right]^k, \end{aligned} \quad (1.1.5)$$

$$\begin{aligned}\oplus_{B,8}^k &= \left[\left(\sum_{i=1}^{p}B_{x_i}\right)^8 - \left(\sum_{j=p+1}^{p+q}B_{x_j}\right)^8\right]^k\\ &= \left[\left(\sum_{i=1}^{p}B_{x_i}\right)^4 - \left(\sum_{j=p+1}^{p+q}B_{x_j}\right)^4\right]^k\left[\left(\sum_{i=1}^{p}B_{x_i}\right)^4 + \left(\sum_{j=p+1}^{p+q}B_{x_j}\right)^4\right]^k. \end{aligned} \quad (1.1.6)$$

Let us denote the operator

$$\circledcirc_B^k = \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 + \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2\right]^k.$$

By (1.1.11) and (1.1.12) we obtain

$$\begin{aligned}\circledcirc_B^k &= \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 + \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2\right]^k \\ &= \left[\left(\frac{\triangle_B + \Box_B}{2}\right)^2 + \left(\frac{\triangle_B - \Box_B}{2}\right)^2\right]^k \\ &= \left(\frac{\triangle_B^2 + \Box_B^2}{2}\right)^k, \qquad (1.1.7)\end{aligned}$$

$$\begin{aligned}\circledcirc_{B,4}^k &= \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^4 + \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^4\right]^k \\ &= \left[\left(\frac{\triangle_B^2 + \Box_B^2}{2}\right)^2 + \left(\frac{\triangle_B^2 - \Box_B^2}{2}\right)^2\right]^k \\ &= \left(\frac{\triangle_B^4 + \Box_B^4}{2}\right)^k. \qquad (1.1.8)\end{aligned}$$

Thus, (1.1.5) and (1.1.6) can be written as

$$\oplus_B^k = \diamondsuit_B^k \circledcirc_B^k = \circledcirc_B^k \diamondsuit_B^k \qquad (1.1.9)$$

and

$$\oplus_{B,8}^k = \oplus_B^k \circledcirc_{B,4}^k = \circledcirc_{B,4}^k \oplus_B^k. \qquad (1.1.10)$$

For $k = 1$ the operator $\diamondsuit_B$ can be expressed in the form $\diamondsuit_B = \triangle_B \Box_B = \Box_B \triangle_B$ where $\Box_B$ is the Bessel ultra-hyperbolic operator,

$$\Box_B = B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - B_{x_{p+2}} - \cdots - B_{x_{p+q}}, \qquad (1.1.11)$$

where $p + q = n$ and $\triangle_B$ is the Laplace Bessel operator,

$$\triangle_B = B_{x_1} + B_{x_2} + \cdots + B_{x_p} + B_{x_{p+1}} + B_{x_{p+2}} + \cdots + B_{x_{p+q}}. \qquad (1.1.12)$$

From (1.1.5) with $q = 0$ and $k = 1$, we obtain

$$\oplus_B = \triangle_B^4 \qquad (1.1.13)$$

where

$$\triangle_B = B_{x_1} + B_{x_2} + \cdots + B_{x_p}. \tag{1.1.14}$$

We can find the elementary solution $u(x)$ of the operator $\oplus_B^k$; that is,

$$\oplus_B^k u(x) = \delta, \tag{1.1.15}$$

where δ is the Dirac-delta distribution. Moreover, we found that $u(x)$ relates to the elementary solution of the Laplace Bessel operator defined by (1.1.12) depending on the conditions of q and k of (1.1.5) with $q = 0$ and $k = 1$. In finding the elementary solution of (1.1.15), we use the method of convolutions of the generalized function.

Let $x = (x_1, \ldots, x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$. Consider a nondegenerate quadratic form in n variables of the form

$$P = P(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{1.1.16}$$

where $p + q = n$. The distributions $(P \pm i0)^\lambda$ are defined by

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \{P \pm i\varepsilon |x|^2\}^\lambda$$

where $\varepsilon > 0, |x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2, \lambda \in \mathbb{C}$.

Moreover the distribution $(P \pm i0)^\lambda$ are analytic in λ every where except at $\lambda = -\frac{n}{2} - k, k = 0, 1, \ldots$ where they have simple poles.

Similarly, the distribution $(m^2 + P \pm i0)^\lambda$ is denote by

$$(m^2 + P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \{m^2 + P \pm i\varepsilon |x|^2\}^\lambda$$

where m is a real positive number. ([8], p.289)

Following Trione ([18], p.32) by causal (anticausal) distributions we mean distributions of the form $T(P \pm i0, \lambda), P = P(x), T(P \pm i0, \lambda) = (P \pm i0)^\lambda f(P \pm i0, \lambda), f(z, \lambda)$ an entire function in the variables z, λ .

Let

$$G_\alpha(P \pm i0, m, n) = H_\alpha(m, n)(P \pm i0)^{\frac{1}{2}(\frac{\alpha - n}{2})} K_{(\frac{n-\alpha}{2})}(\sqrt{m^2(P \pm i0)}) \tag{1.1.17}$$

where m is a real positive real number $\alpha \in \mathbb{C}$, K_ν designates the modified Bessel function of the third kind

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sec \pi\nu}, I_\nu(z) = \sum_{m=0}^{\infty} \frac{(\frac{z}{2})^{2m+r}}{m!\Gamma(m+\nu+1)}$$

and

$$H_\alpha(m,n) = \frac{2^{\frac{1-(\alpha+n)}{2}}(m^2)^{(\frac{1}{2})(\frac{\alpha-n}{2})}e^{\frac{\pi}{2}qi}}{\pi^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}$$

We introduce an auxiliary weight function

$$\lambda_\alpha(P \pm i0, m, n) = e^{iq\frac{\pi}{2}}2^{\frac{1-(\alpha+n)}{2}}(m^2)^{(\frac{1}{2})(\frac{\alpha-n}{2})}(P \pm i0)^{\frac{(n+\alpha)}{4}}K_{\frac{(n+\alpha)}{2}}\sqrt{m^2(P \pm i0)}$$

that is a causal (anticausal) analoque to the auxiliary weight function introduce by Rubin ([5], p. 1247).

Gelfand and Shilov [8] have first introduced the elementary solution of the n-Dimensional Classical Diamond Operator, and have defined the distribution $(P \pm i0)^\lambda$ as

$$(P \pm i0)^\lambda = \lim_{\varepsilon\to 0}\{P \pm i\varepsilon|x|^2\}^\lambda$$

where $\varepsilon > 0, |x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2, \lambda \in \mathbb{C}$. The distributions $(P \pm i0)^\lambda$ are an important contribution of Gelfand and Shilov.

Moreover the distribution $(P \pm i0)^\lambda$ are analytic in λ every where except at $\lambda = -\frac{n}{2} - k, k = 0, 1, \ldots$ where they have simple poles.

CHAPTER II

BASIC CONCEPTS AND PRELIMINARIES

In this chapter, we studied some properties of the test function, the distribution, the gamma function , causal and anticausal solution of the operator $\oplus_B^k$ and $\oplus_{B,8}^k$ which will be used in later chapters.

2.1 Test functions

Let $\mathbb{R}^n$ be a real n-dimensional space in which we have a Cartesian system of coordinates such that a point P is denoted by $x = (x_1, x_2, \ldots, x_n)$ and the distance r, of P from the origin, is $r = |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. Let k be an n-tuple of nonnegative integer, $k = (k_1, k_2, \ldots, k_n)$, the so-called *multiindex* of order n; then we define

$$|k| = k_1 + k_2 + \cdots + k_n, \quad x^k = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

and

$$D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}} = \frac{\partial^{k_1+k_2+\cdots+k_n}}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}} = D_1^{k_1} D_2^{k_2} \cdots D_n^{k_n},$$

where $D_j = \partial/\partial x_j, j = 1, 2, \ldots, n$. For the one-dimensional case, D^k reduces to d/dx. Furthermore, if any component of k is zero, the differentiation with respect to the corresponding variable is omitted.

