CHAPTER 2
PRELIMINARIES

In this chapter, we will briefly review some concepts and results of Semigroup Theory.

2.1 Semigroups

A *semigroup* S is a nonempty set S together with a binary operation $\cdot : S \times S \to S$ which satisfies the associative property:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \text{ for all } a, b, c \in S.$$

Let S be a semigroup. An element e of S is called *left identity* if $e \cdot a = a$ for all $a \in S$, and *right identity* if $a \cdot e = a$ for all $a \in S$. If e is both left identity and right identity, then it is called two-sided *identity*, or simply an identity.

Every semigroup has at most one identity element. A semigroup with identity is called a *monoid*. A semigroup without identity S may be embedded into a monoid simply by adjoining an element $1 \notin S$ to S and defining $1 \cdot s = s = s \cdot 1$ for all $s \in S \cup \{1\}$. The notation S^1 denotes a monoid obtained from S by adjoining an identity if necessary ($S^1 = S$ for a monoid).

2.2 Groups of Units

Let S be a semigroup with identity 1. An element $a \in S$ is called a *unit* of S if there exists $b \in S$ such that $ab = 1 = ba$.

Lemma 2.2.1. *Let S be a semigroup with identity 1 and*

$$G = \{a \in S : a \text{ is a unit of } S\}.$$

Then G is a maximal subgroup of S having 1 as the identity.

Proof. Let $a, b \in G$. Then there exist $a^{'}, b^{'} \in S$ such that $aa^{'} = 1 = a^{'}a$ and $bb^{'} = 1 = b^{'}b$. So $(ab)(b^{'}a^{'}) = 1 = (b^{'}a^{'})(ab)$, that is $ab \in G$. It is clear that $1 \in G$, thus G is a monoid. Let $c \in G$. Then $cc^{'} = 1 = c^{'}c$ for some $c^{'} \in S$, it follows that $c^{'} \in G$ and $c^{'}$ is the inverse of c. Thus G is a subgroup of S. Let $G^{'}$ be a subgroup of S containing 1 and $d \in G^{'}$. So there exists $d^{-1} \in G^{'}$ such that $dd^{-1} = 1 = d^{-1}d$, which implies that d is a unit of S and thus $G^{'} \subseteq G$. Therefore, G is a maximal subgroup of S having 1 as the identity.

We call the subgroup G of S in Lemma 2.2.1 the *group of units* of S. □

2.3 Regularity of Semigroups

An element a of a semigroup S is called *regular* if there exists x in S such that $a = axa$.

An element a of a semigroup S is called *unit regular* if there exists x which is a unit of S such that $a = axa$.

An element a of a semigroup S is called *left* [*right*] *regular* if there exists x in S such that $a = xa^2$ [$a = a^2x$].

An element a of a semigroup S is called *intra-regular* if there exists x and y in S such that $a = xa^2y$.

An element a of a semigroup S is called *completely regular* if there exists x in S such that $a = axa$ and $ax = xa$.

Theorem 2.3.1. *Let S be a semigroup and $a \in S$. Then a is completely regular if and only if a is left regular and right regular.*

Proof. Assume that a is completely regular. Then $a = axa$ and $ax = xa$ for some $x \in S$. So $a = axa = xa^2$ and $a = axa = a^2x$. Thus a is left regular and right regular.

Conversely, assume that a is left and right regular. Then there are $x, y \in S$ such that $a = xa^2$ and $a = a^2y$. So

$$aya = (xa^2)ya = x(a^2y)a = xaa = a,$$
$$axa = ax(a^2y) = a(xa^2)y = aay = a,$$

and $ay = xa^2y = xa$. Then

$$a(xay)a = (axa)ya = aya = a, \text{ and}$$
$$a(xay) = (axa)y = ay = xa = x(aya) = (xay)a.$$

Thus a is completely regular. $\square$

2.4 Green's Relations

A nonempty subset A of a semigroup S is called *a left ideal* if $SA \subseteq A$, *a right ideal* if $AS \subseteq A$, and a (two-sided) *ideal* if it is both a left and a right ideal. Then, for any element a in S,

the smallest left ideal of S containing a is $Sa \cup \{a\} = S^1a$,

the smallest right ideal of S containing a is $aS \cup \{a\} = aS^1$,

the smallest ideal of S containing a is $SaS \cup aS \cup Sa \cup \{a\} = S^1aS^1$,

which we call the *principal left ideal*, *principal right ideal* and *principal ideal generated by a*, respectively.

