INTERPOLATION THEOREMS FOR THE CONNECTIVITY AND THE DOMINATION NUMBERS OF CONNECTED GRAPHS

Miss Aunyarat Bunyawat

A Thesis Submitted Partial Fulfillment of the Requirements
for the Degree of Master of Science in Mathematics
Faculty of Science and Technology
Thammasat University
Academic Year 2006

Thammasat University

Faculty of Science and Technology

Thesis

by

Miss Aunyarat Bunyawat

Thesis Title

INTERPOLATION THEOREMS FOR THE CONNECTIVITY AND THE DOMINATION NUMBERS OF CONNECTED GRAPHS

Accepted by the Faculty of Science and Technology in Partial Fulfillment of the Requirements for the Master's Degree.

Given on 22 December 2006

Chairman	....
	(Assistant Professor Tipaval Phattanangkul, Ph.D.)
Committee Member and Thesis Advisor	....
	(Assistant Professor Araya Chaemchan, Ph.D.)
Committee Member and Thesis Advisor	....
	(Associate Professor Narong Punnim, Ph.D.)
Committee Member	....
	(Associate Professor Wanida Hemakul, Ph.D.)
Committee Member	....
	(Varaporn Saenpholphat, Ph.D.)
Dean of the Faculty of Science and Technology	....
	(Associate Professor Somchai Viriyayudhakorn)

ABSTRACT

In this thesis we study interpolation and extremal problems for the connectivity and the domination numbers of connected graphs of order n and size m.

Let $\kappa(G)$ be the vertex-connectivity of a graph G,

$\lambda(G)$ be the edge-connectivity of a graph G,

$\gamma(G)$ be the domination number of a graph G,

$\gamma'(G)$ be the edge domination number of a graph G,

and $\mathcal{CG}(m,n)$ be the class of all nonisomorphic connected graphs of order n and size m.

We show that for $f \in \{\kappa,\ \lambda,\ \gamma,\ \gamma'\}$, the values of $f(G)$ where $G \in \mathcal{CG}(m,n)$ completely cover a line segment $[a,b]$ of positive integers. Then we say that f is an interpolation graph parameter with respect to $\mathcal{CG}(m,n)$. Thus for a graph parameter f, two invariants $a(f)$ and $b(f)$ where

$a(f) = \min\{f(G) : G \in \mathcal{CG}(m,n)\}$ and

$b(f) = \max\{f(G) : G \in \mathcal{CG}(m,n)\}$, arise naturally.

The extremal values $a(f)$ and $b(f)$ are obtained for all $f \in \{\kappa,\ \lambda,\ \gamma,\ \gamma'\}$.

We also find the minimum and maximum values for all $f \in \{\kappa,\ \lambda,\ \gamma,\ \gamma'\}$.

ACKNOWLEDGEMENTS

I would like to thank Assistant Professor Dr. Araya Chaemchan and Associate Professor Dr. Narong Punnim, my thesis advisors, for their numerous valuable suggestions and constant supports during this thesis which gave me a better perspective on my own results. I am also thankful to Assistant Professor Dr. Tipaval Phattanangkul, the chairman, Associate Professor Dr. Wanida Hemakul and Dr. Varaporn Saenpholphat, committee members, for their comments and assistances. Also, I would like to thank all teachers who have taught me all along, especially, Dr. Paolo Bertozzini, my advisor, for a chance to study here.

Of course, I am grateful to my parents for their love and inspiration throughout graduate study. Without them I may not have a chance. I wish to thank my family for all the good times we had together.

Finally, I had the pleasure of meeting many schoolmates and the staff in Department of Mathematics and Statistics, Thammasat University. They are wonderful people and friendly encouragement. I thank them for this.

Miss Aunyarat Bunyawat
Thammasat University
2006

CONTENTS

CHAPTER 1
INTRODUCTION

In 1980, Chartrand raised the following problem: If a graph G possesses a spanning tree having m end-vertices and another having M end-vertices, where $M > m$, does G possess a spanning tree having k end-vertices for every k between m and M ? This question was answered affirmatively and it led to a number of papers studying the interpolation properties of invariants of spanning trees of given graph.