Example 2.1.1. In $\mathbb{R}^3$, with $k = (3, 0, 4)$, we have

$$D^k = \partial^7/\partial x_1^3 \partial x_3^4 = D_1^3 D_3^4.$$

Definition 2.1.2. A function $f(x)$ is *locally integrable* in $\mathbb{R}^n$ if $\int_R |f(x)|dx$ exists for every bounded region R in $\mathbb{R}^n$. A function $f(x)$ is locally integrable on a hypersurface in $\mathbb{R}^n$ if $\int_S |f(x)|dS$ exists for every bounded region S in $\mathbb{R}^{n-1}$.

Definition 2.1.3. The *support* of a function $f(x)$ is the closure of the set of all points x such that $f(x) \neq 0$. We shall denote the support of f by supp f.

Example 2.1.4. For $f(x) = \sin x, x \in \mathbb{R}$, the support of $f(x)$ consists of the whole real line, even though $\sin x$ vanishes at $x = n\pi$.

Definition 2.1.5. ([11]). If supp f is a bounded set, then f is said to have a *compact support.*

We have observed that an operational quantity such as $\delta(x)$ becomes meaningful if it is first multiplied by a sufficiently smooth auxiliary function and then integrated over the entire space. This point of view is also taken as the basis for the definition of an arbitrary generalized function. Accordingly, consider the space D consisting of real-valued functions $\phi(x) = \phi(x_1, x_2, \ldots, x_n)$, such that the following hold:

(1) $\phi(x)$ is an infinitely differentiable function defined at every point of $\mathbb{R}^n$. This means that $D^k\phi$ exists for all multiindices k. Such a function is also called a C^∞ function.

(2) There exists a number A such that $\phi(x)$ vanishes for $r > A$. This means that $\phi(x)$ has a compact support. Then $\phi(x)$ is called a test function.

Example 2.1.6. The support of the function

$$f(x) = \begin{cases} 0, & \text{for } -\infty < x \leq -1 \\ x+1, & \text{for } -1 < x < 0 \\ 1-x, & \text{for } 0 \leq x < 1 \\ 0, & \text{for } 1 \leq x < \infty \end{cases}$$

is $[-1, 1]$, which is compact.

Example 2.1.7. The prototype of a test function belonging to D is

$$\phi(x, a) = \begin{cases} \exp\left(-\frac{a^2}{a^2 - r^2}\right), & \text{for } r < a \\ 0, & \text{for } r > a. \end{cases} \tag{2.1.1}$$

Its support is clearly $r \leq a$.

The following properties of the test functions are evident.

(1) If ϕ_1 and ϕ_2 are in D, then so is $c_1\phi_1 + c_2\phi_2$, where c_1 and c_2 are real numbers. Thus D is a linear space.

(2) If $\phi \in D$, then so is $D^k\phi$.

(3) For a C^∞ function $f(x)$ and $\phi \in D, f\phi \in D$.

(4) If $\phi(x_1, x_2, \ldots, x_m)$ is an m-dimensional test function and $\psi(x_{m+1}, x_{m+2}, \ldots, x_n)$ is an $(n-m)$ -dimensional test function, then $\phi\psi$ is an n-dimensional test function in the variables $x_1, x_2, \ldots, x_n$.

Definition 2.1.8. The *Schwartz space* or *space of rapidly decreasing functions* S on $\mathbb{R}^n$ is the function space

$$S(\mathbb{R}^n) = \{ \ f \in C^\infty(\mathbb{R}^n) \mid \|f\|_{\alpha,\beta} < \infty \forall \alpha, \beta \ \},$$

where α, β are multi-indices, $C^\infty(\mathbb{R}^n)$ is the set of smooth functions from $\mathbb{R}^n$ to $\mathbb{C}$, and

$$\|f\|_{\alpha,\beta} = \|x^\alpha D^\beta f\|_\infty.$$

Here, $\|\cdot\|_\infty$ is the supremum norm, and we use multi-index notation.

Example 2.1.9. If i is a multi-index, and is a positive real number, then

$$x^i e^{-ax^2} \in S(\mathbb{R}).$$

Any smooth function f with compact support is in S. This is clear since any derivative of f is continuous, so $(x^\alpha D^\beta)f$ has a maximum in $\mathbb{R}^n$.

Definition 2.1.10. A sequence $\{\phi_m\}, m = 1, 2, \ldots,$ where $\phi_m \in D$, converges to ϕ_0 if the following two conditions are satisfied:

(1) All ϕ_m as well as ϕ_0 vanish outside a common region.

(2) $D^k\phi_m \to D^k\phi_0$ uniformly over $\mathbb{R}^n$ as $m \to \infty$ for all multiindices k.

It is not difficult to show that $\phi_0 \in D$ and hence that D is closed (or is complete) with respect to this definition of convergence. For the special case $\phi_0 = 0$, the sequence $\{\phi_m\}$ is called a null sequence.

Example 2.1.11. The sequence

$$\{(1/m)\phi(x,a)\}, \tag{2.1.2}$$

where $\phi(x,a)$ is defined by (2.1.1), is a null sequence. However, the sequence $(1/m)\phi(x/m,a)$ is not a convergent sequence, because the support of the function $\phi(x/m,a)$ is the sphere with radius ma, which is unique for each m.

In addition to the space D of test functions, we shall use certain subspaces of D. For a region R in $\mathbb{R}^n$, the space D_R contains those test functions whose support lies in R, that is,

$$D_R \equiv \{\phi : \phi \in D, \qquad \text{supp}\ \phi \subset \mathbb{R}\}. \tag{2.1.3}$$

It is clearly a linear subspace of D.

Example 2.1.12. D_x and D_y are two one-dimensional subspaces of test functions $\phi(x)$ and $\phi(y)$ and are contained in D_{xy}, which is the space of test functions $\phi(x,y)$ in $\mathbb{R}^2$. The convergence in D_R is defined in the same manner as that in the space D.

2.2 Distributions

Definition 2.2.1. A linear functional t on the space D of test functions is an operation (or a rule) by which we assign to every test function $\phi(x)$ a real number denoted $\langle t, \phi\rangle$, such that

$$\langle t, c_1\phi_1 + c_2\phi_2\rangle = c_1 \langle t, \phi_1\rangle + c_2 \langle t, \phi_2\rangle \tag{2.2.1}$$

for arbitrary test functions ϕ_1 and ϕ_2 and real numbers c_1 and c_2.

Definition 2.2.2. A linear functional on D is *continuous* if and only if the sequence of numbers $\langle t, \phi_m\rangle$ converges to $\langle t, \phi\rangle$ when the sequence of test functions $\{\phi_m\}$ converges to the test function ϕ. Thus

$$\lim_{m\to\infty} \langle t, \phi_m\rangle = \left\langle t, \lim_{m\to\infty} \phi_m \right\rangle.$$

In physical problems, one often encounters idealized concepts such as a force concentrated at a point ξ or an impulsive force that acts instantaneously. These forces are described by the Dirac-delta function $\delta(x-\xi)$, which has several significant properties:

$$\delta(x-\xi) = 0, x \neq \xi, \tag{2.2.2}$$

$$\int_a^b \delta(x-\xi)dx = \begin{cases} 0, & \text{for } a, b < \xi \text{ or } \xi < a, b \\ 1, & \text{for } a \leq \xi \leq b, \end{cases} \tag{2.2.3}$$

and

$$\int_{-\infty}^{\infty} \delta(x-\xi)dx = 1. \tag{2.2.4}$$

Equation (2.2.4) is a special case of the general formula

$$\int_{-\infty}^{\infty} \delta(x-\xi)f(x)dx = f(\xi), \tag{2.2.5}$$

where $f(x)$ is a sufficiently smooth function. Relation (2.2.5) is called the *sifting property* or the *reproducing property* of the delta function, and (2.2.4) is obtained from it by putting $f(x) = 1$.