In 1951, J.A. Green defined the equivalence relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ on S by the rules that, for $a, b \in S$,

$$a\mathcal{L}b \text{ if and only if } S^1a = S^1b,$$
$$a\mathcal{R}b \text{ if and only if } aS^1 = bS^1 \text{ and}$$
$$a\mathcal{J}b \text{ if and only if } S^1aS^1 = S^1bS^1.$$

Then he defined the equivalence relations

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R} \text{ and } \mathcal{D} = \mathcal{L} \circ \mathcal{R},$$

and obtained that the composition of $\mathcal{L}$ and $\mathcal{R}$ is commutative. This follows that $\mathcal{D}$ is the join $\mathcal{L} \vee \mathcal{R}$, that is, $\mathcal{D}$ is the smallest equivalence relation containing $\mathcal{L} \cup \mathcal{R}$. Moreover, $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$ and $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$. But, in commutative semigroups, we have $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J}$. The relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ are called *Green's relations* on S. For each $a \in S$, we denote $\mathcal{L}-$class, $\mathcal{R}-$class, $\mathcal{H}-$class, $\mathcal{D}-$class and $\mathcal{J}-$class containing a by L_a, R_a, H_a, D_a and J_a, respectively.

Lemma 2.4.1. *Let $a, b \in S$. Then the following conditions hold.*

1. *$a\mathcal{L}b$ if and only if $a = xb$ and $b = ya$ for some $x, y \in S^1$.*

2. $a\mathcal{R}b$ *if and only if* $a = bx$ *and* $b = ay$ *for some* $x, y \in S^1$.

3. $a\mathcal{J}b$ *if and only if* $a = xby$ *and* $b = uav$ *for some* $x, y, u, v \in S^1$.

2.5 Cardinality

The *cardinality* of a set is a measure of the number of elements of the set. For example, the set $A = \{2, 4, 6\}$ contains 3 elements, and therefore A has a cardinality of 3.

The cardinality of a set A is denoted by $|A|$.

The formal definitions of cardinality depends on the notion mappings between sets:

(1) Two sets A and B have the same cardinality if there exists a bijection, that is, an injective and surjective function, from A to B. Symbolically, we write $|A| = |B|$.

(2) A has cardinality less than or equal to the cardinality of B if there exists an injective function from A into B. Symbolically, we write $|A| \leq |B|$.

(3) A has cardinality strictly less than the cardinality of B if there is an injective function, but no bijective function, from A to B. Symbolically, we write $|A| < |B|$.

2.6 Semigroups of Transformations

Let X be a set, we denote the set of all mappings from X into X by $T(X)$ and it is a semigroup under the composition of mappings: if $\alpha, \beta \in T(X)$, then $\alpha \circ \beta \in T(X)$ is defined by

$$x(\alpha \circ \beta) = (x\alpha)\beta, \ x \in X.$$

For abbreviation, we always write $\alpha\beta$ for $\alpha \circ \beta$.

It is well known that $T(X)$ is a regular semigroup, that is for each $\alpha \in T(X)$ there exists $\beta \in T(X)$ such that $\alpha = \alpha\beta\alpha$.

For a nonempty subset A of X, we let id_A denote the identity map on A. Then it is clear that id_X is the identity element of $T(X)$.