There are a number of graph parameters which have been studies the interpolation properties on the class of regular graphs such as:

$\chi(G)$, the chromatic number of a graph G,

$\omega(G)$, the clique number of a graph G,

$\alpha_0(G)$, the independence number of a graph G,

$\alpha_1(G)$, the matching number of a graph G,

$\beta_0(G)$, the vertex covering number of a graph G,

$\beta_1(G)$, the edge covering number of a graph G,

$\gamma(G)$, the domination number of a graph G,

and $I(G)$, the order of maximum induced forest in G.

We introduce the notation and terminology used throughout this thesis in section 1.1. A review of basic concepts in graph theory is given in section 1.2. We also describe a background of interpolation theorems relevant to our work in section 1.3.

1.1 Notation and Terminology

In this thesis, we consider only finite simple graphs. In most part, our notation and terminology follows that of Chartrand and Zhang [6]. Let $G = (V, E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We use $|S|$ to denote the cardinality of a set S. We define $n = |V|$ to be the *order* of G and $m = |E|$ to be the *size* of G. We simply write $e = uv$ for the *edge* e that joins the vertices u and v. If $e = uv$ is an edge of G, then u and v are *adjacent vertices*. We also say u and v are *joined* by the edge e. The vertices u and v are referred to as *neighbors* of each other. In this case, the vertex u and the edge e (as well as v and e) are said to be *incident* with each other. Distinct edges incident with a common vertex are *adjacent edges*.

Since the vertex set of every graph is nonempty, the order of every graph is at least 1. A graph with exactly one vertex is called a *trivial graph*, implying that the order of a *nontrivial graph* is at least 2.

A graph H is called a *subgraph* of a graph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We also say that G contains H as a subgraph. If $H \subseteq G$ and either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then H is a *proper subgraph* of G. So the graph H of Figure 1.1 is a subgraph of the graph G; indeed, H is a proper subgraph of G. If a subgraph of a graph G has the same vertex set as G, then it is a *spanning subgraph* of G.

A subgraph F of a graph G is called an *induced subgraph* of G if whenever u and v are vertices of F and uv is an edge of G, then uv is an edge of F as well. Therefore the graph H of Figure 1.1 is not an induced subgraph of G of Figure 1.1 since, for example, $v, x \in V(H)$ and $vx \in E(G)$, but $vx \notin E(H)$. On the other hand, the graph F of Figure 1.1 is an induced subgraph of G. If S is a nonempty set of vertices of a graph G, then the *subgraph of G induced by S* is the induced subgraph with vertex set S. This induced subgraph is denoted by $\langle S \rangle$. To emphasize that this is an induced subgraph of G, we sometimes denote this subgraph by $\langle S \rangle_G$. For a nonempty set X of edges, the *subgraph $\langle X \rangle$ induced by X* has edge set X and consists of all vertices that are incident with at least one edge in X. This subgraph is called an *edge-induced subgraph* of G. Sometimes $G[S]$ and $G[X]$ are used for $\langle S \rangle$ and $\langle X \rangle$, respectively. The graph F' of Figure 1.1 is an edge-induced subgraph of G of that figure; indeed, $F' = \langle X' \rangle$, where $X' = \{e, e'\}$.