We now have all the tools for defining the concept of distributions.

Definition 2.2.3. A continuous linear functional on the space D of test functions is called a *distribution*.

Example 2.2.4. The Heaviside distribution in $\mathbb{R}^n$ is $\langle H_R, \phi \rangle = \int_R \phi(x)dx$, where

$$H_R(x) = \begin{cases} 1 & \text{for } x \in R \\ 0 & \text{for } x \notin R. \end{cases} \tag{2.2.6}$$

For $\mathbb{R}$, (2.2.6) becomes

$$\langle H, \phi \rangle = \int_0^{\infty} \phi(x)dx. \tag{2.2.7}$$

Example 2.2.5. The Dirac delta distribution in $\mathbb{R}^n$ is

$$\langle \delta(x-\xi), \phi(x) \rangle = \phi(\xi) \tag{2.2.8}$$

for ξ is a fixed point in $\mathbb{R}^n$. Linearity of this functional follows from the relation

$$\langle \delta, c_1\phi_1(x) + c_2\phi_2(x) \rangle = c_1\phi_1(\xi) + c_2\phi_2(\xi) = c_1 \langle \delta, \phi_1 \rangle + c_2 \langle \delta, \phi_2 \rangle, \tag{2.2.9}$$

where c_1 and c_2 are arbitrary real constants.

Definition 2.2.6. A distribution E is said to be an *elementary solution* for the differential operator L if

$$LE = \delta.$$

Example 2.2.7. The function $R_{2k}(u)$ is the elementary solution of the operator $\Box^k$, where $\Box^k$ is defined by

$$\Box^k = \left[\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k$$

and $R_{2k}(u)$ is defined by (4.1.2) with $\alpha = 2k$. That is, $\Box^k R_{2k}(u) = \delta$, see([9], p.147)

2.3 Gamma functions

Definition 2.3.1. The gamma function is denoted by Γ and is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \tag{2.3.1}$$

where z is a complex number with $Re\ z > 0$

Example 2.3.2. Show that $\Gamma(1) = 1$.

Proof. By definition 2.3.1, we obtain

$$\begin{aligned} \Gamma(1) &= \int_0^\infty e^{-t} dt \\ &= \lim_{a\to\infty} (-e^{-t}|_0^a) \\ &= 1. \end{aligned}$$

□

Proposition 2.3.3. ([6]) Let z be a complex number. Then

(1) $z\Gamma(z) = \Gamma(z+1), \quad z \neq 0, -1, -2, \ldots$

(2) $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}, \quad z \neq 0, \pm 1, \pm 2, \ldots$

(2) $\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z), \quad z \neq 0, -1, -2, \ldots$

2.4 Properties of the convolution of distributions

Definition 2.4.1. The convolution $f * g$ of two functions $f(t)$ and $g(t)$, both in $\mathbb{R}^n$, is defined as

$$f * g = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau. \tag{2.4.1}$$

Example 2.4.2. Let

$$f(t) = \begin{cases} l^{-t}, & \text{for } t \geq 0 \\ 0, & \text{for } t \geq 0 \end{cases}$$

and

$$g(t) = \begin{cases} \sin t, & \text{for } 0 \leq t \leq \frac{\pi}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$f * g = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau,$$

$$f * g = \begin{cases} \int_0^t l^{-t} \sin(t-\tau)d\tau, & \text{for } 0 < t < \frac{\pi}{2} \\ \int_{t-\frac{\pi}{2}}^t l^{-t} \sin(t-\tau)d\tau, & \text{for } t \geq \frac{\pi}{2} \\ 0, & \text{for } t < 0. \end{cases}$$

Properties of the Convolution of Distributions

Property 1. Commutativity.

$$s * t = t * s \tag{2.4.2}$$

Property 2. Associativity.

$$(s * t) * u = s * (t * u) \tag{2.4.3}$$

if the supports of the two of these three distributions are bounded or if the supports of all three distributions are bounded on the same side.

Proposition 2.4.3. ([11]). If the convolution $s*t$ exists, then the convolutions $(D^k s) * t$ and $s * (D^k t)$ exist, and

$$(D^k s) * t = D^k(s * t) = s * (D^k t). \tag{2.4.4}$$

If L is a differential operator with constant coefficients, we find from (2.4.4) that

$$(Ls) * t = L(s * t) = s * (Lt). \tag{2.4.5}$$

2.5 Causal (anticausal) distributions

Let $x = (x_1, \ldots, x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$. Consider a nondegenerate quadratic form in n variables of the form

$$P = P(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \qquad (2.5.1)$$

where $p + q = n$. The distributions $(P \pm i0)^\lambda$ are defined by

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \{P \pm i\varepsilon |x|^2\}^\lambda$$

where $\varepsilon > 0, |x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2, \lambda \in \mathbb{C}$.

Moreover the distribution $(P \pm i0)^\lambda$ are analytic in λ every where except at $\lambda = -\frac{n}{2} - k, k = 0, 1, \ldots$ where they have simple poles. Similarly, the distribution $(m^2 + P \pm i0)^\lambda$ is denote by

$$(m^2 + P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \{m^2 + P \pm i\varepsilon |x|^2\}^\lambda$$

where m is a real positive number. ([8], p.289)

Following Trione ([18], p.32) by causal (anticausal) distributions we mean distributions of the form $T(P \pm i0, \lambda), P = P(x), T(P \pm i0, \lambda) = (P \pm i0)^\lambda f(P \pm i0, \lambda), f(z, \lambda)$ an entire function in the variables z, λ .

Let

$$G_\alpha(P \pm i0, m, n) = H_\alpha(m, n)(P \pm i0)^{\frac{1}{2}(\frac{\alpha - n}{2})} K_{(\frac{n-\alpha}{2})}(\sqrt{m^2(P \pm i0)}) \qquad (2.5.2)$$

where m is a real positive real number $\alpha \in \mathbb{C}$, K_ν designates the modified Bessel function of the third kind

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sec \pi\nu}, I_\nu(z) = \sum_{m=0}^{\infty} \frac{(\frac{z}{2})^{2m+r}}{m! \Gamma(m + \nu + 1)}$$

and

$$H_\alpha(m, n) = \frac{2^{\frac{1-(\alpha+n)}{2}} (m^2)^{(\frac{1}{2})(\frac{\alpha-n}{2})} e^{\frac{\pi}{2} qi}}{\pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})}$$

We introduce an auxiliary weight function

$$\lambda_\alpha(P \pm i0, m, n) = e^{iq\frac{\pi}{2}} 2^{\frac{1-(\alpha+n)}{2}} (m^2)^{(\frac{1}{2})(\frac{\alpha-n}{2})} (P \pm i0)^{\frac{(n+\alpha)}{4}} K_{\frac{(n+\alpha)}{2}} \sqrt{m^2(P \pm i0)}$$

that is a causal (anticausal) analoque to the auxiliary weight function introduce by Rubin ([5], p. 1247).

