CHAPTER 3

MAIN RESULTS

In this chapter, we present the characterizations of left regular elements, right regular elements, intra-regular elements, unit regular elements and completely regular elements on $S(X,Y)$. We also consider the relationships of these elements. Moreover, we count the numbers of left regular, right regular, intra-regular and unit regular elements of $S(X,Y)$ when X is a finite set.

3.1 Left Regular, Right Regular and Completely Regular Elements

In this section, we give necessary and sufficient conditions for elements in $S(X,Y)$ to be left regular, right regular and completely regular.

Theorem 3.1.1. *Let $\alpha \in S(X,Y)$. Then the following statements are equivalent:*

(1) *α is left regular.*

(2) *$X\alpha = X\alpha^2$ and $Y\alpha = Y\alpha^2$.*

(3) *$\alpha^2 \in L_\alpha$.*

Proof. (1) $\Rightarrow$ (2) Assume that α is left regular. Then $\alpha = \beta\alpha^2$ for some $\beta \in S(X,Y)$. We have $X\alpha^2 = (X\alpha)\alpha \subseteq X\alpha$ and $Y\alpha^2 = (Y\alpha)\alpha \subseteq Y\alpha$. We prove that $X\alpha \subseteq X\alpha^2$ and $Y\alpha \subseteq Y\alpha^2$. Since $\alpha = \beta\alpha^2$, we obtain $X\alpha = (X\beta)\alpha^2 \subseteq X\alpha^2$. So $X\alpha \subseteq X\alpha^2$. Thus $X\alpha^2 = X\alpha$. Similarly, we can prove that $Y\alpha \subseteq Y\alpha^2$. Therefore, $X\alpha = X\alpha^2$ and $Y\alpha = Y\alpha^2$.

(2) $\Rightarrow$ (3) Assume that $X\alpha = X\alpha^2$ and $Y\alpha = Y\alpha^2$. Then by Lemma 2.7.2, we obtain $\alpha\mathcal{L}\alpha^2$, that is $\alpha^2 \in L_\alpha$.

(3) $\Rightarrow$ (1) Assume that $\alpha^2 \in L_\alpha$. Then $\alpha\mathcal{L}\alpha^2$ and hence $\alpha = \beta\alpha^2$ for some $\beta \in S(X,Y)^1 = S(X,Y)$. Thus α is left regular. $\square$

Theorem 3.1.2. *Let $\alpha \in S(X,Y)$. Then the following statements are equivalent:*

(1) *α is right regular.*

(2) *$\pi_\alpha = \pi_{\alpha^2}$ and $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$.*

(3) *$\alpha^2 \in R_\alpha$.*

Proof. (1) $\Rightarrow$ (2) Assume that α is right regular. Then $\alpha = \alpha^2\beta$ for some $\beta \in S(X,Y)$. We first prove that $\pi_\alpha = \pi_{\alpha^2}$.

Let $A \in \pi_\alpha$. Then $A = x\alpha^{-1}$ for some $x \in X\alpha$. For each $z \in A$, $z\alpha = x$ and hence $x\alpha = z\alpha^2 \in X\alpha^2$. So $z \in x\alpha(\alpha^2)^{-1} \in \pi_{\alpha^2}$. Set $B = x\alpha(\alpha^2)^{-1}$. Thus $A \subseteq B$. Hence π_α refines π_{α^2}. Let $C \in \pi_{\alpha^2}$. Then $C = x(\alpha^2)^{-1}$ for some $x \in X\alpha^2$. For each $z \in C$, $z\alpha^2 = x$ and hence $x\beta = z\alpha^2\beta = z\alpha \in X\alpha$. So $z \in (x\beta)\alpha^{-1} \in \pi_\alpha$. Set $D = (x\beta)\alpha^{-1}$. Thus $C \subseteq D$. Hence π_{α^2} refines π_α.

Now, we prove that $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$. Let $A \in \pi_\alpha(Y)$. Then $A = y\alpha^{-1}$ for some $y \in X\alpha \cap Y$. For each $z \in A$, $z\alpha = y$ and hence $y\alpha = z\alpha^2 \in X\alpha^2 \cap Y$. So $z \in y\alpha(\alpha^2)^{-1} \in \pi_{\alpha^2}(Y)$. Set $B = y\alpha(\alpha^2)^{-1}$. Thus $A \subseteq B$. Hence $\pi_\alpha(Y)$ refines $\pi_{\alpha^2}(Y)$. Let $C \in \pi_{\alpha^2}(Y)$. Then $C = y(\alpha^2)^{-1}$ for some $y \in X\alpha^2 \cap Y$. For each $z \in C$, $z\alpha^2 = y$ and hence $y\beta = z\alpha^2\beta = z\alpha \in X\alpha \cap Y$. So $z \in (y\beta)\alpha^{-1} \in \pi_\alpha(Y)$. Set $D = (y\beta)\alpha^{-1}$. Thus $C \subseteq D$. Hence $\pi_{\alpha^2}(Y)$ refines $\pi_\alpha(Y)$.

(2) $\Rightarrow$ (3) Assume that $\pi_\alpha = \pi_{\alpha^2}$ and $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$. Then by Lemma 2.7.3, we obtain $\alpha\mathcal{R}\alpha^2$, that is $\alpha^2 \in R_\alpha$.

(3) $\Rightarrow$ (1) Assume that $\alpha^2 \in R_\alpha$. Then $\alpha\mathcal{R}\alpha^2$ and hence $\alpha = \alpha^2\beta$ for some $\beta \in S(X,Y)^1 = S(X,Y)$. Thus α is right regular. □

Theorem 3.1.3. *Let $\alpha \in S(X,Y)$. Then the following statements are equivalent:*

(1) *α is completely regular.*

(2) *α is left regular and α is right regular.*

(3) *$X\alpha = X\alpha^2, Y\alpha = Y\alpha^2$ and $\pi_\alpha = \pi_{\alpha^2}$, $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$.*

(4) *$\alpha^2 \in H_\alpha$.*

Proof. (1) $\Rightarrow$ (2) Assume that α is completely regular. Then by Theorem 2.3.1, we obtain α is left regular and α is right regular.

(2) $\Rightarrow$ (3) Assume that α is left regular and α is right regular. Then by Theorem 3.1.1 and 3.1.2, we have that $X\alpha = X\alpha^2, Y\alpha = Y\alpha^2$, $\pi_\alpha = \pi_{\alpha^2}$ and

$\pi_{\alpha}(Y) = \pi_{\alpha^2}(Y)$.

(3) $\Rightarrow$ (4) Assume that $X\alpha = X\alpha^2, Y\alpha = Y\alpha^2$ and $\pi_{\alpha} = \pi_{\alpha^2}$, $\pi_{\alpha}(Y) = \pi_{\alpha^2}(Y)$. Then by Corollary 2.7.4, we obtain $\alpha\mathcal{H}\alpha^2$, that is $\alpha^2 \in H_{\alpha}$.

(4) $\Rightarrow$ (1) Assume that $\alpha^2 \in H_{\alpha}$. Then $\alpha\mathcal{H}\alpha^2$, that is $\alpha\mathcal{L}\alpha^2$ and $\alpha\mathcal{R}\alpha^2$ and hence $\alpha = \alpha^2\beta$ and $\alpha = \gamma\alpha^2$ for some $\beta, \gamma \in S(X,Y)^1 = S(X,Y)$. Thus α is left regular and right regular. Hence by Theorem 2.3.1, we obtain α is completely regular. □

For convenience, from now on if $\alpha \in S(X,Y)$, the notations Y' and X' are for $Y\alpha$ and $X\alpha \setminus Y\alpha$ respectively. To prove that $\alpha \in S(X,Y)$ is left regular if and only if α is right regular when $X\alpha$ is finite, we begin with the following three lemmas.

Lemma 3.1.4. *Let $\alpha \in S(X,Y)$. If $X\alpha = X\alpha^2$ is finite and $Y\alpha = Y\alpha^2$, then*

(1) $(X\backslash Y)\alpha \subseteq (X\backslash Y) \cup Y'$;

(2) $X' \subseteq X\backslash Y'\alpha^{-1}$;

(3) $|y'\alpha^{-1} \cap Y'| = 1$ *for all* $y' \in Y'$;

(4) $|x'\alpha^{-1} \cap X'| = 1$ *for all* $x' \in X'$.

Proof. Let $\alpha \in S(X,Y)$. Assume that $X\alpha = X\alpha^2$ is finite and $Y\alpha = Y\alpha^2$, so we can write

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_n & B_1 & B_2 & \dots & B_m & C_1 & C_2 & \dots & C_t \\ a_1 & a_2 & \dots & a_n & b_1 & b_2 & \dots & b_m & c_1 & c_2 & \dots & c_t \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$, $B_j, C_k \subseteq X \setminus Y$; $a_i, b_j \in Y, c_k \in X \setminus Y$ for all $i = 1, 2, ..., n$, $j = 1, 2, ..., m$ and $k = 1, 2, ..., t$. Then $Y' = \{a_1, a_2, ..., a_n\}$ and $X' = \{b_1, b_2, ..., b_m, c_1, c_2, ..., c_t\}$.

(1) We prove that $J = \emptyset$ where $J = \{1, 2, ..., m\}$.