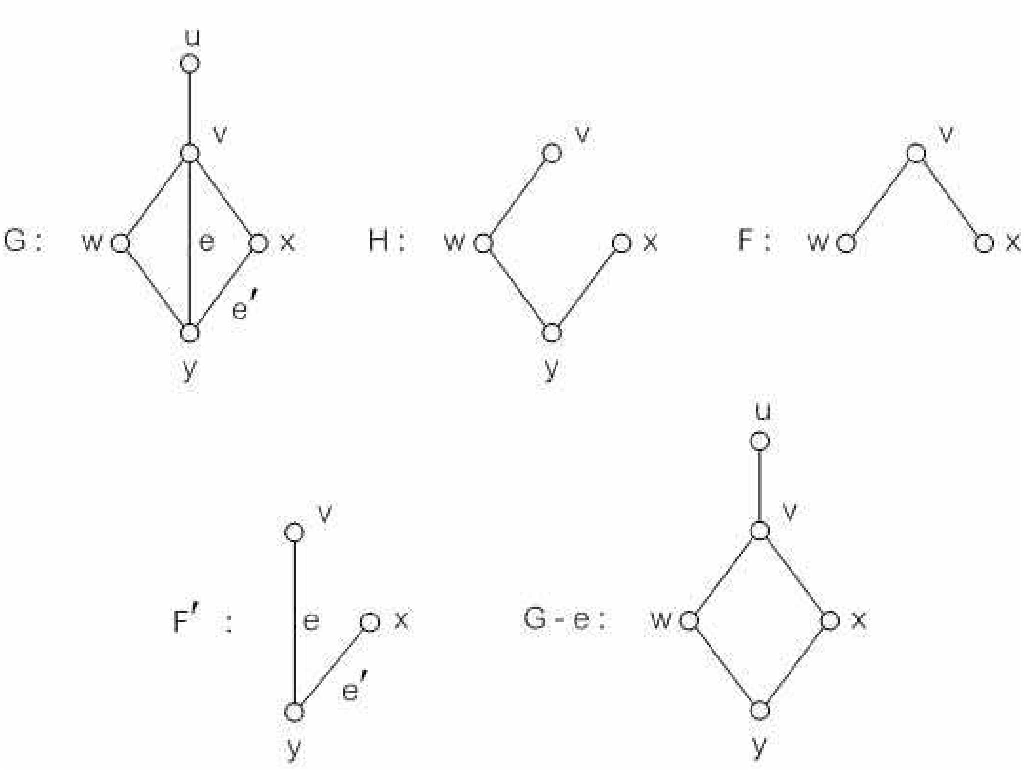

Figure 1.1: Subgraphs of a graph G.

Any proper subgraph of a graph G can be obtained by removing vertices and edges from G. For an edge e of G, we write $G - e$ for the spanning subgraph of G whose edge set consists of all edges of G except e. More generally, if X is a set of edges of G, then $G - X$ is the spanning subgraph of G with $E(G-X) = E(G) - X$. For the graph G of Figure 1.1 and $e = vy$, the subgraph $G - e$ is shown. If $X = \{e_1, e_2, \ldots, e_k\}$, then we also write $G - X$ as $G - e_1 - e_2 - \ldots - e_k$.

For a vertex v of a nontrivial graph G, the subgraph $G - v$ consists of all vertices of G except v and all edges of G except those incident with v. For a proper subset U of $V(G)$, the subgraph $G - U$ has vertex set $V(G) - U$ and its edge set consists of all edges of G joining two vertices in $V(G) - U$.

If u and v are nonadjacent vertices of a graph G, then $e = uv \notin E(G)$. By $G + e$, we mean the graph with vertex set $V(G)$ and edge set $E(G) \cup \{e\}$. Thus G is a spanning subgraph of $G + e$.

A $u - v$ *walk* W in G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in the sequence are adjacent, that is, we can express W as

$$W : u = v_0, v_1, \ldots, v_k = v,$$

where $k \geq 0$ and v_i and v_{i+1} are adjacent for $i = 0, 1, 2, \ldots, k - 1$. Each vertex v_i $(0 \leq i \leq k)$ and each edge $v_i v_{i+1}$ is said to lie on or belong to W. If $u = v$, then the walk W is closed; while if $u \neq v$, then W is open. For the graph G of Figure 1.2,

$$W : x, y, w, y, v, w$$

is a $x - w$ walk.

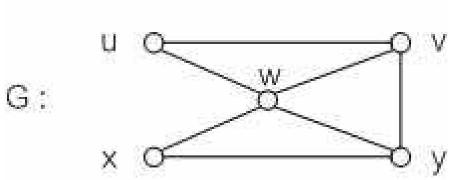

Figure 1.2: Illustrating walks in a graph.

A $u - v$ *trail* in a graph G to be a $u - v$ walk in which no edge is traversed more than one. In the graph G of Figure 1.2,

$$T : u, w, y, x, w, v$$

is a $u - v$ trail.