Let us define the n-dimensional ultrahyperbolic Klein-Gordon operator iterated k-times

$$(\Box + m^2)^k = \left[\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right]^k$$

, The distributional function $G_{2k}(P \pm i0, m, n)$ where n is an integer ≥ 2 and $k = 1, 2, 3, \ldots$ are elementary causal (anticausal) solutions of the ultrahyperbolic Klein-Gordon operator iterated k-times

$$(\Box + m^2)^k G_{2k}(P \pm i0, m, n) = \delta.$$

CHAPTER III

CAUSAL AND ANTICAUSAL SOLUTION OF THE OPERATOR $\oplus_B^k$

In this chapter, we study the causal and anticausal solution of the operator $\oplus_B^k$. Moreover, such a solution is unique.

3.1 Main results

Lemma 3.1.1. *Given the equation $\Box_B^k(P \pm i0) = \delta$ for $x \in \mathbb{R}_n^+$, where $\Box_B^k$ is the Bessel-ultra hyperbolic operator iterated k-times defined by (1.1.11) . Then $(P \pm i0) = R_{2k}(P \pm i0)$ is an elementary solution of the operator $\Box_B^k$, where*

$$R_{2k}(P \pm i0) = \frac{(P \pm i0)^{\frac{2k-n-|v|}{2}}}{K_n(2k)} = \frac{(P \pm i0)^{\left(\frac{2k-n-|v|}{2}\right)}}{K_n(2k)} \tag{3.1.1}$$

for

$$P = P(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2 \tag{3.1.2}$$

and

$$K_n(2k) = \frac{\pi^{\frac{n+2|v|-1}{2}}\Gamma\left(\frac{2+2k-n-2|v|}{2}\right)\Gamma\left(\frac{1-2k}{2}\right)\Gamma(2k)}{\Gamma\left(\frac{2+2k-p-2|v|}{2}\right)\Gamma\left(\frac{p-2k}{2}\right)}. \tag{3.1.3}$$

Lemma 3.1.2. *Given the equation $\triangle_B^k(P \pm i0) = \delta$ for $x \in \mathbb{R}_n^+$, where $\triangle_B^k$ is the Laplace Bessel operator iterated k-times defined by (1.1.12) . Then $(P \pm i0) = (-1)^k S_{2k}(P' \pm i0)$ is an elementary solution of the operator $\triangle_B^k$, where*

$$S_\alpha = S_\alpha(P' \pm i0) = \frac{e^{\frac{i\pi\alpha}{2}} e^{\pm\frac{i\frac{\pi}{2}q}{2}}\Gamma(\frac{n-\alpha}{2})}{2^\alpha \pi^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}(P' \pm i0)^{\frac{\alpha-n-2|v|}{2}} \tag{3.1.4}$$

where $\alpha \in \mathbb{C}$,

$$P' = P'(x) = x_1^2 - x_2^2 - \cdots - x_n^2$$

and q is the number of negative terms of the quadratic form P . The distributional functions S_α are the causal (anticausal) analogues of the elliptic kernel of M. Riesz ([12], pp.16-21), and have analogous properties ([19]).

$$S_{2k}(P' \pm i0) = \frac{|x|^{2k-n-2|v|}}{w_n(2k)}, \qquad p+q=n, \tag{3.1.5}$$

$$|x| = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{\frac{1}{2}} \tag{3.1.6}$$

$$P' = x_1^2 - x_2^2 - \cdots - x_n^2 \tag{3.1.7}$$

and

$$w_n(2k) = \frac{\Pi_{i=1}^n 2^{v_i - \frac{1}{2}}\Gamma\left(v_i + \frac{1}{2}\right)\Gamma(k)}{2^{n+2|v|-4k}\Gamma\left(\frac{n+2|v|-2k}{2}\right)}. \tag{3.1.8}$$

Lemma 3.1.3. *The convolution $R_{2k}(P \pm i0) * (-1)^k S_{2k}(P' \pm i0)$ is an elementary solution for the operator $\diamondsuit_B^k$ iterated k-times and is defined by (1.1.1).*

Lemma 3.1.4. *$R_{2k}(P \pm i0)$ and $S_{2k}(P' \pm i0)$ are homogeneous distributions of order $(2k - n - 2|v|)$.*

We need to show that $R_{2k}(P \pm i0)$ and $(-1)^k S_{2k}(P' \pm i0)$ satisfy the Euler equation; that is,

$$(2k - n - 2|v|)\, R_{2k}(P \pm i0) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} R_{2k}(P \pm i0),$$

$$(2k - n - 2|v|)\, S_{2k}(P' \pm i0) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} S_{2k}(P' \pm i0).$$

Lemma 3.1.5. *(The B-convolution of tempered distribution). $R_{2k}(P \pm i0) * S_{2k}(P' \pm i0)$ exists and is a tempered distribution.*

For the proof of Lemma 4.1.1- Lemma 4.1.5, see ([7], p.378-383).

Lemma 3.1.6. *The function $R_{-2k}(P \pm i0)$ and $(-1)^k S_{-2k}(P' \pm i0)$ are the inverse in the convolution algebra of $R_{2k}(P \pm i0)$ and $(-1)^k S_{2k}(P' \pm i0)$, respectively. That is,*

$$R_{-2k}(P \pm i0) * R_{2k}(P \pm i0) = R_{-2k+2k}(P \pm i0) = R_0(P \pm i0) = \delta,$$

$$(-1)^k S_{-2k}(P' \pm i0) * (-1)^k S_{2k}(P' \pm i0) = S_{-2k+2k}(P' \pm i0) = S_0(P' \pm i0) = \delta$$

Lemma 3.1.7. *(The B-convolution of $R_{2k}(P \pm i0)$ and $S_{2k}(P' \pm i0)$). Let $R_{2k}(P \pm i0)$ and $S_{2k}(P' \pm i0)$ defined by (4.1.1) and (4.1.4) respectively, then we obtain:*

(1) $S_{2k}(P' \pm i0) * S_{2m}(P' \pm i0) = S_{2k+2m}(P' \pm i0)$, where k and m are non-negative integers.

(2) $R_{2k}(P \pm i0) * R_{2m}(P \pm i0) = R_{2k+2m}(P \pm i0)$, where k and m are nonnegative integers.

For the proof of Lemma 4.1.6 and Lemma 4.1.7, see [10].

Theorem 3.1.8. *Given the equation*

$$\circledcirc_B^k (P \pm i0)^* = \delta \tag{3.1.9}$$

for $x \in \mathbb{R}_n^+$, where $\circledcirc_B^k$ is the operator iterated k-times is defined by (1.1.7) . Then we obtain $G(P \pm i0)$ is an elementary solution of (4.1.21), where

$$(P \pm i0)^* = (R_{4k}(P \pm i0) * (-1)^{2k} S_{4k}(P' \pm i0)) * (C^{*k}(P \pm i0))^{*-1} \tag{3.1.10}$$

where

$$C(P \pm i0) = \frac{1}{2} R_4(P \pm i0) + \frac{1}{2}(-1)^2 S_4(P' \pm i0). \tag{3.1.11}$$

Here $C^{*k}(P \pm i0)$ denotes the convolution of $C(P \pm i0)$ itself k-times, $(C^{*k}(P \pm i0))^{*-1}$ denotes the inverse of $C^{*k}(P \pm i0)$ in the convolution algebra. Moreover $(P \pm i0)^*$ is a tempered distribution.