Suppose that there is $j \in J$ such that $B_j\alpha = \{b_j\}$ where $b_j \in Y$. Then $B_j\alpha^2 = (B_j\alpha)\alpha = \{b_j\}\alpha$. Assume that $b_j \in A_{i_0}$ for some i_0, then $B_j\alpha^2 = \{b_j\}\alpha = \{a_{i_0}\}$. Since $Y\alpha^2 = Y\alpha = \{a_1, a_2, ..., a_n\}$, there is A_k such that $A_k\alpha^2 = \{a_{i_0}\}$. That is $A_k\alpha^2 = B_j\alpha^2$. But $A_k\alpha \neq B_j\alpha$, so $|X\alpha^2| \leq n + (m-1) + t < |X\alpha|$, which implies that $X\alpha^2 \subsetneq X\alpha$, a contradiction. Thus $J = \emptyset$. So we can write

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_n & C_1 & C_2 & \dots & C_t \\ a_1 & a_2 & \dots & a_n & c_1 & c_2 & \dots & c_t \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$, $C_k \subseteq X \setminus Y$; $a_i \in Y, c_k \in X \setminus Y$ for all $i = 1, 2, ..., n$ and $k = 1, 2, ..., t$. Then $Y' = \{a_1, a_2, ..., a_n\}$ and $X' = \{c_1, c_2, ..., c_t\}$. Therefore, $(X \setminus Y)\alpha \subseteq (X \setminus Y) \cup Y'$.

(2) Now, we have $X' = \{c_1, c_2, ..., c_t\} \subseteq X \setminus Y$ and $X \setminus Y'\alpha^{-1} = C_1 \cup C_2 \cup ... \cup C_t$. Suppose that there is $c_p \in X' \cap Y'\alpha^{-1}$. Then $c_p \in A_{i_0}$ for some i_0. Thus $C_p\alpha^2 = (C_p\alpha)\alpha = \{c_p\}\alpha = \{a_{i_0}\}$. Since $Y\alpha^2 = Y\alpha$, there is A_q such that $A_q\alpha^2 = \{a_{i_0}\}$, so we get $C_p\alpha^2 = \{a_{i_0}\} = A_q\alpha^2$ for some q, which gives $|X\alpha^2| \leq n + (t-1) < |X\alpha|$ which implies $X\alpha^2 \subsetneq X\alpha$, a contradiction. Therefore, $X' \cap Y'\alpha^{-1} = \emptyset$, hence $X' \subseteq X \setminus Y'\alpha^{-1}$.

(3) Assume that $|y'\alpha^{-1} \cap Y'| \neq 1$ for some $y' \in Y'$. Suppose that there are a_i and a_j both belong to A_l for some $l \in \{1, 2, ..., n\}$ where $a_i \neq a_j \in Y'$. Then $A_i\alpha^2 = (A_i\alpha)\alpha = \{a_i\}\alpha = \{a_l\}$ and $A_j\alpha^2 = (A_j\alpha)\alpha = \{a_j\}\alpha = \{a_l\}$. Thus $A_i\alpha^2 = A_j\alpha^2$, so $|Y\alpha^2| \leq n-1$ where $|Y\alpha| = n$, a contradiction. So a_i and a_j belong to different A_l for all $i \neq j$...(∗). If there exists A_p such that $Y' \cap A_p = \emptyset$, then there are $a_i, a_j \in Y'$ such that $a_i \neq a_j$ and $a_i, a_j \in A_l$ for some l which contradicts (∗). Therefore $Y' \cap A_l \neq \emptyset$ for all l and that $|y'\alpha^{-1} \cap Y'| = 1$ for all $y' \in Y'$.

(4) Assume that $|x'\alpha^{-1} \cap X'| \neq 1$ for some $x' \in X'$. Suppose that there are c_u and $c_v \in C_k$ for some $k \in \{1, 2, ..., t\}$ where $c_u \neq c_v \in X'$. Then $C_u\alpha^2 = (C_u\alpha)\alpha = \{c_u\}\alpha = \{c_k\}$ and $C_v\alpha^2 = (C_v\alpha)\alpha = \{c_v\}\alpha = \{c_k\}$ and thus $C_u\alpha^2 = C_v\alpha^2$, a contradiction, since $X\alpha^2 = X\alpha$. So c_u and c_v belong to different C_k for all $u \neq v$...(∗∗). If there exist C_q such that $X' \cap C_q = \emptyset$, then there are $c_u, c_v \in X'$ such that $c_u \neq c_v$ and $c_u, c_v \in C_k$ for some k which contradicts (∗∗). Hence $X' \cap C_k \neq \emptyset$ for all k. So $|x'\alpha^{-1} \cap X'| = 1$ for all $x' \in X'$. □

The following examples show that Lemma 3.1.4 does not hold if $X\alpha$ is an infinite set.

Example 3.1.5. Let X be the set of all natural numbers and $Y = \{1, 2, 3, 4\}$.

Let α be defined by

$$n\alpha = \begin{cases} n & , \ n \in Y \setminus \{4\}; \\ 3 & , \ n = 4; \\ n-1 & , \ \text{otherwise.} \end{cases}$$

Then $\alpha \in S(X, Y)$ and

$$n\alpha^2 = \begin{cases} n & , \ n \in \{1,2,3\}; \\ 3 & , \ n \in \{4,5\}; \\ n-2 & , \ \text{otherwise.} \end{cases}$$

Thus we have $X\alpha = X\alpha^2$ is the set of all natural numbers, $Y\alpha = \{1,2,3\} = Y\alpha^2$, $X \setminus Y = \{n \in X : n \geq 5\}$ and $Y' = \{1,2,3\}$. We see that $5 \in X \setminus Y$ such that $5\alpha = 4$, that is, there is $x \in X \setminus Y$ such that $x\alpha \notin (X \setminus Y) \cup Y'$.

Example 3.1.6. Let X be the set of all natural numbers and $Y = \{1,2,3,4\}$. Let α be defined by

$$n\alpha = \begin{cases} n & , \ n \in \{1,2\}; \\ 4 & , \ n = 3; \\ 3 & , \ n \in \{4,5\}; \\ n-1 & , \ \text{otherwise.} \end{cases}$$

Then $\alpha \in S(X, Y)$ and

$$n\alpha^2 = \begin{cases} n & , \ n \in Y; \\ 4 & , \ n = 5; \\ 3 & , \ n = 6; \\ n-2 & , \ \text{otherwise.} \end{cases}$$

Thus we have $X\alpha = X\alpha^2$ is the set of all natural numbers, $Y\alpha = \{1,2,3,4\} = Y\alpha^2$, $Y' = \{1,2,3,4\}$ and $X' = \{n \in X : n \geq 5\}$. Thus $Y'\alpha^{-1} = \{1,2,3,4,5\}$. We see that $5 \in 3\alpha^{-1}$, that is, there is $x' \in X'$ such that $x' \notin X \setminus Y'\alpha^{-1}$.

Example 3.1.7. Let X be the set of all natural numbers and Y the set of all positive even integers. Let α be defined by

$$n\alpha = \begin{cases} n & , \ n \in \{1,2,3,5\}; \\ 2 & , \ n = 4; \\ 5 & , \ n = 7; \\ n-2 & , \ \text{otherwise.} \end{cases}$$

Then $\alpha \in S(X,Y)$ and

$$n\alpha^2 = \begin{cases} n & , \ n \in \{1,2,3,5\}; \\ 2 & , \ n \in \{4,6\}; \\ 5 & , \ n \in \{7,9\}; \\ n-4 & , \ \text{otherwise.} \end{cases}$$

Thus we have $X\alpha = X\alpha^2$ is the set of all natural numbers, $Y\alpha = Y\alpha^2$ is the set of all positive even integers, $Y' = Y$ and X' is the set of all positive odd integers. We see that $2\alpha^{-1} = \{2,4\}$ such that $2\alpha^{-1} \cap Y' = \{2,4\}$, that is, there is $y' \in Y'$ such that $|y'\alpha^{-1} \cap Y'| \geq 2$. And $5\alpha^{-1} = \{5,7\}$ such that $5\alpha^{-1} \cap X' = \{5,7\}$, that is, there is $x' \in X'$ such that $|x'\alpha^{-1} \cap X'| \geq 2$.

Lemma 3.1.8. *Let $\alpha \in S(X,Y)$. If $\pi_\alpha = \pi_{\alpha^2}$ is finite and $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$, then $\pi_\alpha(Y\alpha) = \pi_{\alpha^2}(Y\alpha)$.*

Proof. Assume that $\pi_\alpha = \pi_{\alpha^2}$ is finite and $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$. We have $\pi_\alpha(Y\alpha) = \{y\alpha^{-1} : y \in X\alpha \cap Y\alpha\}$ and $\pi_{\alpha^2}(Y\alpha) = \{y(\alpha^2)^{-1} : y \in X\alpha^2 \cap Y\alpha\}$. Since π_α is finite, we can write

$$\alpha = \begin{pmatrix} A_1 \ A_2 & \ldots & A_n & B_1 \ B_2 & \ldots & B_m & C_1 \ C_2 & \ldots & C_t \\ a_1 \ a_2 & \ldots & a_n & b_1 \ b_2 & \ldots & b_m & c_1 \ c_2 & \ldots & c_t \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$, $B_j, C_k \subseteq X \backslash Y$; $a_i, b_j \in Y, c_k \in X \backslash Y$ for all $i = 1,2,...,n$, $j = 1,2,...,m$ and $k = 1,2,...,t$. Since $\pi_\alpha = \pi_{\alpha^2}$ and $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$, we have

$$\alpha^2 = \begin{pmatrix} A_1 \ A_2 & \ldots & A_n & B_1 \ B_2 & \ldots & B_m & C_1 \ C_2 & \ldots & C_t \\ a_1' \ a_2' & \ldots & a_n' & b_1' \ b_2' & \ldots & b_m' & c_1' \ c_2' & \ldots & c_t' \end{pmatrix}$$

where $a_i^{'}, b_j^{'} \in Y$ and $c_k^{'} \in X \setminus Y$. So $Y\alpha^2 = \{a_1^{'}, a_2^{'}, ..., a_n^{'}\}$, and we have $Y\alpha = \{a_1, a_2, ..., a_n\}$. Since $Y\alpha \subseteq Y$, we obtain $Y\alpha^2 \subseteq Y\alpha$. So $\{a_1^{'}, a_2^{'}, ..., a_n^{'}\} \subseteq \{a_1, a_2, ..., a_n\}$. Since $Y\alpha$ is finite, $\{a_1^{'}, a_2^{'}, ..., a_n^{'}\} = \{a_1, a_2, ..., a_n\}$. Thus $X\alpha \cap Y\alpha = X\alpha^2 \cap Y\alpha$. Therefore, $\pi_\alpha(Y\alpha) = \{A_1, A_2 ..., A_n\} = \pi_{\alpha^2}(Y\alpha)$. □