A $u - v$ walk in a graph in which no vertices are repeated is a $u - v$ *path*. In the graph G of Figure 1.2,

$$P : u, w, y, v$$

is a $u - v$ path.

A *circuit* in a graph G is a closed trail of length 3 or more. For example, in the graph G of Figure 1.2,

$$C : y, w, u, v, w, x, y, \quad \text{or} \quad C : x, y, w, u, v, w, x$$

is a circuit.

A circuit that repeats no vertex, except for the first and last, is a *cycle*. In the graph G of Figure 1.2,

$$C' : x, y, v, w, x$$

is a cycle.

If G contains a $u - v$ path, then u and v are said to be *connected* and u *is connected to* v (and v is connected to u). A graph G is connected if every two vertices of G are connected, that is, if G contains a $u - v$ path for every pair u, v of distinct vertices of G. A graph G that is not connected is called *disconnected*. A connected subgraph of G that is not a proper subgraph of any other connected subgraph of G is a *component* of G. A graph G is then connected if and only if it has exactly one component.

A graph G is *complete* if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n. Therefore, K_n has the maximum possible size for a graph with n vertices. Since every two distinct vertices of K_n are joined by an edge, the number of pairs of vertices in K_n is $\binom{n}{2} = \frac{n(n-1)}{2}$.

The *complement* $\overline{G}$ of a graph G is that graph whose vertex set is $V(G)$ and such that for each pair u, v of vertices of G, uv is an edge of $\overline{G}$ if and only if uv is not an edge of G. Observe that if G is a graph of order n and size m, then $\overline{G}$ is a graph of order n and size $\binom{n}{2} - m$. The graph $\overline{K}_n$ has n vertices and no edge; it is called the *empty graph* of order n. Therefore, empty graphs have empty edge sets. A graph G and its complement are shown in Figure 1.3.

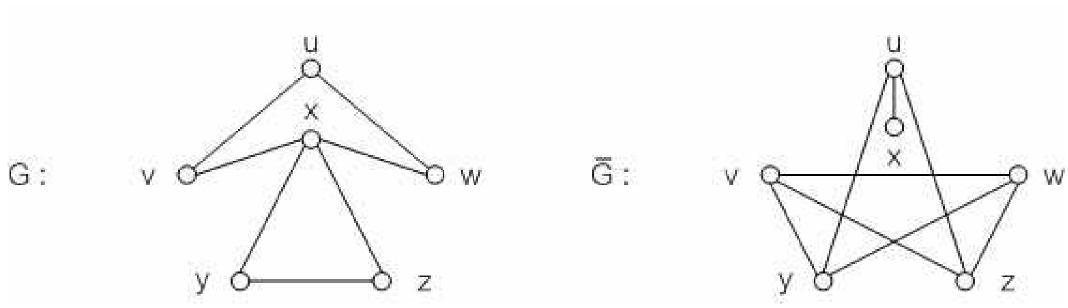

Figure 1.3: A graph and its complement.

A graph G is a *bipartite graph* if $V(G)$ can be partitioned into two subsets U and W, called *partite set*, such that every edge of G joins a vertex of U and a vertex of W. For example, the connected graph G of Figure 1.4 is bipartite, as every edge of G joins a vertex of $U = \{u, x, y\}$ and a vertex of $W = \{v, w\}$.