Proof. We have

$$\circledcirc_B^k (P \pm i0)^* = \left(\frac{\triangle_B^2 + \Box_B^2}{2} \right)^k (P \pm i0)^* = \delta$$

or we can write

$$\left(\frac{1}{2} \triangle_B^2 + \frac{1}{2} \Box_B^2 \right) \left(\frac{1}{2} \triangle_B^2 + \frac{1}{2} \Box_B^2 \right)^{k-1} (P \pm i0)^* = \delta.$$

Convolving both sides of the above equation by $R_4(P \pm i0) * (-1)^2 S_4(P \pm i0)$,

$$\left(\frac{1}{2} \triangle_B^2 + \frac{1}{2} \Box_B^2 \right) * \left(R_4(P \pm i0) * (-1)^2 S_4(P' \pm i0) \right) \left(\frac{1}{2} \triangle_B^2 + \frac{1}{2} \Box_B^2 \right)^{k-1} (P \pm i0)^*$$
$$= \delta * R_4(P \pm i0) * (-1)^2 S_4(P' \pm i0)$$

or

$$\left(\frac{1}{2}\triangle_B^2\left(R_4(P\pm i0)*(-1)^2S_4(P\pm i0)\right)+\frac{1}{2}\Box_B^2\left(R_4(P\pm i0)*(-1)^2S_4(P'\pm i0)\right)\right)\cdot$$
$$\left(\frac{1}{2}\triangle_B^2+\frac{1}{2}\Box_B^2\right)^{k-1}(P\pm i0)^*=\delta*R_4(P\pm i0)*(-1)^2S_4(P'\pm i0).$$

By properties of convolutions,

$$\left(\frac{1}{2}\triangle_B^2\left((-1)^2S_4(P'\pm i0)\right)*R_4(P\pm i0)+\frac{1}{2}\Box_B^2\left(R_4(P\pm i0)\right)*(-1)^2S_4(P'\pm i0)\right)$$
$$\left(\frac{1}{2}\triangle_B^2+\frac{1}{2}\Box_B^2\right)^{k-1}(P\pm i0)^*=\delta*R_4(P\pm i0)*(-1)^2S_4(P'\pm i0).$$

By Lemma 2.1 and 2.2, we obtain

$$\left(\frac{1}{2}\delta*R_4(P\pm i0)+\frac{1}{2}\delta*(-1)^2S_4(P'\pm i0)\right)\left(\frac{1}{2}\triangle_B^2+\frac{1}{2}\Box_B^2\right)^{k-1}(P\pm i0)^*$$
$$=\delta*R_4(P\pm i0)*(-1)^2S_4(P'\pm i0).$$

or

$$\left(\frac{1}{2}R_4(P\pm i0)+\frac{1}{2}(-1)^2S_4(P'\pm i0)\right)\left(\frac{1}{2}\triangle_B^2+\frac{1}{2}\Box_B^2\right)^{k-1}(P\pm i0)^*$$
$$=R_4(P\pm i0)*(-1)^2S_4(P'\pm i0).$$

Keeping on convolving both sides of the above equation by $R_4(P\pm i0)*(-1)^2S_4(P'\pm i0)$ up to $k-1$ times, we obtain

$$C^{*k}(P\pm i0)*(P\pm i0)^*=\left(R_4(P\pm i0)*(-1)^2S_4(P'\pm i0)\right)^{*k} \tag{3.1.12}$$

the symbol $*k$ denotes the convolution of itself k-times. By properties of $R_{2k}(P\pm i0)$ and $S_{2k}(P'\pm i0)$ in Lemma 2.7, we have

$$\left(R_4(P\pm i0)*(-1)^2S_4(P'\pm i0)\right)^{*k}(P\pm i0)^*=R_{4k}(P\pm i0)*(-1)^{2k}S_{4k}(P'\pm i0).$$

Thus (4.1.10) becomes,

$$C^{*k}(P\pm i0)*(P\pm i0)^*=R_{4k}(P\pm i0)*(-1)^{2k}S_{4k}(P'\pm i0)$$

or

$$(P\pm i0)^*=\left(R_{4k}(P\pm i0)*(-1)^{2k}S_{4k}(P'\pm i0)\right)*\left(C^{*k}(P\pm i0)\right)^{*-1} \tag{3.1.13}$$

is an elementary solution of (4.1.21). We consider the function $C^{*k}(P \pm i0)$, since $R_4(P \pm i0) * (-1)^2 S_4(P' \pm i0)$ is a tempered distribution. Thus $C(P \pm i0)$ defined by (4.1.23) is tempered distribution, we obtain $C^{*k}(P \pm i0)$ is tempered distribution.

Now, $R_{4k}(P \pm i0) * (-1)^{2k} S_{4k}(P' \pm i0) \in S'$, the space of tempered distribution. Choose $S' \subset D'_R$, where D'_R is the right-side distribution which is a subspace of D' of distribution. Thus $R_{4k}(P \pm i0) * (-1)^{2k} S_{4k}(P' \pm i0) \in D'_R$. It follow that $R_{4k}(P \pm i0) * (-1)^{2k} S_{4k}(P' \pm i0)$ is an element of convolution algebra, since D'_R is a convolution algebra. Hence Zemanian [2], (4.1.8) has a unique solution

$$(P \pm i0)^* = \left(R_{4k}(P \pm i0) * (-1)^{2k} S_{4k}(P' \pm i0)\right) * \left(C^{*k}(P \pm i0)\right)^{*-1},$$

where $\left(C^{*k}(P \pm i0)\right)^{*-1}$ is an inverse of $C^{*k}(P \pm i0)$ in the convolution algebra. $(P \pm i0)^*$ is called the Green function of the operator $\circledcirc_B^k$.

Since $R_{4k}(P \pm i0) * (-1)^{2k} S_{4k}(P' \pm i0)$ and $\left(C^{*k}(P \pm i0)\right)^{*-1}$ are lies in S', then by ([2], p.152) again, we have $\left(R_{4k}(P \pm i0) * (-1)^{2k} S_{4k}(P' \pm i0)\right) * \left(C^{*k}(P \pm i0)\right)^{*-1} \in S'$. Hence $(P \pm i0)^*$ is a tempered distribution. □

Theorem 3.1.9. *Given the equation*

$$\oplus_B^k (P \pm i0) = \delta, \tag{3.1.14}$$

where $\oplus_B^k$ is the operator iterated k-times defined by (1.1.5) , δ is the Dirac-delta distribution, $x \in \mathbb{R}_n^+$ and k is a nonnegative integer. Then we obtain

$$(P \pm i0) = \left(R_{2k}(P \pm i0) * (-1)^k S_{2k}(P' \pm i0)\right) * (P \pm i0)^* \tag{3.1.15}$$

or

$$(P \pm i0) = \left(R_{6k}(P \pm i0) * (-1)^{3k} S_{6k}(P' \pm i0)\right) * \left(C^{*k}(P \pm i0)\right)^{*-1} \tag{3.1.16}$$

is a Green's function or an elementary solution for the operator $\oplus_B^k$ iterated k-times where $\oplus_B^k$ is defined by (1.1.5) , and $(P \pm i0)$ defined by (4.1.8) . For $q = 0$, then (4.1.12) becomes

$$\triangle_B^{4k} (P \pm i0) = \delta, \tag{3.1.17}$$

we obtain

$$(P \pm i0) = S_{8k}(P' \pm i0)$$

is an elementary solution of (4.1.27), where $\triangle_B^{4k}$ is the Laplace Bessel operator of p-dimension, iterated 4k-times and is defined by (1.1.14) . Moreover, we obtain

$$R_{-4k}(P \pm i0) * (-1)^{3k} S_{-6k}(P' \pm i0) * \left(C^{*k}(P \pm i0)\right) * (P \pm i0) = R_{2k}(P \pm i0) \tag{3.1.18}$$

as an elementary solution of the Bessel ultra-hyperbolic operator iterated k-times is defined by (1.1.11).