Lemma 3.1.9. *Let* $\alpha \in S(X, Y)$. *If* $\pi_\alpha = \pi_{\alpha^2}$ *is finite and* $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$, *then*

(1) $(X \setminus Y)\alpha \subseteq (X \setminus Y) \cup Y^{'}$;

(2) $X^{'} \subseteq X \setminus Y^{'}\alpha^{-1}$;

(3) $|y^{'}\alpha^{-1} \cap Y^{'}| = 1$ *for all* $y^{'} \in Y^{'}$;

(4) $|x^{'}\alpha^{-1} \cap X^{'}| = 1$ *for all* $x^{'} \in X^{'}$.

Proof. Let $\alpha \in S(X, Y)$ and $\pi_\alpha = \pi_{\alpha^2}$ is finite. So we can write

$$\alpha = \begin{pmatrix} A_1 & A_2 & ... & A_n & B_1 & B_2 & ... & B_m & C_1 & C_2 & ... & C_t \\ a_1 & a_2 & ... & a_n & b_1 & b_2 & ... & b_m & c_1 & c_2 & ... & c_t \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$, $B_j, C_k \subseteq X \setminus Y$; $a_i, b_j \in Y, c_k \in X \setminus Y$ for all $i = 1, 2, ..., n$, $j = 1, 2, ..., m$ and $k = 1, 2, ..., t$. Then $Y^{'} = \{a_1, a_2, ..., a_n\}$ and $X^{'} = \{b_1, b_2, ..., b_m, c_1, c_2, ..., c_t\}$. Since $\pi_\alpha = \pi_{\alpha^2}$ is finite and $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$, we have by Lemma 3.1.8 that $\pi_\alpha(Y\alpha) = \pi_{\alpha^2}(Y\alpha)$. So $\pi_{\alpha^2}(Y\alpha) = \{A_1, ..., A_n\}$.

(1) We prove that $J = \emptyset$ where $J = \{1, 2, ..., m\}$.

Suppose that there is $j \in J$ such that $B_j\alpha = \{b_j\}$ where $b_j \in Y$. Then $B_j\alpha^2 = (B_j\alpha)\alpha = \{b_j\}\alpha$. Assume that $b_j \in A_{i_0}$ for some i_0, then $B_j\alpha^2 = \{b_j\}\alpha = \{a_{i_0}\}$. Since $a_{i_0} \in X\alpha^2$ and $a_{i_0} \in Y\alpha$, we obtain $a_{i_0} \in X\alpha^2 \cap Y\alpha$, so $a_{i_0}(\alpha^2)^{-1} \in \pi_{\alpha^2}(Y\alpha) = \{A_1, ..., A_n\}$. Thus there is $A_k \in \pi_{\alpha^2}(Y\alpha)$ such that $A_k\alpha^2 = \{a_{i_0}\}$. Hence $B_j\alpha^2 = \{a_{i_0}\} = A_k\alpha^2$ and therefore $B_j \cup A_k \subseteq a_{i_0}(\alpha^2)^{-1}$. That means $\pi_{\alpha^2}(Y) \neq \pi_\alpha(Y)$ which is a contradiction. Then $J = \emptyset$. So we can write

$$\alpha = \begin{pmatrix} A_1 & A_2 & ... & A_n & C_1 & C_2 & ... & C_t \\ a_1 & a_2 & ... & a_n & c_1 & c_2 & ... & c_t \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$, $C_k \subseteq X \setminus Y$; $a_i \in Y, c_k \in X \setminus Y$ for all $i = 1, 2, ..., n$ and $k = 1, 2, ..., t$. Then $Y^{'} = \{a_1, a_2, ..., a_n\}$ and $X^{'} = \{c_1, c_2, ..., c_t\}$. Therefore, $(X \setminus Y)\alpha \subseteq (X \setminus Y) \cup Y^{'}$.

(2) We have $X^{'} = \{c_1, c_2, ..., c_t\} \subseteq X \backslash Y$ and $X \backslash Y^{'}\alpha^{-1} = C_1 \cup C_2 \cup ... \cup C_t$. Suppose that there is $c_p \in X^{'} \cap Y^{'}\alpha^{-1}$. So $c_p \in A_{i_0}$ for some i_0. Then $C_p\alpha^2 = (C_p\alpha)\alpha = \{c_p\}\alpha = \{a_{i_0}\}$. Since $a_{i_0} \in X\alpha^2$ and $a_{i_0} \in Y\alpha$, we obtain $a_{i_0} \in X\alpha^2 \cap Y\alpha$, so $a_{i_0}(\alpha^2)^{-1} \in \pi_{\alpha^2}(Y\alpha) = \{A_1, A_2, ..., A_n\}$. Thus there is $A_k \in \pi_{\alpha^2}(Y\alpha)$ such that $A_k\alpha^2 = \{a_{i_0}\}$. Hence $C_p\alpha^2 = \{a_{i_0}\} = A_k\alpha^2$ and therefore $C_p \cup A_k \subseteq a_{i_0}(\alpha^2)^{-1}$, which implies that $\pi_{\alpha^2}(Y) \neq \pi_\alpha(Y)$, a contradiction. Hence $X^{'} \cap Y^{'}\alpha^{-1} = \emptyset$. Therefore, $X^{'} \subseteq X \backslash Y^{'}\alpha^{-1}$.

(3) Assume that $|y^{'}\alpha^{-1} \cap Y^{'}| \neq 1$ for some $y^{'} \in Y^{'}$. Suppose that there are a_i and a_j both belong to A_l for some $l \in \{1, 2, ..., n\}$ where $a_i \neq a_j \in Y^{'}$. Then $A_i\alpha^2 = (A_i\alpha)\alpha = \{a_i\}\alpha = \{a_l\}$ and $A_j\alpha^2 = (A_j\alpha)\alpha = \{a_j\}\alpha = \{a_l\}$. Thus $A_i\alpha^2 = \{a_l\} = A_j\alpha^2$, so $A_i \cup A_j \subseteq a_l(\alpha^2)^{-1}$. Since $a_l \in X\alpha^2 \cap Y$, we obtain $a_l(\alpha^2)^{-1} \in \pi_{\alpha^2}(Y)$, so $|\pi_{\alpha^2}(Y)| \leq n-1$ where $|\pi_\alpha(Y)| = n$. Thus $\pi_\alpha(Y) \neq \pi_{\alpha^2}(Y)$, which is a contradiction. So a_i and a_j belong to different A_l for all $i \neq j ...(*)$. If there exists A_t such that $Y^{'} \cap A_t = \emptyset$, then there are $a_i, a_j \in Y^{'}$ such that $a_i \neq a_j$ and $a_i, a_j \in A_l$ for some l which contradicts $(*)$. Therefore $Y^{'} \cap A_l \neq \emptyset$ for all l and that $|y^{'}\alpha^{-1} \cap Y^{'}| = 1$ for all $y^{'} \in Y^{'}$.

(4) Assume that $|x^{'}\alpha^{-1} \cap X^{'}| \neq 1$ for some $x^{'} \in X^{'}$. Suppose that there are c_u and $c_v \in C_k$ for some $k \in \{1, 2, ...t\}$ where $c_u \neq c_v \in X^{'}$. Then $C_u\alpha^2 = (C_u\alpha)\alpha = \{c_u\}\alpha = \{c_k\}$ and $C_v\alpha^2 = (C_v\alpha)\alpha = \{c_v\}\alpha = \{c_k\}$. So $C_u\alpha^2 = \{c_k\} = C_v\alpha^2$ and thus $C_u \cup C_v \subseteq c_k(\alpha^2)^{-1}$. This implies that $|\pi_{\alpha^2}| \leq n + (t-1) < |\pi_\alpha|$ which is a contradiction. So c_u and c_v belong to different C_k for all $u \neq v ...(**)$. If there exists C_q such that $X^{'} \cap C_q = \emptyset$, then there are $c_u, c_v \in X^{'}$ such that $c_u \neq c_v$ and $c_u, c_v \in C_k$ for some k which contradicts $(**)$. Therefore $X^{'} \cap C_k \neq \emptyset$ for all k and that $|x^{'}\alpha^{-1} \cap X^{'}| = 1$ for all $x^{'} \in X^{'}$. □

The following examples show that Lemma 3.1.9 dose not hold if π_α is an infinite set.