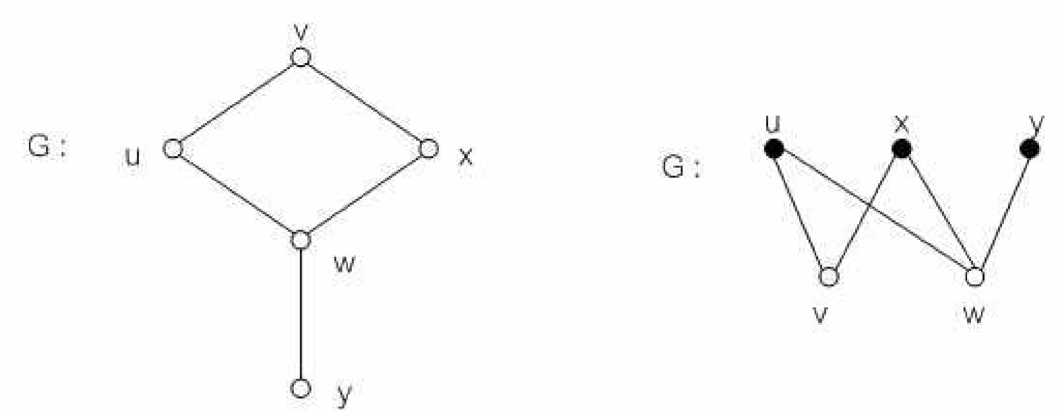

Figure 1.4: Bipartite graphs.

We call G a *complete bipartite graph* if every vertex of U is adjacent to every vertex of W. A complete bipartite graph with $|U| = s$ and $|W| = t$ is denoted by $K_{s,t}$ or $K_{t,s}$. If either $s = 1$ or $t = 1$, then $K_{s,t}$ is a *star*. Several complete bipartite graphs are shown in Figure 1.5, including the star $K_{1,3}$.

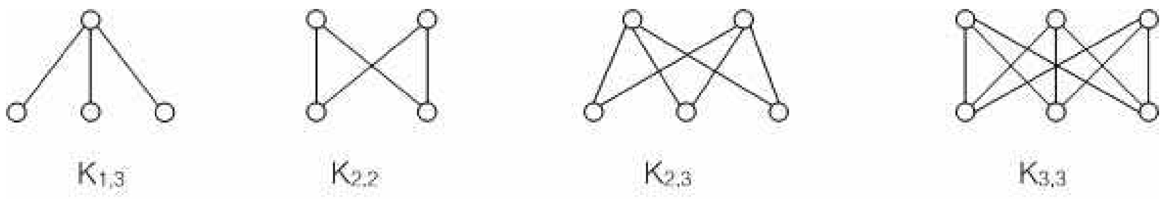

Figure 1.5: Complete bipartite graphs.

The *degree of a vertex* v in a graph G is the number of edges incident with v and is denoted by $deg_G\ v$, or simply by $deg\ v$ if the graph G is clear from the context. A vertex of degree 0 is referred to as an *isolated vertex* and a vertex of degree 1 is an *end-vertex* (or a *leaf*). The *minimum degree* of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$; the *maximum degree* of G is denoted by $\Delta(G)$. So if G is a graph of order n and v is any vertex of G, then

$$0 \leq \delta(G) \leq deg\ v \leq \Delta(G) \leq n-1.$$

If $\delta(G) = \Delta(G)$, then the vertices of G have the same degree and G is called *regular*. If $deg\ v = r$ for every vertex v of G, where $0 \leq r \leq n-1$, then G is *r-regular* or *regular of degree r*.

The *degree of an edge* $e = uv$ of G is defined by $deg\ e = deg\ u + deg\ v - 2$.

Recall that two graphs G and H are equal if $V(G) = V(H)$ and $E(G) = E(H)$. We call two graphs G_1 and G_2 are *isomorphic* if there exists a one-to-one correspondence ϕ from $V(G_1)$ to $V(G_2)$ such that $u_1v_1 \in E(G_1)$ if and only if $\phi(u_1)\phi(v_1) \in E(G_2)$. In this case, ϕ is called an *isomorphism* from G_1 to G_2. Thus, if G_1 and G_2 are isomorphic graphs, then we say that G_1 *is isomorphic to* G_2 and we write $G_1 \cong G_2$. The graphs G_1 and G_2 are shown in Figure 1.6. If two graphs G_1 and G_2 are not isomorphic, then they are called *nonisomorphic graphs* and we write $G_1 \ncong G_2$. Figure 1.6 illustrates that $G_1 \cong G_2$ where $\phi : V(G_1) \rightarrow V(G_2)$ is defined by $\phi(u_i) = v_i; i = 1, 2, \ldots, 5$.