Proof. From (1.1.9) and (4.1.12), we have

$$\oplus_B^k (P \pm i0) = \left(\diamondsuit_B^k \circledcirc_B^k\right)(P \pm i0) = \delta. \tag{3.1.19}$$

Convolving both sides of (4.1.17) by $\left(R_{2k}(P \pm i0) * (-1)^k S_{2k}(P' \pm i0)\right) * (P \pm i0)^*$, we obtain

$$\begin{aligned}&\left(R_{2k}(P \pm i0) * (-1)^k S_{2k}(P' \pm i0)\right) * (P \pm i0)^* * (P \pm i0) * \left(\diamondsuit_B^k \circledcirc_B^k\right)(P \pm i0)\\ &= \delta * \left(R_{2k}(P \pm i0) * (-1)^k S_{2k}(P' \pm i0)\right) * (P \pm i0)^*.\end{aligned}$$

By properties of convolution

$$\begin{aligned}&\diamondsuit_B^k \left(R_{2k}(P \pm i0) * (-1)^k S_{2k}(P' \pm i0)\right) * \circledcirc_B^k \left((P \pm i0)^*\right) * (P \pm i0)\\ &= \left(R_{2k}(P \pm i0) * (-1)^k S_{2k}(P' \pm i0)\right) * (P \pm i0)^*.\end{aligned}$$

By Lemma 2.3 and Theorem 3.1, we obtain,

$$\delta * \delta * (P \pm i0) = (P \pm i0) = \left(R_{2k}(P \pm i0) * (-1)^k S_{2k}(P' \pm i0)\right) * (P \pm i0)^*.$$

By Lemma 2.7 and (4.1.8), we obtain,

$$(P \pm i0) = \left(R_{6k}(P \pm i0) * (-1)^{3k} S_{6k}(P' \pm i0)\right) * \left(C^{*k}(P \pm i0)\right)^{*-1} \tag{3.1.20}$$

is an elementary solution or Green's function of $\oplus_B^k$ operator. Now, for $q = 0$ the (4.1.12) becomes

$$\triangle_B^{4k}(P \pm i0) = \delta, \tag{3.1.21}$$

where $\triangle_B^{4k}$ is Laplace Bessel operator of p-dimension iterated $4k$-times. By Lemma 2.2, we have

$$(P \pm i0) = (-1)^{4k} S_{8k}(P' \pm i0) = S_{8k}(P' \pm i0)$$

is an elementary solution of (4.1.27). On the other hand, we can also find $(P \pm i0)$ from (4.1.18). Since $q = 0$, we have $R_{2k}(P \pm i0)$ reduces to $(-1)^k S_{2k}(P' \pm i0)$. Thus, by (4.1.18) for $q = 0$, we obtain

$$\begin{aligned}(P \pm i0) &= \big(S_{6k}(P \pm i0) * (-1)^{6k} S_{6k}(P' \pm i0)\big) * \big((-1)^{2k} S_{4k}(P' \pm i0)\big)^{*-1} \\ &= (-1)^{6k} S_{6k+6k}(P' \pm i0) \big((-1)^{2k} S_{4k}(P' \pm i0)\big)^{*-1} \\ &= S_{8k}(P' \pm i0).\end{aligned}$$

From (4.1.18), we have

$$(P \pm i0) = \big(R_{6k}(P \pm i0) * (-1)^{3k} S_{6k}(P' \pm i0)\big) * \big(C^{*k}(P \pm i0)\big)^{*-1}.$$

Convolving the above equation by $R_{-4k}(P \pm i0) * (-1)^{3k} S_{-6k}(P' \pm i0) * \big(C^{*k}(P \pm i0)\big)$. By Lemma 2.6 and 2.7, we obtain

$$\begin{aligned}&R_{-4k}(P \pm i0) * (-1)^{3k} S_{-6k}(P' \pm i0) * \big(C^{*k}(P \pm i0)\big) * (P \pm i0) \\ &\quad = R_0(P \pm i0) * S_0(P' \pm i0) * \delta * R_{2k}(P \pm i0)\end{aligned}$$

or

$$R_{-4k}(P \pm i0) * (-1)^{3k} S_{-6k}(P' \pm i0) * \big(C^{*k}(P \pm i0)\big) * (P \pm i0) = \delta * \delta * \delta * R_{2k}(P \pm i0).$$

It follows that

$$R_{-4k}(P \pm i0) * (-1)^{3k} S_{-6k}(P' \pm i0) * \big(C^{*k}(P \pm i0)\big) * (P \pm i0) = R_{2k}(P \pm i0) \tag{3.1.22}$$

as an elementary solution of the operator $\Box_B^k$ iterated k-times defined by (1.1.11)

□

CHAPTER IV

The Solution of the n-Dimensional $\oplus_B^k$ and $\oplus_{B,8}^k$ Operator

In this chapter, we study the elementary solution of the operator $\oplus_B^k$ and $\oplus_{B,8}^k$ and after that, we apply such an elementary solution to solve for the solution of the operator $\oplus_B^k$ and $\oplus_{B,8}^k$. The work in this chapter will appear in [15].

4.1 Main results

Lemma 4.1.1. *Given the equation* $\Box_B^k u(x) = \delta(x)$ *for* $x \in \mathbb{R}_n^+$*, where* $\Box_B^k$ *is the Bessel-ultra hyperbolic operator iterated k-times defined by (1.1.11) . Then* $u(x) = R_{2k}(x)$ *is an elementary solution of the operator* $\Box_B^k$*, where*

$$R_{2k}(x) = \frac{V^{\frac{2k-n-|v|}{2}}}{K_n(2k)} = \frac{\left(x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2\right)^{\left(\frac{2k-n-|v|}{2}\right)}}{K_n(2k)} \tag{4.1.1}$$

for

$$V = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2 \tag{4.1.2}$$

and

$$K_n(2k) = \frac{\pi^{\frac{n+2|v|-1}{2}} \Gamma\left(\frac{2+2k-n-2|v|}{2}\right) \Gamma\left(\frac{1-2k}{2}\right) \Gamma(2k)}{\Gamma\left(\frac{2+2k-p-2|v|}{2}\right) \Gamma\left(\frac{p-2k}{2}\right)}. \tag{4.1.3}$$

Lemma 4.1.2. *Given the equation* $\triangle_B^k u(x) = \delta(x)$ *for* $x \in \mathbb{R}_n^+$*, where* $\triangle_B^k$ *is the Laplace Bessel operator iterated k-times defined by (1.1.12) . Then* $u(x) = (-1)^k S_{2k}(x)$ *is an elementary solution of the operator* $\triangle_B^k$*, where*

$$S_{2k}(x) = \frac{|x|^{2k-n-2|v|}}{w_n(2k)}, \qquad p + q = n, \tag{4.1.4}$$

$$|x| = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{\frac{1}{2}} \tag{4.1.5}$$

and

$$w_n(2k) = \frac{\Pi_{i=1}^{n} 2^{v_i - \frac{1}{2}} \Gamma\left(v_i + \frac{1}{2}\right) \Gamma(k)}{2^{n+2|v|-4k} \Gamma\left(\frac{n+2|v|-2k}{2}\right)}. \tag{4.1.6}$$

Lemma 4.1.3. *The convolution $R_{2k}(x) * (-1)^k S_{2k}(x)$ is an elementary solution for the operator $\diamondsuit_B^k$ iterated k-times and is defined by (1.1.1).*

Lemma 4.1.4. *$R_{2k}(x)$ and $S_{2k}(x)$ are homogeneous distributions of order $(2k - n - 2|v|)$.*

We need to show that $R_{2k}(x)$ and $(-1)^k S_{2k}(x)$ satisfy the Euler equation; that is,

$$(2k - n - 2|v|)\, R_{2k}(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} R_{2k}(x), (2k - n - 2|v|)\, S_{2k}(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} S_{2k}(x).$$

Lemma 4.1.5. *(The B-convolution of tempered distribution). $R_{2k}(x) * S_{2k}(x)$ exists and is a tempered distribution.*

For the proof of Lemma 4.1.1- Lemma 4.1.5, see ([7], p.378-383).