Example 3.1.10. Let X be the set of all natural numbers and Y the set of all positive even integers. Let α be defined by

$$n\alpha = 2n \text{ for all } n \in X.$$

Then $\alpha \in S(X, Y)$ and

$$n\alpha^2 = 4n \text{ for all } n \in X.$$

Thus we have $\pi_\alpha = \{\{n\}\} = \pi_{\alpha^2}$, $\pi_\alpha(Y) = \{\{n\}\} = \pi_{\alpha^2}(Y)$, $Y' = \{4n\}$, $X' = \{2, 4n+2\}$ and $Y'\alpha^{-1} = Y$. We see that $(X \backslash Y)\alpha = \{2, 4n+2\} \nsubseteq (X \setminus Y) \cup Y'$ and there is $2 \in X'$ such that $2 \in Y'\alpha^{-1}$, so $2 \notin X \setminus Y'\alpha^{-1}$.

Example 3.1.11. Let X be the set of all natural numbers and Y the set of all positive even integers. Let α be defined by

$$n\alpha = n + 2 \text{ for all } n \in X.$$

Then $\alpha \in S(X, Y)$ and

$$n\alpha^2 = n + 4 \text{ for all } n \in X.$$

Thus we have $\pi_\alpha = \{\{n\}\} = \pi_{\alpha^2}$, $\pi_\alpha(Y) = \{\{2n\}\} = \pi_{\alpha^2}(Y)$ and $Y' = \{2n+2\}$. We see that $4 \in Y'$ such that $4\alpha^{-1} \cap \{2n+2\} = \{2\} \cap \{2n+2\} = \emptyset$, so $|y'\alpha^{-1} \cap Y'| \neq 1$.

Example 3.1.12. Let X be the set of all natural numbers and Y the set of all positive even integers. Let α be defined by

$$n\alpha = \begin{cases} 2n & , n \in Y; \\ 2 & , \text{otherwise}. \end{cases}$$

Then $\alpha \in S(X, Y)$ and

$$n\alpha^2 = \begin{cases} 4n & , n \in Y; \\ 4 & , \text{otherwise}. \end{cases}$$

Thus we have $\pi_\alpha = \{\{2n\}, X \setminus Y\} = \pi_{\alpha^2}$, $\pi_\alpha(Y) = \{\{2n\}, X \setminus Y\} = \pi_{\alpha^2}(Y)$ and $X' = \{2\}$. We see that $2 \in X'$ such that $2\alpha^{-1} \cap \{2\} = (X \setminus Y) \cap \{2\} = \emptyset$, so $|x'\alpha^{-1} \cap X'| \neq 1$.

Corollary 3.1.13. *Let X be a finite set and $\alpha \in S(X, Y)$. If α is left regular, then α is regular.*

Proof. Assume that α is left regular. Then $X\alpha = X\alpha^2$ and $Y\alpha = Y\alpha^2$. Since $Y\alpha \subseteq Y$ and $Y\alpha \subseteq X\alpha$, we obtain $Y\alpha \subseteq X\alpha \cap Y$. Now, we prove that $X\alpha \cap Y \subseteq Y\alpha$ by letting $y \in X\alpha \cap Y$. If $y \notin Y\alpha$, then $y \in (X \setminus Y)\alpha \subseteq (X \setminus Y) \cup Y'$ by Lemma 3.1.4(1). Thus $y \in X \setminus Y$ or $y \in Y' = im\ \alpha|_Y$. Since $y \in Y$, it follows that

$y \in Y' = Y\alpha$ which is a contradiction. Therefore, $y \in Y\alpha$ and that $X\alpha \cap Y \subseteq Y\alpha$. So $X\alpha \cap Y = Y\alpha$ which implies that α is regular by Theorem 2.7.1. □

The condition X is a finite set in Corollary 3.1.13 is necessary as shown in the example below.

Example 3.1.14. Let X be the set of all natural numbers and $Y = \{1, 2, 3, 4\}$. Let α be defined by

$$n\alpha = \begin{cases} n & , \; n \in \{1, 2, 3\}; \\ 3 & , \; n = 4; \\ n - 1 & , \; \text{otherwise}. \end{cases}$$

Then $\alpha \in S(X, Y)$ and

$$n\alpha^2 = \begin{cases} n & , \; n \in \{1, 2, 3\}; \\ 3 & , \; n \in \{4, 5\}; \\ n - 2 & , \; \text{otherwise}. \end{cases}$$

Thus we have $X\alpha = X\alpha^2$ is the set of all natural numbers and $Y\alpha = \{1, 2, 3\} = Y\alpha^2$. So α is left regular. But α is not regular, since $X\alpha \cap Y = \{1, 2, 3, 4\} \neq \{1, 2, 3\} = Y\alpha$.

The following example shows that the converse of Corollary 3.1.13 does not hold.

Example 3.1.15. Let $X = \{1, 2, 3, 4\}$ and $Y = \{1, 2\}$. Let α be defined by

$$\alpha = \begin{pmatrix} 1 & \{2, 4\} & 3 \\ 2 & 1 & 4 \end{pmatrix}.$$

Then $\alpha \in S(X, Y)$ and

$$\alpha^2 = \begin{pmatrix} \{1, 3\} & \{2, 4\} \\ 1 & 2 \end{pmatrix}.$$

Thus we have $X\alpha \cap Y = \{1, 2\} = Y\alpha$. So α is regular. But α is not left regular, since $X\alpha = \{1, 2, 4\} \neq \{1, 2\} = X\alpha^2$.

Corollary 3.1.16. *Let X be a finite set and $\alpha \in S(X,Y)$. If α is right regular, then α is regular.*

Proof. Assume that α is right regular. Then $\pi_\alpha = \pi_{\alpha^2}$ is finite and $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$. By using the same proof as given for Corollary 3.1.13 and Lemma 3.1.9(1), we obtain $X\alpha \cap Y = Y\alpha$, hence α is regular as required. □

The condition X is a finite set in Corollary 3.1.16 is necessary as shown in the example below.

Example 3.1.17. Let X be the set of all natural numbers and Y the set of all positive even integers. Let α be defined by

$$n\alpha = \begin{cases} 2 & , \ n = 2; \\ 2n & , \ n \in Y \setminus \{2\}; \\ 4 & , \ \text{otherwise.} \end{cases}$$

Then $\alpha \in S(X,Y)$ and

$$n\alpha^2 = \begin{cases} 2 & , \ n = 2; \\ 4n & , \ n \in Y \setminus \{2\}; \\ 8 & , \ \text{otherwise.} \end{cases}$$

Thus we have $\pi_\alpha = \{\{2n\}, X \setminus Y\} = \pi_{\alpha^2}$ and $\pi_\alpha(Y) = \{\{2n\}, X \setminus Y\} = \pi_{\alpha^2}(Y)$. So α is right regular. But α is not regular, since $X\alpha \cap Y = \{4n\} \cup \{2\} \neq (\{4n\} \setminus \{4\}) \cup \{2\} = Y\alpha$.

The following example shows that the converse of Corollary 3.1.16 does not hold.

Example 3.1.18. Let $X = \{1,2,3,4\}$ and $Y = \{1,2\}$. Let α be defined by

$$\alpha = \begin{pmatrix} 1 & \{2,3\} & 4 \\ 2 & 1 & 3 \end{pmatrix}.$$

Then $\alpha \in S(X,Y)$ and

$$\alpha^2 = \begin{pmatrix} \{1,4\} & \{2,3\} \\ 1 & 2 \end{pmatrix}.$$

Thus we have $X\alpha \cap Y = \{1,2\} = Y\alpha$. So α is regular. But α is not right regular, since $\pi_\alpha = \{\{1\},\{2,3\},\{4\}\} \neq \{\{1,4\},\{2,3\}\} = \pi_{\alpha^2}$.

Theorem 3.1.19. *Let $\alpha \in S(X,Y)$ be such that $X\alpha$ is finite. Then α is left regular if and only if α is right regular.*

Proof. Assume that α is left regular. Then $X\alpha = X\alpha^2$ and $Y\alpha = Y\alpha^2$. Since $X\alpha$ is finite, we may write $X\alpha = \{a_1, a_2, ..., a_n, c_1, c_2, ..., c_t\}$ where $Y' = Y\alpha = \{a_1, a_2, ..., a_n\}$. By Lemma 3.1.4(1), we can write

$$\alpha = \begin{pmatrix} A_1 & A_2 & ... & A_n & C_1 & C_2 & ... & C_t \\ a_1 & a_2 & ... & a_n & c_1 & c_2 & ... & c_t \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$, $C_k \subseteq X \setminus Y$; $a_i \in Y, c_k \in X \setminus Y$ for all $i = 1,2,...,n$ and $k = 1,2,...,t$. Since $|y'\alpha^{-1} \cap Y'| = |A_i \cap \{a_1, a_2, ..., a_n\}| = 1$ for all $i = 1,2,...,n$, there is a permutation σ on the set $\{1,2,...,n\}$ such that $a_i \in A_{i\sigma}$ for all i. So we obtain

$$A_i\alpha^2 = (A_i\alpha)\alpha = \{a_i\}\alpha = \{a_{i\sigma}\}.$$

Similarly, since $|x'\alpha^{-1} \cap X'| = |C_k \cap \{c_1, c_2, ..., c_t\}| = 1$ for all $k = 1,2,...,t$, we obtain there is a permutation δ on the set $\{1,2,...,t\}$ such that $c_k \in C_{k\delta}$ for all k. Thus

$$C_k\alpha^2 = (C_k\alpha)\alpha = \{c_k\}\alpha = \{c_{k\delta}\}.$$

So

$$\alpha^2 = \begin{pmatrix} A_1 & A_2 & ... & A_n & C_1 & C_2 & ... & C_t \\ a_{1\sigma} & a_{2\sigma} & ... & a_{n\sigma} & c_{1\delta} & c_{2\delta} & ... & c_{t\delta} \end{pmatrix}.$$

Thus $\pi_\alpha(Y) = \{A_1, A_2, ..., A_n\} = \pi_{\alpha^2}(Y)$ and $\pi_\alpha = \{A_1, A_2, ..., A_n, C_1, C_2, ..., C_t\} = \pi_{\alpha^2}$. Hence α is right regular.