2.7 Semigroups of Transformations with Invariant Sets

Let X be a set and Y a nonempty subset of X. We defined

$$S(X,Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}.$$

For $\alpha, \beta \in S(X,Y)$, we have $Y\alpha \subseteq Y$ and $Y\beta \subseteq Y$. Then $Y\alpha\beta \subseteq Y\beta \subseteq Y$, so $\alpha\beta \in S(X,Y)$. Therefore $S(X,Y)$ is a subsemigroup of $T(X)$. From [5], we have $S(X,Y)$ is regular if and only if $X = Y$ or $|Y| = 1$. That is, in general $S(X,Y)$ is not a regular semigroup. Let

$$\text{Reg } S(X,Y) = \{\alpha \in S(X,Y) : X\alpha \cap Y = Y\alpha\}.$$

Then Reg $S(X,Y)$ is the set of all regular elements of $S(X,Y)$ by [5].

As in A. H. Clifford and G.B. Preston [1], we shall use the notation

$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix}$$

to mean that $\alpha \in T(X)$ and take as understood that the subscript i belongs to some (unmentioned) index set I, the abbreviation $\{a_i\}$ denotes $\{a_i : i \in I\}$, and that $im\ \alpha = X\alpha = \{a_i\}$ and $a_i\alpha^{-1} = X_i$ for all $i \in I$.

With the above notation, for any $\alpha \in S(X,Y)$ we can write

$$\alpha = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix},$$

where $A_i \cap Y \neq \emptyset$, $B_j, C_k \subseteq X \setminus Y$, $\{a_i\} \subseteq Y$, $\{b_j\} \subseteq Y \setminus \{a_i\}$ and $\{c_k\} \subseteq X \setminus Y$. Here, I is a nonempty set, but J or K can be empty. For examples: If $\alpha \in$ Reg $S(X,Y)$, then J is an empty set. And if $\alpha \in S(X,Y) \setminus$ Reg $S(X,Y)$, then both I and J are nonempty but K can be an empty set.

Throughout the thesis, the set X that we consider can be finite or infinite and $X = A \dot{\cup} B$ means X is a disjoint union of A and B.

We note that for any $\alpha \in S(X,Y)$, the symbol π_α will denote the decomposition of X induced by the map α, namely

$$\pi_\alpha = \{x\alpha^{-1} : x \in X\alpha\}.$$

If $Z \subseteq X$, we will denote $\pi_\alpha(Z)$ by

$$\pi_\alpha(Z) = \{x\alpha^{-1} : x \in X\alpha \cap Z\}.$$

Thus $\pi_\alpha(Y) = \{y\alpha^{-1} : y \in X\alpha \cap Y\}$. For $\alpha, \beta \in S(X,Y)$, we say that π_β refines π_α if for each $A \in \pi_\beta$ there exists $B \in \pi_\alpha$ such that $A \subseteq B$.

Also, we say that $\pi_\beta(Z)$ refines $\pi_\alpha(Z)$ if for each $A \in \pi_\beta(Z)$ there exists $B \in \pi_\alpha(Z)$ such that $A \subseteq B$.

The following theorem is given by S. Nenthein, P. Youngkhong and Y. Kemprasit in [5].

Theorem 2.7.1. *The following statements hold for the semigroup $S(X,Y)$.*

(1) *For $\alpha \in S(X,Y)$, α is a regular element of $S(X,Y)$ if and only if $X\alpha \cap Y = Y\alpha$.*

(2) *$S(X,Y)$ is regular if and only if either $Y = X$ or $|Y| = 1$.*

Green's relations on $S(X,Y)$ are given by P. Honyam and J. Sanwong [4], which are needed in characterizing left regular elements, right regular elements, intra-regular elements, unit regular elements and completely regular elements on $S(X,Y)$.

Lemma 2.7.2. *Let $\alpha, \beta \in S(X,Y)$. Then $\alpha = \gamma\beta$ for some $\gamma \in S(X,Y)$ if and only if $X\alpha \subseteq X\beta$ and $Y\alpha \subseteq Y\beta$. Consequently, $\alpha\mathcal{L}\beta$ if and only if $X\alpha = X\beta$ and $Y\alpha = Y\beta$.*

Lemma 2.7.3. *Let $\alpha, \beta \in S(X,Y)$. Then $\alpha = \beta\gamma$ for some $\gamma \in S(X,Y)$ if and only if π_β refines π_α and $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$. Consequently, $\alpha\mathcal{R}\beta$ if and only if $\pi_\beta = \pi_\alpha$ and $\pi_\beta(Y) = \pi_\alpha(Y)$.*