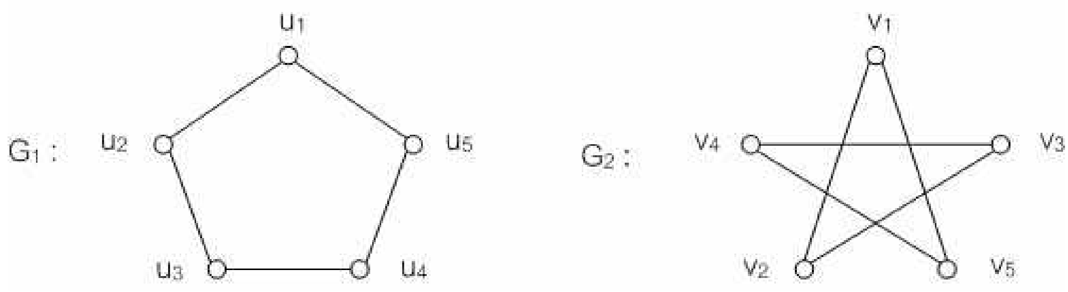

Figure 1.6: Isomorphic graphs.

An edge $e = uv$ of a connected graph G is called a *bridge* of G if $G - e$ is disconnected.

A graph G is called *acyclic* if it has no cycles. A *tree* is an acyclic connected graph. A spanning subgraph H of a connected graph G such that H is a tree is called a *spanning tree* of G. For the connected graph G of Figure 1.7, two different spanning trees T_1 and T_2 of G are also shown in Figure 1.7.

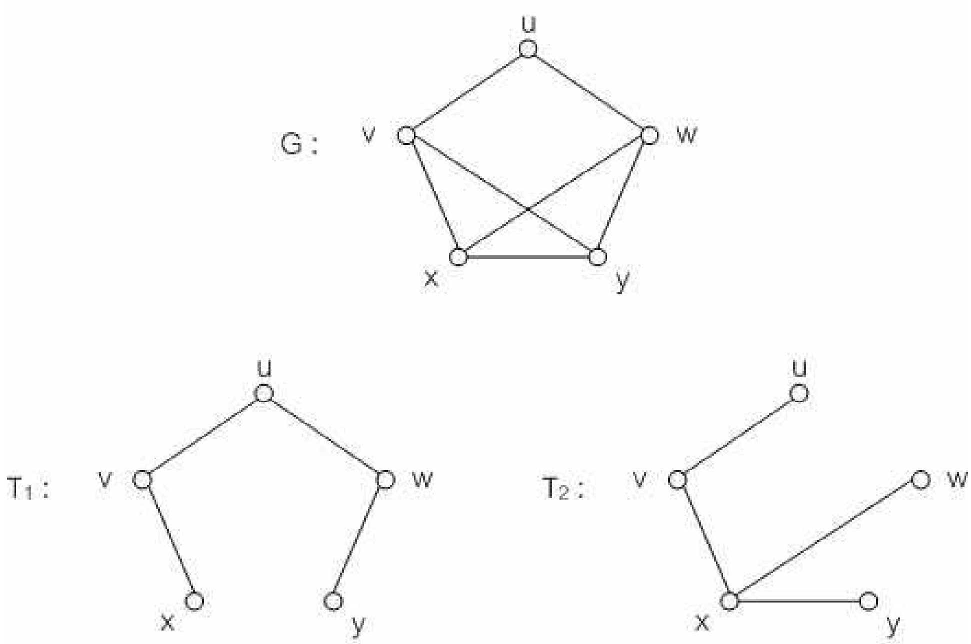

Figure 1.7: Two spanning trees in a graph.

A *vertex-cut* in a graph G is a set U of vertices of G such that $G-U$ is disconnected. A vertex-cut of minimum cardinality in G is called a minimum vertex-cut. For a graph G that is not complete, the *vertex-connectivity* (connectivity) $\kappa(G)$ of G is defined as the cardinality of a minimum vertex-cut of G; if $G \cong K_n$ for some positive integer n, then $\kappa(G)$ is defined to be $n-1$. Therefore, for every graph G of order n, $0 \leq \kappa(G) \leq n-1$.