Conversely, assume that α is right regular. Then $\pi_\alpha = \pi_{\alpha^2}$ and $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$. Since $X\alpha$ is finite, we obtain $\pi_\alpha = \pi_{\alpha^2}$ is finite. Then by Lemma 3.1.9(1), we can write

$$\alpha = \begin{pmatrix} A_1 & A_2 & ... & A_n & C_1 & C_2 & ... & C_t \\ a_1 & a_2 & ... & a_n & c_1 & c_2 & ... & c_t \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$, $C_k \subseteq X \setminus Y$; $a_i \in Y, c_k \in X \setminus Y$ for all $i = 1, 2, ..., n$ and $k = 1, 2, ..., t$. Since $|y^{'}\alpha^{-1} \cap Y^{'}| = 1$ for all $y^{'} \in Y^{'}$ and $|x^{'}\alpha^{-1} \cap X^{'}| = 1$ for all $x^{'} \in X^{'}$. So by the same proof as given above

$$\alpha^2 = \begin{pmatrix} A_1 & A_2 & ... & A_n & C_1 & C_2 & ... & C_t \\ a_{1\sigma} & a_{2\sigma} & ... & a_{n\sigma} & c_{1\delta} & c_{2\delta} & ... & c_{t\delta} \end{pmatrix}$$

where σ is a permutation on the set $\{1, 2, ..., n\}$ and δ is a permutation on the set $\{1, 2, ..., t\}$. Thus $Y\alpha = \{a_1, a_2, ..., a_n\} = Y\alpha^2$ and $X\alpha = \{a_1, a_2, ..., a_n, c_1, c_2, ..., c_t\} = X\alpha^2$. Hence α is left regular. □

If $X\alpha$ is not a finite set, then Theorem 3.1.19 may not hold as shown in the following examples.

Example 3.1.20. Let X be the set of all natural numbers and $Y = \{1, 2, 3, 4\}$. Let α be defined by

$$n\alpha = \begin{cases} n & , n \in \{1, 2\}; \\ 4 & , n = 3; \\ 3 & , n \in \{4, 5\}; \\ n - 1 & , \text{otherwise.} \end{cases}$$

Then $\alpha \in S(X, Y)$ and

$$n\alpha^2 = \begin{cases} n & , n \in Y; \\ 4 & , n = 5; \\ 3 & , n = 6; \\ n - 2 & , \text{otherwise.} \end{cases}$$

Thus we have $X\alpha = X\alpha^2$ is the set of all natural numbers and $Y\alpha = \{1, 2, 3, 4\} = Y\alpha^2$. So α is left regular. But α is not right regular, since

$$\pi_\alpha(Y) = \{\{1\}, \{2\}, \{3\}, \{4, 5\}\} \neq \{\{1\}, \{2\}, \{3, 6\}, \{4, 5\}\} = \pi_{\alpha^2}(Y).$$

Example 3.1.21. Let X be the set of all natural numbers and Y the set of all positive even integers. Let α be defined by

$$n\alpha = n + 2 \text{ for all } n \in X.$$

Then $\alpha \in S(X,Y)$ and

$$n\alpha^2 = n+4 \text{ for all } n \in X.$$

Thus we have

$$\pi_\alpha(Y) = \{\{2n\}\} = \pi_{\alpha^2}(Y) \text{ and } \pi_\alpha = \{\{n\}\} = \pi_{\alpha^2}.$$

So α is right regular. But α is not left regular, since

$$Y\alpha = \{2n+2\} \neq \{2n+4\} = Y\alpha^2.$$

3.2 Intra-Regular Elements

In this section, we give a necessary and sufficient condition for elements in $S(X,Y)$ to be intra-regular.

Theorem 3.2.1. *Let $\alpha \in S(X,Y)$. Then the following statements are equivalent:*

(1) α *is intra-regular.*

(2) $|X\alpha| = |X\alpha^2|, |Y\alpha| = |Y\alpha^2|$ *and* $|X\alpha \setminus Y| = |X\alpha^2 \setminus Y|$.

(3) $\alpha^2 \in J_\alpha$.

Proof. (1) $\Rightarrow$ (2) Assume that α is intra-regular. Then $\alpha = \beta\alpha^2\gamma$ for some $\beta,\gamma \in S(X,Y)$. Since $X\alpha^2 = (X\alpha)\alpha \subseteq X\alpha$, we obtain $|X\alpha^2| \le |X\alpha|$. Similarly, we have $|Y\alpha^2| \le |Y\alpha|$. Since $X\alpha^2 \subseteq X\alpha$, we get $X\alpha^2 \setminus Y \subseteq X\alpha \setminus Y$, that is $|X\alpha^2 \setminus Y| \le |X\alpha \setminus Y|$. Now, we show that $|X\alpha| \le |X\alpha^2|$, $|Y\alpha| \le |Y\alpha^2|$ and $|X\alpha \setminus Y| \le |X\alpha^2 \setminus Y|$ as follows:

$$|X\alpha| = |X\beta\alpha^2\gamma| = |(X\beta\alpha^2)\gamma| \le |X\beta\alpha^2| = |(X\beta)\alpha^2| \le |X\alpha^2|,$$

$$|Y\alpha| = |Y\beta\alpha^2\gamma| = |(Y\beta\alpha^2)\gamma| \le |Y\beta\alpha^2| = |(Y\beta)\alpha^2| \le |Y\alpha^2|, \text{ and}$$

$$\begin{aligned} |X\alpha \setminus Y| = |X\beta\alpha^2\gamma \setminus Y| &= |(X\beta)\alpha^2\gamma \setminus Y| \le |X\alpha^2\gamma \setminus Y| \\ &= |(X\alpha^2)\gamma \setminus Y|, \\ &= |[(X\alpha^2 \setminus Y) \cup (X\alpha^2 \cap Y)]\gamma \setminus Y|, \\ &= |[(X\alpha^2 \setminus Y)\gamma \cup (X\alpha^2 \cap Y)\gamma] \setminus Y|, \\ &= |[(X\alpha^2 \setminus Y)\gamma \setminus Y] \cup [(X\alpha^2 \cap Y)\gamma \setminus Y]|, \\ &= |(X\alpha^2 \setminus Y)\gamma \setminus Y| \le |(X\alpha^2 \setminus Y)\gamma| \le |X\alpha^2 \setminus Y|. \end{aligned}$$

(2) $\Rightarrow$ (3) Assume that $|X\alpha| = |X\alpha^2|$, $|Y\alpha| = |Y\alpha^2|$ and $|X\alpha \setminus Y| =$

$|X\alpha^2 \setminus Y|$. Then by Theorem 2.7.6, we obtain $\alpha\mathcal{J}\alpha^2$, that is $\alpha^2 \in J_\alpha$.

(3) $\Rightarrow$ (1) Assume that $\alpha^2 \in J_\alpha$. Then $\alpha\mathcal{J}\alpha^2$ and hence $\alpha = \beta\alpha^2\gamma$ for some $\beta, \gamma \in S(X,Y)^1 = S(X,Y)$. Thus α is intra-regular. □

The following two corollaries are direct consequences of Theorem 3.1.1, Theorem 3.1.2 and Theorem 3.2.1.

Corollary 3.2.2. *Let $\alpha \in S(X,Y)$. If α is left regular, then α is intra-regular.*

Proof. Assume that α is left regular. Then by Theorem 3.1.1, $\alpha^2 \in L_\alpha$. Since $\mathcal{L} \subseteq \mathcal{J}$, we may have $\alpha^2 \in L_\alpha \subseteq J_\alpha$ and thus α is intra-regular by Theorem 3.2.1. □

Corollary 3.2.3. *Let $\alpha \in S(X,Y)$. If α is right regular, then α is intra-regular.*

Proof. Assume that α is right regular. Then by Theorem 3.1.2, $\alpha^2 \in R_\alpha$. Since $\mathcal{R} \subseteq \mathcal{J}$, we may have $\alpha^2 \in R_\alpha \subseteq J_\alpha$ and thus α is intra-regular by Theorem 3.2.1. □

Theorem 3.2.4. *Let $\alpha \in S(X,Y)$ be such that $X\alpha$ is a finite set. Then the following statements are equivalent:*

(1) *α is left regular.*

(2) *α is right regular.*

(3) *α is intra-regular.*

Proof. (1) $\Longleftrightarrow$ (2) follows from Theorem 3.1.19.

(2) $\Rightarrow$ (3) is Corollary 3.2.3.

(3) $\Rightarrow$ (1) Assume that α is intra-regular. Then $|X\alpha| = |X\alpha^2|$ and $|Y\alpha| = |Y\alpha^2|$. Since $X\alpha^2 \subseteq X\alpha$, $|X\alpha^2| = |X\alpha|$ and $X\alpha$ is finite, we obtain $X\alpha = X\alpha^2$. Similarly, we have $Y\alpha = Y\alpha^2$. Therefore, α is left regular by Theorem 3.1.1. □

Remark 3.2.5. Since left regular, right regular and intra-regular elements of $S(X,Y)$ are the same when X is a finite set, and by Theorem 3.1.3, we obtain the numbers of left regular elements, right regular elements, intra-regular elements and completely regular elements are equal.

