CHAPTER 4
CONCLUSION

In this thesis, we obtain the following results.

4.1 Left Regular, Right Regular and Completely Regular Elements

1. Let $\alpha \in S(X,Y)$. Then the following statements are equivalent:

   (1) α is left regular.

   (2) $X\alpha = X\alpha^2$ and $Y\alpha = Y\alpha^2$.

   (3) $\alpha^2 \in L_\alpha$.

2. Let $\alpha \in S(X,Y)$. Then the following statements are equivalent:

   (1) α is right regular.

   (2) $\pi_\alpha = \pi_{\alpha^2}$ and $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$.

   (3) $\alpha^2 \in R_\alpha$.

3. Let $\alpha \in S(X,Y)$. Then the following statements are equivalent:

   (1) α is completely regular.

   (2) α is left regular and α is right regular.

   (3) $X\alpha = X\alpha^2, Y\alpha = Y\alpha^2$ and $\pi_\alpha = \pi_{\alpha^2}, \pi_\alpha(Y) = \pi_{\alpha^2}(Y)$.

   (4) $\alpha^2 \in H_\alpha$.

4. Let $\alpha \in S(X,Y)$ be such that $X\alpha$ is finite. Then α is left regular if and only if α is right regular.

4.2   Intra-Regular Elements

1. Let $\alpha \in S(X, Y)$. Then the following statements are equivalent:

   (1) α is intra-regular.

   (2) $|X\alpha| = |X\alpha^2|, |Y\alpha| = |Y\alpha^2|$ and $|X\alpha \setminus Y| = |X\alpha^2 \setminus Y|$.

   (3) $\alpha^2 \in J_\alpha$.

2. Let $\alpha \in S(X, Y)$ be such that $X\alpha$ is a finite set. Then the following statements are equivalent:

   (1) α is left regular.

   (2) α is right regular.

   (3) α is intra-regular.

4.3   Unit Regular Elements

1. Let $\alpha \in S(X, Y)$ and $C = \emptyset$. Then α is unit regular if and only if $\alpha \in G(X, Y)$.

2. Let $\alpha \in S(X, Y)$ and $C \neq \emptyset$. If α is unit regular, then the following three conditions holds.

   (1) α is regular.

   (2) $|D_Y(\alpha) \cup A| = |\underset{a \in A}{\cup} a\alpha^{-1} \cap Y|$.

   (3) $|D_{X \setminus Y}(\alpha) \cup B| = |\underset{a \in C}{\cup} a\alpha^{-1} \cap (X \setminus Y)|$.

3. Let X be a finite set and $\alpha \in S(X, Y)$. Then α is unit regular if and only if α is regular.

4.4   The Numbers of Left Regular, Right Regular and Intra-Regular Elements

1. The number of left regular elements in $S(X, Y)$ is
$$\sum_{m=0}^{n-r} \sum_{k=1}^{r} \binom{r}{k} k! k^{r-k} \binom{n-r}{m} m!(k+m)^{n-r-m}, \quad \text{where } |X| = n, |Y| = r.$$