Lemma 4.1.6. *The function $R_{-2k}(x)$ and $(-1)^k S_{-2k}(x)$ are the inverse in the convolution algebra of $R_{2k}(x)$ and $(-1)^k S_{2k}(x)$, respectively. That is,*

$$R_{-2k}(x) * R_{2k}(x) = R_{-2k+2k}(x) = R_0(x) = \delta(x),$$
$$(-1)^k S_{-2k}(x) * (-1)^k S_{2k}(x) = S_{-2k+2k}(x) = S_0(x) = \delta(x)$$

Lemma 4.1.7. *(The B-convolution of $R_{2k}(x)$ and $S_{2k}(x)$). Let $R_{2k}(x)$ and $S_{2k}(x)$ defined by (4.1.1) and (4.1.4) respectively, then we obtain:*

(1) $S_{2k}(x) * S_{2m}(x) = S_{2k+2m}(x)$, where k and m are nonnegative integers.

(2) $R_{2k}(x) * R_{2m}(x) = R_{2k+2m}(x)$, where k and m are nonnegative integers.

For the proof of Lemma 4.1.6 and Lemma 4.1.7, see [10].

Theorem 4.1.8. *Given the equation*

$$\circledcirc_B^k G(x) = \delta(x) \tag{4.1.7}$$

for $x \in \mathbb{R}_n^+$*, where* $\circledcirc_B^k$ *is the operator iterated k-times is defined by (1.1.7) . Then we obtain* $G(x)$ *is an elementary solution of (4.1.21), where*

$$G(x) = (R_{4k}(x) * (-1)^{2k} S_{4k}(x)) * (C^{*k}(x))^{*-1} \tag{4.1.8}$$

where

$$C(x) = \frac{1}{2} R_4(x) + \frac{1}{2}(-1)^2 S_4(x). \tag{4.1.9}$$

Here $C^{*k}(x)$ *denotes the convolution of* $C(x)$ *itself k-times,* $(C^{*k}(x))^{*-1}$ *denotes the inverse of* $C^{*k}(x)$ *in the convolution algebra. Moreover* $G(x)$ *is a tempered distribution.*

Proof. We have

$$\circledcirc_B^k G(x) = \left(\frac{\triangle_B^2 + \Box_B^2}{2}\right)^k G(x) = \delta(x)$$

or we can write

$$\left(\frac{1}{2} \triangle_B^2 + \frac{1}{2}\Box_B^2\right)\left(\frac{1}{2} \triangle_B^2 + \frac{1}{2}\Box_B^2\right)^{k-1} G(x) = \delta(x).$$

Convolving both sides of the above equation by $R_4(x) * (-1)^2 S_4(x)$,

$$\left(\frac{1}{2} \triangle_B^2 + \frac{1}{2}\Box_B^2\right) * \left(R_4(x) * (-1)^2 S_4(x)\right)\left(\frac{1}{2} \triangle_B^2 + \frac{1}{2}\Box_B^2\right)^{k-1} G(x)$$
$$= \delta(x) * R_4(x) * (-1)^2 S_4(x)$$

or

$$\left(\frac{1}{2} \triangle_B^2 \left(R_4(x) * (-1)^2 S_4(x)\right) + \frac{1}{2}\Box_B^2 \left(R_4(x) * (-1)^2 S_4(x)\right)\right)\left(\frac{1}{2} \triangle_B^2 + \frac{1}{2}\Box_B^2\right)^{k-1} G(x)$$
$$= \delta(x) * R_4(x) * (-1)^2 S_4(x).$$

By properties of convolutions,

$$\left(\frac{1}{2} \triangle_B^2 \left((-1)^2 S_4(x)\right) * R_4(x) + \frac{1}{2}\Box_B^2 \left(R_4(x)\right) * (-1)^2 S_4(x)\right)\left(\frac{1}{2} \triangle_B^2 + \frac{1}{2}\Box_B^2\right)^{k-1} G(x)$$
$$= \delta(x) * R_4(x) * (-1)^2 S_4(x).$$

By Lemma 2.1 and 2.2, we obtain

$$\left(\frac{1}{2}\delta * R_4(x) + \frac{1}{2}\delta * (-1)^2 S_4(x)\right)\left(\frac{1}{2}\triangle_B^2 + \frac{1}{2}\Box_B^2\right)^{k-1} G(x) = \delta(x) * R_4(x) * (-1)^2 S_4(x).$$

or

$$\left(\frac{1}{2}R_4(x) + \frac{1}{2}(-1)^2 S_4(x)\right)\left(\frac{1}{2}\triangle_B^2 + \frac{1}{2}\Box_B^2\right)^{k-1} G(x) = R_4(x) * (-1)^2 S_4(x).$$

Keeping on convolving both sides of the above equation by $R_4(x) * (-1)^2 S_4(x)$ up to $k-1$ times, we obtain

$$C^{*k}(x) * G(x) = \left(R_4(x) * (-1)^2 S_4(x)\right)^{*k} \tag{4.1.10}$$

the symbol $*k$ denotes the convolution of itself k-times. By properties of $R_{2k}(x)$ and $S_{2k}(x)$ in Lemma 2.7, we have

$$\left(R_4(x) * (-1)^2 S_4(x)\right)^{*k}(x) = R_{4k}(x) * (-1)^{2k} S_{4k}(x).$$

Thus (4.1.10) becomes,

$$C^{*k}(x) * G(x) = R_{4k}(x) * (-1)^{2k} S_{4k}(x)$$

or

$$G(x) = \left(R_{4k}(x) * (-1)^{2k} S_{4k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} \tag{4.1.11}$$

is an elementary solution of (4.1.21). We consider the function $C^{*k}(x)$, since $R_4(x) * (-1)^2 S_4(x)$ is a tempered distribution. Thus $C(x)$ defined by (4.1.23) is tempered distribution, we obtain $C^{*k}(x)$ is tempered distribution.

Now, $R_{4k}(x) * (-1)^{2k} S_{4k}(x) \in S'$, the space of tempered distribution. Choose $S' \subset D'_R$, where D'_R is the right-side distribution which is a subspace of D' of distribution. Thus $R_{4k}(x) * (-1)^{2k} S_{4k}(x) \in D'_R$. It follow that $R_{4k}(x) * (-1)^{2k} S_{4k}(x)$ is an element of convolution algebra, since D'_R is a convolution algebra. Hence Zemanian [2], (4.1.8) has a unique solution

$$G(x) = \left(R_{4k}(x) * (-1)^{2k} S_{4k}(x)\right) * \left(C^{*k}(x)\right)^{*-1},$$

where $\left(C^{*k}(x)\right)^{*-1}$ is an inverse of $C^{*k}(x)$ in the convolution algebra. $G(x)$ is called the Green function of the operator $\circledcirc_B^k$.