The following examples show that the finiteness of $X\alpha$ in Theorem 3.2.4 is necessary.

Example 3.2.6. Let X be the set of all natural numbers and Y the set of all positive even integers. Let α be defined by

$$n\alpha = n + 4 \text{ for all } n \in X.$$

Then $\alpha \in S(X, Y)$ and

$$n\alpha^2 = n + 8 \text{ for all } n \in X.$$

Thus we have

$$|X\alpha| = \aleph_0 = |X\alpha^2|,\ |Y\alpha| = \aleph_0 = |Y\alpha^2| \text{ and } |X\alpha \setminus Y| = \aleph_0 = |X\alpha^2 \setminus Y|.$$

So α is intra-regular. But α is not left regular, since

$$Y\alpha = \{2n + 4\} \neq \{2n + 8\} = Y\alpha^2.$$

Example 3.2.7. Let X be the set of all natural numbers and Y the set of all positive even integers. Let α be defined by

$$n\alpha = \begin{cases} n & , \ n \in \{1, 2, 3, 5\}; \\ 2 & , \ n = 4; \\ n - 2 & , \ \text{otherwise.} \end{cases}$$

Then $\alpha \in S(X, Y)$ and

$$n\alpha^2 = \begin{cases} n & , \ n \in \{1, 2, 3, 5\}; \\ 2 & , \ n \in \{4, 6\}; \\ 5 & , \ n \in \{7, 9\}; \\ n - 4 & , \ \text{otherwise.} \end{cases}$$

Thus we have

$$|X\alpha| = \aleph_0 = |X\alpha^2|,\ |Y\alpha| = \aleph_0 = |Y\alpha^2| \text{ and } |X\alpha \setminus Y| = \aleph_0 = |X\alpha^2 \setminus Y|.$$

So α is intra-regular. But α is not right regular, since $\pi_\alpha(Y) = \{\{2, 4\}, \{2n+4\}\} \neq \{\{2, 4, 6\}, \{2n + 6\}\} = \pi_{\alpha^2}(Y)$.

3.3 Unit Regular Elements

In this section, we give a necessary and sufficient condition for elements in $S(X,Y)$ to be unit regular. For each $\alpha \in S(X,Y)$, let

$$\begin{aligned} A &= \{a \in X\alpha \cap Y : |a\alpha^{-1}| \geq 2\}, \\ B &= \{a \in X\alpha \cap (X \setminus Y) : |a\alpha^{-1}| \geq 2\}, \\ \text{and}\quad C &= \{a \in X\alpha : |a\alpha^{-1}| \geq 2\}. \end{aligned}$$

Then $C = A \dot{\cup} B$ the disjoint union of A and B. Now, let $D_Z(\alpha) = (X \setminus X\alpha) \cap Z$ where $Z \subseteq X$. Thus

$$\begin{aligned} &D_Y(\alpha) = (X \setminus X\alpha) \cap Y, \text{ and} \\ &D_{X \setminus Y}(\alpha) = (X \setminus X\alpha) \cap (X \setminus Y). \end{aligned}$$

We begin this section with the following simple result.

Lemma 3.3.1. *Let $\alpha \in S(X,Y)$ be such that $\alpha = \alpha\beta\alpha$ for some $\beta \in S(X,Y)$. Then*

(1) $a\beta \in a\alpha^{-1}$ *for all* $a \in X\alpha \cap Y$;

(2) $c\beta \in c\alpha^{-1}$ *for all* $c \in X\alpha \cap (X \setminus Y)$.

Proof. Since $\alpha = \alpha\beta\alpha$ is a regular element in $S(X,Y)$, we get $X\alpha \cap Y = Y\alpha$ and thus we can write

$$\alpha = \begin{pmatrix} A_i & C_k \\ a_i & c_k \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset, C_k \subseteq X \setminus Y$; $a_i \in Y, c_k \in X \setminus Y$. So $X\alpha \cap Y = \{a_i\}$ and for each a_i there exists $u_i \in A_i \cap Y$ such that

$$a_i = u_i\alpha = u_i\alpha\beta\alpha = a_i\beta\alpha \text{ for all } i \in I.$$

So $a_i\beta \in a_i\alpha^{-1}$ for all $i \in I$. Thus $a\beta \in a\alpha^{-1}$ for all $a \in X\alpha \cap Y$. Similarly, we can show that $c\beta \in c\alpha^{-1}$ for all $c \in X\alpha \cap (X \setminus Y)$. □

Note that, every element in $G(X,Y)$ is unit regular, since for each $\alpha \in G(X,Y)$ there is $\alpha^{-1} \in G(X,Y)$ such that $\alpha = \alpha\alpha^{-1}\alpha$.

Theorem 3.3.2. *Let $\alpha \in S(X,Y)$ and $C = \emptyset$. Then α is unit regular if and only if $\alpha \in G(X,Y)$.*

Proof. Since $C = \emptyset$, we obtain that α is injective. If α is unit regular, then there is $\beta \in G(X, Y)$ such that $\alpha = \alpha\beta\alpha$. Let

$$\alpha = \begin{pmatrix} a_i & c_k \\ a_i^{'} & c_k^{'} \end{pmatrix}$$

where $\{a_i\} = Y, \{c_k\} = X \setminus Y$ and $\{a_i\} \cup \{c_k\} = X$. We have $X\alpha \cap Y = \{a_i^{'}\}$ and $X\alpha \cap (X \setminus Y) = \{c_k^{'}\}$. Then Lemma 3.3.1 gives that

$$\beta = \begin{pmatrix} a_i^{'} & c_k^{'} & b_j^{'} \\ a_i & c_k & b_j \end{pmatrix}$$

where $\{b_j^{'}\} = X \setminus X\alpha$. If $J \neq \emptyset$, then there is $b_j \in X = \{a_i\} \cup \{c_k\}$ such that $b_j^{'}\beta = a_i = a_i^{'}\beta$ or $b_j^{'}\beta = c_k = c_k^{'}\beta$. So β is not injective which is a contradiction. Thus $J = \emptyset$, that is $X \setminus X\alpha = \emptyset$, and α is surjective. Therefore, $\alpha \in G(X)$. If $Y\alpha \subsetneq Y$, then there exist $y \in Y \setminus Y\alpha$, that is there exist $x \in X \setminus Y$ such that $x\alpha = y$ which contradicts $X\alpha \cap Y = Y\alpha$. So $Y\alpha = Y$, hence $\alpha|_Y \in G(Y)$. Thus $\alpha \in G(X, Y)$. Now, if $\alpha \in G(X, Y)$, then it is clear that α is unit regular, since $G(X, Y)$ is a group. □

Theorem 3.3.3. *Let $\alpha \in S(X, Y)$ and $C \neq \emptyset$. If α is unit regular, then the following three conditions hold.*

(1) *α is regular.*

(2) $|D_Y(\alpha) \cup A| = |\underset{a \in A}{\cup} a\alpha^{-1} \cap Y|$.

(3) $|D_{X \setminus Y}(\alpha) \cup B| = |\underset{a \in C}{\cup} a\alpha^{-1} \cap (X \setminus Y)|$.

Proof. Assume that α is unit regular. Then α is regular and $\alpha = \alpha\beta\alpha$ for some $\beta \in G(X, Y)$. Let

$$\alpha = \begin{pmatrix} a_i & A_j & c_k & C_l \\ a_i^{'} & a_j^{'} & c_k^{'} & c_l^{'} \end{pmatrix}$$

where $a_i, a_i^{'} \in Y$; $c_k, c_k^{'} \in X \setminus Y$ and $A = \{a_j^{'}\}, B = \{c_l^{'}\}$. Suppose that $D_Y(\alpha) = \{y_p^{'}\}$ and $D_{X \setminus Y}(\alpha) = \{x_q^{'}\}$. Then by Lemma 3.3.1, we have

$$\beta = \begin{pmatrix} a_i^{'} & a_j^{'} & c_k^{'} & c_l^{'} & y_p^{'} & x_q^{'} \\ a_i & a_j & c_k & c_l & y_p & x_q \end{pmatrix}$$

where $a_j \in A_j \cap Y, c_l \in C_l, y_p \in \underset{a\in A}{\cup} a\alpha^{-1} \cap Y$ and $x_q \in \underset{a\in C}{\cup} a\alpha^{-1} \cap (X \setminus Y)$. Since β is bijective, we obtain $(\{a_j'\} \cup \{y_p'\})\beta = \underset{a\in A}{\cup} a\alpha^{-1} \cap Y$. Thus $|\{a_j'\} \cup \{y_p'\}| = |\underset{a\in A}{\cup} a\alpha^{-1} \cap Y|$, so $|D_Y(\alpha) \cup A| = |\underset{a\in A}{\cup} a\alpha^{-1} \cap Y|$. Also, since β is bijective, we obtain $(\{c_l'\} \cup \{x_q'\})\beta = \underset{a\in C}{\cup} a\alpha^{-1} \cap (X \setminus Y)$. Thus $|\{c_l'\} \cup \{x_q'\}| = |\underset{a\in C}{\cup} a\alpha^{-1} \cap (X \setminus Y)|$, so $|D_{X\setminus Y}(\alpha) \cup B| = |\underset{a\in C}{\cup} a\alpha^{-1} \cap (X \setminus Y)|$. □