An *edge-cut* in a nontrivial graph G is a set X of edges of G such that $G-X$ is disconnected. An edge-cut of minimum cardinality in G is called a minimum edge-cut. For a nontrivial graph G, the *edge-connectivity* $\lambda(G)$ of G is defined as the cardinality of a minimum edge-cut of G, while we define $\lambda(K_1) = 0$. For every graph G of order n, $0 \leq \lambda(G) \leq n-1$.

It is useful to describe a class of graphs $H_{r,n}$ for integers r and n with $2 \leq r < n$ such that $H_{r,n}$ has order n and $\kappa(H_{r,n}) = r$.

Let $V(H_{r,n}) = \{v_1, v_2, \ldots, v_n\}$. Suppose first that $r = 2k$ is even. For each integer i ($1 \leq i \leq n$), the vertex v_i is adjacent to $v_{i+1}, v_{i+2}, \ldots, v_{i+k}$ and to $v_{i-1}, v_{i-2}, \ldots, v_{i-k}$. Thus $H_{r,n}$ is an r-regular graph of order n.

Next, suppose that $r = 2k+1$ is odd and $n = 2\ell$ is even. For each integer i ($1 \leq i \leq n$), the vertex v_i is joined to the $2k$ vertices mentioned above as well as to $v_{i+\ell}$. Here too $H_{r,n}$ is an r-regular graph of order n.

Finally, suppose that $r = 2k+1$ and $n = 2\ell+1$ are both odd. In this case, $H_{r,n}$ is obtained from $H_{r-1,n}$ by adding the edge $v_i v_{i+\ell}$ for each i ($1 \leq i \leq n$) and the edge $v_1 v_{\ell+2}$. Therefore, when r and n are both odd, $H_{r,n}$ contains one vertex of degree $r+1$ and $n-1$ vertices of degree r. If r or n is even, then the size of $H_{r,n}$ is $\frac{rn}{2}$; while if r and n are odd, then the size of $H_{r,n}$ is $\frac{rn+1}{2}$. In general, the size of $H_{r,n}$ is $\lceil \frac{rn}{2} \rceil$.

The graphs $H_{4,8}, H_{5,8}$, and $H_{5,9}$ are shown in Figure 1.8. The graphs $H_{r,n}$ are referred to as the Harary graphs.

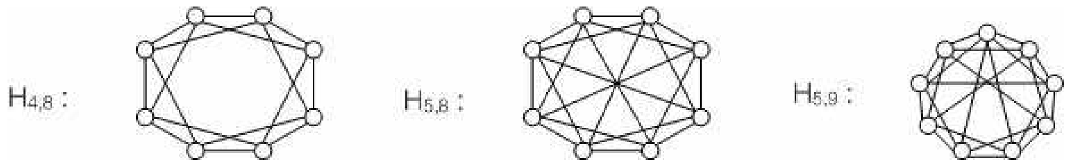

Figure 1.8: The Harary graphs.

For a vertex v of a graph G, recall that a *neighbor* of v is a vertex adjacent to v in G. Also, the *neighborhood* (or open neighborhood) $N(v)$ of v is the set of neighbors of v. The *closed neighborhood* $N[v]$ is defined as $N[v] = N(v) \cup \{v\}$.

A vertex v in a graph G is said to *dominate* itself and each of its neighbors, that is, v dominates the vertices in its closed neighborhood $N[v]$. Therefore, v dominates $1 + deg\ v$ vertices of G. A set S of vertices of G is a *dominating set* of G if every vertex of G is dominated by some vertex in S. Equivalently, a set S of vertices of G is a dominating set of G if every vertex in $V(G) - S$ is adjacent to some vertex in S. A *minimum dominating set* in a graph G is a dominating set of minimum cardinality. The cardinality of a minimum dominating set is called the *domination number* of G and is denoted by $\gamma(G)$.