Since $R_{4k}(x) * (-1)^{2k} S_{4k}(x)$ and $\left(C^{*k}(x)\right)^{*-1}$ are lies in S', then by ([2], p.152) again, we have $\left(R_{4k}(x) * (-1)^{2k} S_{4k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} \in S'$. Hence $G(x)$ is a tempered distribution. □

Theorem 4.1.9. *Given the equation*

$$\oplus_B^k u(x) = \delta(x), \tag{4.1.12}$$

where $\oplus_B^k$ *is the operator iterated* k*-times defined by (1.1.5) ,* $\delta(x)$ *is the Dirac-delta distribution,* $x \in \mathbb{R}_n^+$ *and* k *is a nonnegative integer. Then we obtain*

$$u(x) = \left(R_{2k}(x) * (-1)^k S_{2k}(x)\right) * G(x) \tag{4.1.13}$$

or

$$u(x) = \left(R_{6k}(x) * (-1)^{3k} S_{6k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} \tag{4.1.14}$$

is a Green's function or an elementary solution for the operator $\oplus_B^k$ *iterated* k*-times where* $\oplus_B^k$ *is defined by (1.1.5) , and* $G(x)$ *defined by (4.1.8) . For* $q = 0$*, then (4.1.12) becomes*

$$\triangle_B^{4k} u(x) = \delta(x), \tag{4.1.15}$$

we obtain

$$u(x) = S_{8k}(x)$$

is an elementary solution of (4.1.27), where $\triangle_B^{4k}$ *is the Laplace Bessel operator of p-dimension, iterated* $4k$*-times and is defined by (1.1.14) . Moreover, we obtain*

$$R_{-4k}(x) * (-1)^{3k} S_{-6k}(x) * \left(C^{*k}(x)\right) * u(x) = R_{2k}(x) \tag{4.1.16}$$

as an elementary solution of the Bessel ultra-hyperbolic operator iterated k-times is defined by (1.1.11).

Proof. From (1.1.9) and (4.1.12), we have

$$\oplus_B^k u(x) = \left(\diamondsuit_B^k \circledcirc_B^k\right) u(x) = \delta(x). \tag{4.1.17}$$

Convolving both sides of (4.1.17) by $\left(R_{2k}(x) * (-1)^k S_{2k}(x)\right) * G(x)$, we obtain

$$\left(R_{2k}(x) * (-1)^k S_{2k}(x)\right) * G(x) * \left(\diamondsuit_B^k \circledcirc_B^k\right) u(x) = \delta(x) * \left(R_{2k}(x) * (-1)^k S_{2k}(x)\right) * G(x).$$

By properties of convolution

$$\diamondsuit_B^k \left(R_{2k}(x) * (-1)^k S_{2k}(x)\right) * \circledcirc_B^k \left(G(x)\right) * u(x) = \left(R_{2k}(x) * (-1)^k S_{2k}(x)\right) * G(x).$$

By Lemma 2.3 and Theorem 3.1, we obtain,

$$\delta * \delta * u(x) = u(x) = \left(R_{2k}(x) * (-1)^k S_{2k}(x)\right) * G(x).$$

By Lemma 2.7 and (4.1.8), we obtain,

$$u(x) = \left(R_{6k}(x) * (-1)^{3k} S_{6k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} \tag{4.1.18}$$

is an elementary solution or Green's function of $\oplus_B^k$ operator. Now, for $q = 0$ the (4.1.12) becomes

$$\triangle_B^{4k} u(x) = \delta(x), \tag{4.1.19}$$

where $\triangle_B^{4k}$ is Laplace Bessel operator of p-dimension iterated $4k$-times. By Lemma 2.2, we have

$$u(x) = (-1)^{4k} S_{8k}(x) = S_{8k}(x)$$

is an elementary solution of (4.1.27). On the other hand, we can also find $u(x)$ from (4.1.18). Since $q = 0$, we have $R_{2k}(x)$ reduces to $(-1)^k S_{2k}(x)$. Thus, by (4.1.18) for $q = 0$, we obtain

$$\begin{aligned} u(x) &= \left(S_{6k}(x) * (-1)^{6k} S_{6k}(x)\right) * \left((-1)^{2k} S_{4k}(x)\right)^{*-1} \\ &= (-1)^{6k} S_{6k+6k}(x) \left((-1)^{2k} S_{4k}(x)\right)^{*-1} \\ &= S_{8k}(x). \end{aligned}$$

From (4.1.18), we have

$$u(x) = \left(R_{6k}(x) * (-1)^{3k} S_{6k}(x)\right) * \left(C^{*k}(x)\right)^{*-1}.$$

Convolving the above equation by $R_{-4k}(x) * (-1)^{3k} S_{-6k}(x) * \left(C^{*k}(x)\right)$. By Lemma 2.6 and 2.7, we obtain

$$R_{-4k}(x) * (-1)^{3k} S_{-6k}(x) * \left(C^{*k}(x)\right) * u(x) = R_0(x) * S_0(x) * \delta(x) * R_{2k}(x)$$

or

$$R_{-4k}(x) * (-1)^{3k} S_{-6k}(x) * \left(C^{*k}(x)\right) * u(x) = \delta(x) * \delta(x) * \delta(x) * R_{2k}(x).$$

It follows that

$$R_{-4k}(x) * (-1)^{3k} S_{-6k}(x) * \left(C^{*k}(x)\right) * u(x) = R_{2k}(x) \tag{4.1.20}$$

as an elementary solution of the operator $\Box_B^k$ iterated k-times defined by (1.1.11)

□

Theorem 4.1.10. *Given the equation*

$$\circledcirc_{B,4}^k K(x) = \delta(x) \tag{4.1.21}$$

for $x \in \mathbb{R}_n^+$, where $\circledcirc_{B,4}^k$ is the operator iterated k-times is defined by (1.1.8) . Then we obtain $K(x)$ is an elementary solution of (4.1.21), where

$$K(x) = (R_{8k}(x) * (-1)^{4k} S_{8k}(x)) * (H^{*k}(x))^{*-1} \tag{4.1.22}$$

where

$$H(x) = \frac{1}{2} R_8(x) + \frac{1}{2}(-1)^4 S_8(x). \tag{4.1.23}$$

Here $H^{*k}(x)$ denotes the convolution of $H(x)$ itself k-times, $(H^{*k}(x))^{*-1}$ denotes the inverse of $H^{*k}(x)$ in the convolution algebra. Moreover $K(x)$ is a tempered distribution.

Proof. Similary theorem 4.1.8. □

Theorem 4.1.11. *Given the equation*

$$\oplus_{B,8}^k Y(x) = \delta(x), \tag{4.1.24}$$

where $\oplus_{B,8}^k$ is the operator iterated k-times defined by (1.1.6) , $\delta(x)$ is the Dirac-delta distribution, $x \in \mathbb{R}_n^+$ and k is a nonnegative integer. Then we obtain

$$Y(x) = \left(R_{2k}(x) * (-1)^k S_{2k}(x) * G(x)\right) * K(x) \tag{4.1.25}$$

or

$$Y(x) = \left(R_{10k}(x) * (-1)^{5k} S_{10k}(x)\right) * \left(H^{*k}(x)\right)^{*-1} \tag{4.1.26}$$

is a Green's function or an elementary solution for the operator $\oplus_{B,8}^{k}$ *iterated k-times where* $\oplus_{B,8}^{k}$ *is defined by (1.1.6) , and* $Y(x)$ *the same defined by (4.1.22) . For* $q = 0$*, then (4.1.24) becomes*

$$\triangle_B^{8k} Y(x) = \delta(x), \tag{4.1.27}$$

we obtain

$$Y(x) = S_{16k}(x)$$

is an elementary solution of (4.1.27), where $\triangle_B^{8k}$ *is the Laplace Bessel operator of p-dimension, iterated* $8k$*-times and is defined by (1.1.14) . Moreover, we obtain*

$$R_{-8k}(x) * (-1)^{5k} S_{-10k}(x) * \left(H^{*k}(x)\right) * u(x) = R_{2k}(x) \tag{4.1.28}$$

as an elementary solution of the Bessel ultra-hyperbolic operator iterated k-times is defined by (1.1.11).

Proof. Similary theorem 4.1.9 □

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APPENDIX