Theorem 3.3.4. *Let X be a finite set and $\alpha \in S(X,Y)$. Then α is unit regular if and only if α is regular.*

Proof. If α is unit regular, then it is clear that α is regular. Now, if α is regular and X is finite, so we can write

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_n & C_1 & C_2 & \dots & C_t \\ a_1 & a_2 & \dots & a_n & c_1 & c_2 & \dots & c_t \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$, $C_k \subseteq X \setminus Y$; $a_i \in Y, c_k \in X \setminus Y$ for all $i = 1, 2, ..., n$ and $k = 1, 2, ..., t$. Suppose that $|Y| = m$ and $|X \setminus Y| = s$. Let $Y = \{a_1, a_2, ...a_n, a_{n+1}, ..., a_m\}$ and $X \setminus Y = \{c_1, c_2, ...c_t, c_{t+1}, ..., c_s\}$. Choose

$$\beta = \begin{pmatrix} a_1\ a_2 \dots a_n & a_{n+1} \dots a_m & c_1\ c_2 \dots c_t & c_{t+1} \dots c_s \\ a_1'\ a_2' \dots a_n' & a_{n+1}' \dots a_m' & c_1'\ c_2' \dots c_t' & c_{t+1}' \dots c_s' \end{pmatrix}$$

where $a_i' \in A_i \cap Y, c_k' \in C_k$ for all $i = 1, 2, ..., n$, $k = 1, 2, ..., t$, $a_{n+1}', ..., a_m' \in Y \setminus \{a_1', ..., a_n'\}$ and $c_{t+1}', ..., c_s' \in (X \setminus Y) \setminus \{c_1', ..., c_t'\}$. We see that β is injective. Since β is finite, we obtain that β is surjective. So $\beta \in G(X,Y)$ and

$$\alpha\beta\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_n & C_1 & C_2 & \dots & C_t \\ a_1 & a_2 & \dots & a_n & c_1 & c_2 & \dots & c_t \end{pmatrix} = \alpha.$$

Thus α is unit regular. □

Remark 3.3.5. By Corollary 3.3.4, we have that the number of unit regular elements of $S(X,Y)$ is equal to the number of regular elements of $S(X,Y)$. By Theorem 2.8.1, we obtain that the number of unit regular elements of $S(X,Y)$ is

$$\sum_{r=1}^{m} \binom{m}{r} r!S(m,r)(n-m+r)^{n-m}.$$

where $S(m,r)$ is the Stirling numbers of the second kind, $|X| = n$ and $|Y| = m$.

The following examples show that Corollary 3.3.4 does not hold if X is an infinite set.

Example 3.3.6. Let X be the set of all natural numbers and Y the set of all positive even integers. Let α be defined by

$$n\alpha = n + 2 \text{ for all } n \in X.$$

Then $\alpha \in S(X,Y)$. We have $X\alpha \cap Y = \{2n\} \setminus \{2\} = Y\alpha$, hence α is regular. We see that $\{1,2\} \notin X\alpha$. So α is not surjective, that is $\alpha \notin G(X,Y)$, which contradicts Theorem 3.3.2. Thus α is not unit regular.

Example 3.3.7. Let X be the set of all natural numbers and $Y = \{1,2,3,4\}$. Let α be defined by

$$n\alpha = \begin{cases} n & , n \in \{1,2\}; \\ 4 & , n = 3; \\ 3 & , n \in \{4,5\}; \\ n-1 & , \text{otherwise.} \end{cases}$$

Then $\alpha \in S(X,Y)$. We have $X\alpha \cap Y = Y = Y\alpha$, hence α is regular. We see that $D_{X\setminus Y}(\alpha) \cup B = \emptyset$ and $\underset{a\in C}{\cup} a\alpha^{-1} \cap (X \setminus Y) = \{5\}$, so $|D_{X\setminus Y}(\alpha) \cup B| = 0 \neq 1 = |\underset{a\in C}{\cup} a\alpha^{-1} \cap (X \setminus Y)|$, which contradicts Theorem 3.3.3(3). Thus α is not unit regular.

3.4 The Numbers of Left Regular, Right Regular and Intra-Regular Elements

Throughout this section, X is a finite set with n elements and $Y \subseteq X$ has r elements. To count the number of left regular elements of $S(X,Y)$, we recall that the number of combinations of n distinct things taken r at a time written $\binom{n}{r}$ is given by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} \quad .$$

That is, $\begin{pmatrix} n \\ r \end{pmatrix}$ is the number of ways that r objects can be chosen from n distinct objects.

Lemma 3.4.1. *Let $\alpha \in S(X,Y)$ be a left regular element. Then $\alpha|_Y : Y \to Y$ has $\sum_{k=1}^{r} \begin{pmatrix} r \\ k \end{pmatrix} k!k^{r-k}$ forms.*

Proof. Since $\alpha \in S(X,Y)$ is a left regular element, we obtain $X\alpha = X\alpha^2$ is finite and $Y\alpha = Y\alpha^2$. So $\alpha|_Y : Y \to Y$ since $Y\alpha \subseteq Y$. Suppose that $Y\alpha = Y^{'}$ has k elements. Let $Y^{'} = \{y_1^{'}, y_2^{'}, ..., y_k^{'}\}$. Since $\alpha \in S(X,Y)$, by Lemma 3.1.4 we can write

$$\alpha|_Y = \begin{pmatrix} B_1 & B_2 & ... & B_k \\ y_1^{'} & y_2^{'} & ... & y_k^{'} \end{pmatrix}$$

where $\bigcup_{i=1}^{k} B_i = Y$ and $|B_i \cap Y^{'}| = 1$ for all $i = 1, 2, ..., k$. Since $|B_i \cap Y^{'}| = 1$ for all i, we have

$$\alpha|_{Y'} : Y^{'} \to Y^{'} \text{ is a permutation.}$$

Thus $\alpha|_{Y'}$ can have $k!$ forms. If $Y^{'} = Y$, then $\alpha|_Y$ is a permutation on Y and there are $r!$ distinct forms. If $Y^{'} \subsetneq Y$, then $|Y \setminus Y^{'}| = r - k$. Since $Y\alpha = Y^{'}$, the number of ways of placing $r-k$ distinct elements into k distinct places is k^{r-k}. Hence in this case $\alpha|_Y$ can have $\begin{pmatrix} r \\ k \end{pmatrix} k!k^{r-k}$ forms. Since $1 \leq k \leq r$ the number of the maps $\alpha|_Y$ when α is left regular is

$$\sum_{k=1}^{r} \begin{pmatrix} r \\ k \end{pmatrix} k!k^{r-k}.$$

□

Lemma 3.4.2. *Let $\alpha \in S(X,Y)$ be a left regular element. Let $Y\alpha = Y^{'}$ and $(X \setminus Y^{'}\alpha^{-1})\alpha = X^{'} \neq \emptyset$. Then $\alpha|_{X'} : X^{'} \to X^{'}$ has $\sum_{m=1}^{n-r} \begin{pmatrix} n-r \\ m \end{pmatrix} m!$ forms.*

Proof. Since $\alpha \in S(X,Y)$ is a left regular element, we have by Lemma 3.1.4(4) that $|x^{'}\alpha^{-1} \cap X^{'}| = 1$ for all $x^{'} \in X^{'}$. Then $\alpha|_{X'} : X^{'} \to X^{'}$ is a permutation. Suppose that X' has m elements. Let $X' = \{x_1', x_2', ..., x_m'\}$ and δ is the permutation on the set $\{1, 2, ..., m\}$. So we can write

$$\alpha|_{X'} = \begin{pmatrix} x_1' & x_2' & ... & x_m' \\ x_{1\delta}' & x_{2\delta}' & ... & x_{m\delta}' \end{pmatrix}.$$

Thus $\alpha|_{X'}$ can have $m!$ forms. Since $1 \leq m \leq n-r$, the number of the maps $\alpha|_{X'}$ when α is left regular is

$$\sum_{m=1}^{n-r} \binom{n-r}{m} m!\ .$$

□

Theorem 3.4.3. *The number of left regular elements in $S(X,Y)$ is*

$$\sum_{m=0}^{n-r} \sum_{k=1}^{r} \binom{r}{k} k! k^{r-k} \binom{n-r}{m} m!\ (k+m)^{n-r-m}$$

where $|X| = n, |Y| = r$.

Proof. Let $\alpha \in S(X,Y)$ be a left regular element. Suppose that $Y\alpha = Y'$ has k elements and $(X \setminus Y'\alpha^{-1})\alpha = X'$ has m elements. Let $Y' = \{y_1', y_2', ..., y_k'\}$ and $X' = \{x_1', x_2', ..., x_m'\}$. Then by Lemma 3.1.4, we can write

$$\alpha = \begin{pmatrix} A_1 & A_2 & ... & A_k & C_1 & C_2 & ... & C_m \\ y_1' & y_2' & ... & y_k' & x_1' & x_2' & ... & x_m' \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$, $C_j \subseteq X \setminus Y$ and $|A_i \cap Y'| = 1$, $|C_j \cap X'| = 1$ for all $i = 1, 2, ..., k$ and $j = 1, 2, ..., m$.