Corollary 2.7.4. *Let $\alpha, \beta \in S(X,Y)$. Then $\alpha\mathcal{H}\beta$ if and only if $X\alpha = X\beta$, $Y\alpha = Y\beta$ and $\pi_\alpha = \pi_\beta$, $\pi_\alpha(Y) = \pi_\beta(Y)$.*

Theorem 2.7.5. *Let $\alpha, \beta \in S(X,Y)$. Then $\alpha\mathcal{D}\beta$ if and only if $|Y\alpha| = |Y\beta|$, $|X\alpha \setminus Y| = |X\beta \setminus Y|$ and $|(X\alpha \cap Y) \setminus Y\alpha| = |(X\beta \cap Y) \setminus Y\beta|$.*

Theorem 2.7.6. *Let $\alpha, \beta \in S(X,Y)$. Then $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in S(X,Y)$ if and only if $|X\alpha| \leq |X\beta|$, $|Y\alpha| \leq |Y\beta|$ and $|X\alpha \setminus Y| \leq |X\beta \setminus Y|$. Consequently, $\alpha\mathcal{J}\beta$ if and only if $|X\alpha| = |X\beta|$, $|Y\alpha| = |Y\beta|$ and $|X\alpha \setminus Y| = |X\beta \setminus Y|$.*

We note that for any $\alpha \in S(X,Y)$, the notation $\alpha|_Z : Z \to X$ where $Z \subseteq X$ and $G(X)$ is a permutation group on X. Since $S(X,Y)$ is a semigroup with identity 1_X, the identity map on X, we obtain its group of units is as follows.

Lemma 2.7.7. *Let $G(X,Y) = \{\alpha \in G(X) : \alpha|_Y \in G(Y)\}$. Then $G(X,Y)$ is the group of units of $S(X,Y)$.*

Proof. Let $\alpha \in G(X,Y)$. Then $\alpha|_Y$ and $\alpha|_{X\setminus Y}$ are permutations on Y and $X \setminus Y$ respectively. So $\alpha^{-1} \in G(X,Y)$ and $\alpha\alpha^{-1} = 1_X = \alpha^{-1}\alpha$, that is α is a unit of $S(X,Y)$.

Conversely, assume that α is a unit of $S(X,Y)$. Then there is $\beta \in S(X,Y)$ such that $\alpha\beta = 1_X = \beta\alpha$, which implies that α is one-to-one from X onto X. Since α is injective and $Y\alpha \subseteq Y$, we obtain $\alpha|_Y : Y \to Y$ is injective. We show that $\alpha|_Y$ is surjective. If $\alpha|_Y$ is not surjective, then there exists $y' \in Y$ such that for all $y \in Y$, $y\alpha \neq y'$. Since α is surjective and $y\alpha \neq y'$ for all $y \in Y$, we have there is $z \in X \setminus Y$ such that $z\alpha = y'$. Then $z = z\alpha\beta = y'\beta \in Y$, which is a contradiction since $\alpha\beta = 1_X = \beta\alpha$. Thus $\alpha|_Y$ is surjective. Hence $\alpha|_Y \in G(Y)$. Therefore $\alpha \in G(X,Y)$. □

2.8 Stirling Numbers of the Second Kind

For positive numbers n and r with $n \geq r$, the number of partitions of $\{1, 2, ..., n\}$ into r blocks is denoted by $S(n,r)$ and is called the *Stirling numbers of the second kind.* It is known that

$$S(n,r) = \frac{1}{r!}\sum_{i=0}^{r}(-1)^i \binom{r}{i} (r-i)^n.$$

Hence the number of maps from $\{1, 2, ..., n\}$ onto {1,2,...,r} is $r!S(n,r)$.

Theorem 2.8.1. [5] *If* $|X| = n$ *and* $|Y| = m$*, then the number of regular elements in* $S(X,Y)$ *is*

$$\sum_{r=1}^{m} \binom{m}{r} r!S(m,r)(n-m+r)^{n-m}.$$