If $X' = \emptyset$, then $\bigcup_{i=1}^{m} C_i = \emptyset$, that is

$$\alpha = \begin{pmatrix} A_1 & A_2 & ... & A_k \\ y_1' & y_2' & ... & y_k' \end{pmatrix}.$$

By Lemma 3.4.1, $\alpha|_Y$ has $\sum_{k=1}^{r} \binom{r}{k} k! k^{r-k}$ forms. Since the number of ways of placing $n-r$ elements of the set $X \setminus Y$ into k distinct places is k^{n-r}, in this case the number of left regular elements is

$$\sum_{k=1}^{r} \binom{r}{k} k! k^{r-k} k^{n-r} = \sum_{k=1}^{r} \binom{r}{k} k! k^{n-k}.$$

If $X' \neq \emptyset$, then by Lemma 3.4.2 we have the number of the maps $\alpha|_{X'}$ where α is a left regular is

$$\sum_{m=1}^{n-r} \binom{n-r}{m} m!.$$

And by Lemma 3.4.1, the number of $\alpha|_Y$ is $\sum_{k=1}^{r} \binom{r}{k} k!k^{r-k}$. So, the number of $\alpha|_{Y\cup X'}$ is

$$\left(\sum_{k=1}^{r} \binom{r}{k} k!k^{r-k}\right)\left(\sum_{m=1}^{n-r} \binom{n-r}{m} m!\right)$$

$$= \sum_{m=1}^{n-r}\sum_{k=1}^{r} \binom{r}{k} k!k^{r-k} \binom{n-r}{m} m!.$$

Now, $X \setminus (Y \cup X')$ has $n-r-m$ elements, and there are $(k+m)^{n-r-m}$ ways to placing $n-r-m$ elements into $k+m$ places. Therefore, the number of left regular elements in this case is

$$\sum_{m=1}^{n-r}\sum_{k=1}^{r} \binom{r}{k} k!k^{r-k} \binom{n-r}{m} m!(k+m)^{n-r-m}.$$

Observe that if $m = 0$, then

$$\sum_{k=1}^{r} \binom{r}{k} k!k^{r-k} \binom{n-r}{0} 0!(k+0)^{n-r-0}$$

$$= \sum_{k=1}^{r} \binom{r}{k} k!k^{r-k}(1)(1)(k)^{n-r}$$

$$= \sum_{k=1}^{r} \binom{r}{k} k!k^{n-k}.$$

Therefore, we conclude that the number of left regular elements in $S(X,Y)$ is

$$\sum_{m=0}^{n-r}\sum_{k=1}^{r} \binom{r}{k} k!k^{r-k} \binom{n-r}{m} m!(k+m)^{n-r-m}.$$

□

Example 3.4.4. Let $X = \{1,2,3,4\}$ and $Y = \{1,2\}$. Then $|X| = 4$ and $|Y| = 2$. By Theorem 3.4.3, we have the number of left regular elements in $S(X,Y)$ is

$$\sum_{m=0}^{4-2}\sum_{k=1}^{2}\binom{2}{k}k!k^{2-k}\binom{4-2}{m}m!(k+m)^{4-2-m}$$
$$=\sum_{m=0}^{2}\sum_{k=1}^{2}\binom{2}{k}k!k^{2-k}\binom{2}{m}m!(k+m)^{2-m}$$
$$=\sum_{m=0}^{2}\left[\binom{2}{1}1!\binom{2}{m}m!(1+m)^{2-m}+\binom{2}{2}2!\binom{2}{m}m!(2+m)^{2-m}\right]$$
$$=\sum_{m=0}^{2}\left[2\binom{2}{m}m!(1+m)^{2-m}+2\binom{2}{m}m!(2+m)^{2-m}\right]$$
$$=2\sum_{m=0}^{2}\binom{2}{m}m!(1+m)^{2-m}+2\sum_{m=0}^{2}\binom{2}{m}m!(2+m)^{2-m}$$
$$=2\left[\binom{2}{0}0!(1+0)^{2-0}+\binom{2}{1}1!(1+1)^{2-1}+\binom{2}{2}2!(1+2)^{2-2}\right]$$
$$+\,2\left[\binom{2}{0}0!(2+0)^{2-0}+\binom{2}{1}1!(2+1)^{2-1}+\binom{2}{2}2!(2+2)^{2-2}\right]$$
$$=2[(1)(1)+(2)(2)+(2)(1)]+2[(1)(4)+(2)(3)+(2)(1)]$$
$$=2(7)+2(12)=38.$$

If $X' = \emptyset$, then by Theorem 3.4.3, we have the number of left regular elements is

$$\sum_{k=1}^{2}\binom{2}{k}k!k^{4-k}=\binom{2}{1}1!1^{4-1}+\binom{2}{2}2!2^{4-2}$$
$$=2+8=10.$$

And there are

$$\begin{pmatrix}1 & \{2,3,4\}\\ 1 & 2\end{pmatrix},\begin{pmatrix}1 & \{2,3,4\}\\ 2 & 1\end{pmatrix},\begin{pmatrix}\{1,3,4\} & 2\\ 2 & 1\end{pmatrix},\begin{pmatrix}\{1,3,4\} & 2\\ 1 & 2\end{pmatrix},$$

$\begin{pmatrix}\{1,3\} & \{2,4\}\\ 1 & 2\end{pmatrix}$, $\begin{pmatrix}\{1,3\} & \{2,4\}\\ 2 & 1\end{pmatrix}$, $\begin{pmatrix}\{1,4\} & \{2,3\}\\ 1 & 2\end{pmatrix}$, $\begin{pmatrix}\{1,4\} & \{2,3\}\\ 2 & 1\end{pmatrix}$,
$\begin{pmatrix}\{1,2,3,4\}\\ 1\end{pmatrix}$, $\begin{pmatrix}\{1,2,3,4\}\\ 2\end{pmatrix}$.

If $X' \neq \emptyset$, then by Theorem 3.4.3, we have the number of left regular elements is

$$\sum_{m=1}^{4-2}\sum_{k=1}^{2}\binom{2}{k}k!k^{2-k}\binom{4-2}{m}m!(k+m)^{4-2-m}$$

$$\begin{aligned}
&= \sum_{m=1}^{2}\sum_{k=1}^{2}\binom{2}{k}k!k^{2-k}\binom{2}{m}m!(k+m)^{2-m}\\
&= \sum_{m=1}^{2}\left[\binom{2}{1}1!\binom{2}{m}m!(1+m)^{2-m}\right]\\
&\quad + \sum_{m=1}^{2}\left[\binom{2}{2}2!\binom{2}{m}m!(2+m)^{2-m}\right]\\
&= 2\sum_{m=1}^{2}\binom{2}{m}m!(1+m)^{2-m} + 2\sum_{m=1}^{2}\binom{2}{m}m!(2+m)^{2-m}\\
&= 2\left[\binom{2}{1}1!(1+1)^{2-1} + \binom{2}{2}2!(1+2)^{2-2}\right]\\
&\quad + 2\left[\binom{2}{1}1!(2+1)^{2-1} + \binom{2}{2}2!(2+2)^{2-2}\right]\\
&= 2[(2)(2)+(2)(1)] + 2[(2)(3)+(2)(1)]\\
&= 2(6) + 2(8) = 28.
\end{aligned}$$

They are as follows:

$\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 3 & 4\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 4 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}$,
$\begin{pmatrix}1 & 2 & \{3,4\}\\ 1 & 2 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 2 & \{3,4\}\\ 2 & 1 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 2 & \{3,4\}\\ 1 & 2 & 4\end{pmatrix}$, $\begin{pmatrix}1 & 2 & \{3,4\}\\ 2 & 1 & 4\end{pmatrix}$,

$$\begin{pmatrix}1 & \{2,4\} & 3\\ 2 & 1 & 3\end{pmatrix}, \begin{pmatrix}1 & \{2,4\} & 3\\ 1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & \{2,3\} & 4\\ 1 & 2 & 4\end{pmatrix}, \begin{pmatrix}1 & \{2,3\} & 4\\ 2 & 1 & 4\end{pmatrix},$$

$$\begin{pmatrix}\{1,2\} & 3 & 4\\ 1 & 3 & 4\end{pmatrix}, \begin{pmatrix}\{1,2\} & 3 & 4\\ 1 & 4 & 3\end{pmatrix}, \begin{pmatrix}\{1,2\} & 3 & 4\\ 2 & 3 & 4\end{pmatrix}, \begin{pmatrix}\{1,2\} & 3 & 4\\ 2 & 4 & 3\end{pmatrix},$$

$$\begin{pmatrix}\{1,3\} & 2 & 4\\ 1 & 2 & 4\end{pmatrix}, \begin{pmatrix}\{1,3\} & 2 & 4\\ 2 & 1 & 4\end{pmatrix}, \begin{pmatrix}\{1,4\} & 2 & 3\\ 1 & 2 & 3\end{pmatrix}, \begin{pmatrix}\{1,4\} & 2 & 3\\ 2 & 1 & 3\end{pmatrix},$$

$$\begin{pmatrix}\{1,2,4\} & 3\\ 2 & 3\end{pmatrix}, \begin{pmatrix}\{1,2,4\} & 3\\ 1 & 3\end{pmatrix}, \begin{pmatrix}\{1,2,3\} & 4\\ 1 & 4\end{pmatrix}, \begin{pmatrix}\{1,2,3\} & 4\\ 2 & 4\end{pmatrix},$$

$$\begin{pmatrix}\{1,2\} & \{3,4\}\\ 1 & 3\end{pmatrix}, \begin{pmatrix}\{1,2\} & \{3,4\}\\ 2 & 3\end{pmatrix}, \begin{pmatrix}\{1,2\} & \{3,4\}\\ 2 & 4\end{pmatrix}, \begin{pmatrix}\{1,2\} & \{3,4\}\\ 1 & 4\end{pmatrix}.$$