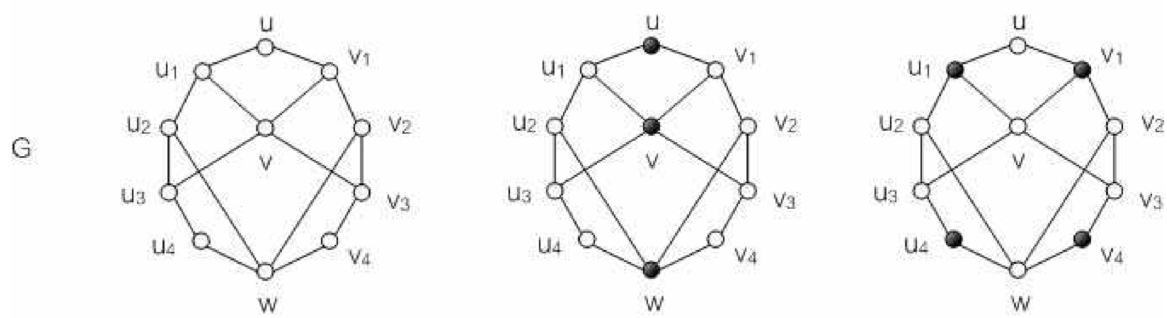

Figure 1.9: Two dominating sets in a graph G.

Consider the graph G of Figure 1.9. The set $S_1 = \{u, v, w\}$ and $S_2 = \{u_1, u_4, v_1, v_4\}$ are both dominating sets in G indicated by solid vertices.

Let's now turn our attention to the graph G of Figure 1.10. If we replace the edge uv by a vertex s of degree 2 and join s to u and v, then we obtain the graph G_1. The graph G_1 is referred to as a subdivision of G. We might also think of producing G_1 by inserting a vertex of degree 2 into the edge uv of G.

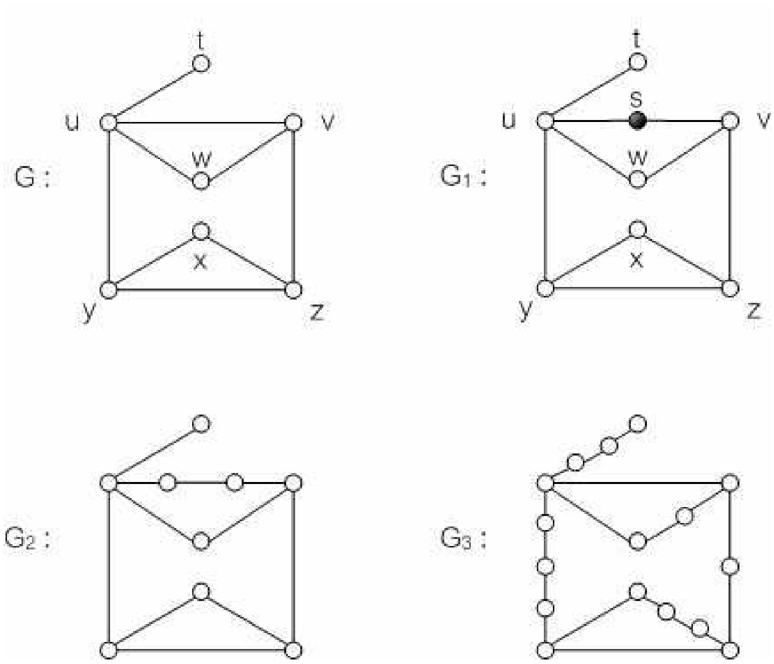

Figure 1.10: Subdivisions of a graph.

More formally, a graph G' is called a *subdivision* of a graph G if one or more vertices of degree 2 are inserted into one or more edges of G. Consequently, all of G_1, G_2, and G_3 are subdivisions of G. In fact, G_2 is a subdivision of G_1 as well.

The concept of edge domination was introduced by Mitchell and Hedetniemi [21]. A subset X of edges in a graph G is called an *edge dominating set* of G if every edge not in X is adjacent to some edge in X. The *edge domination number* $\gamma'(G)$ of G is the minimum cardinality taken over all edge dominating sets of G.

Consider the graph G of Figure 1.11. The set $X_1 = \{uv_1, u_2w, u_3u_4, vv_3\}$ and $X_2 = \{u_1u_2, u_3v, u_4w, v_1v_2, v_2v_3\}$ are both edge dominating sets in G indicated by solid edges.

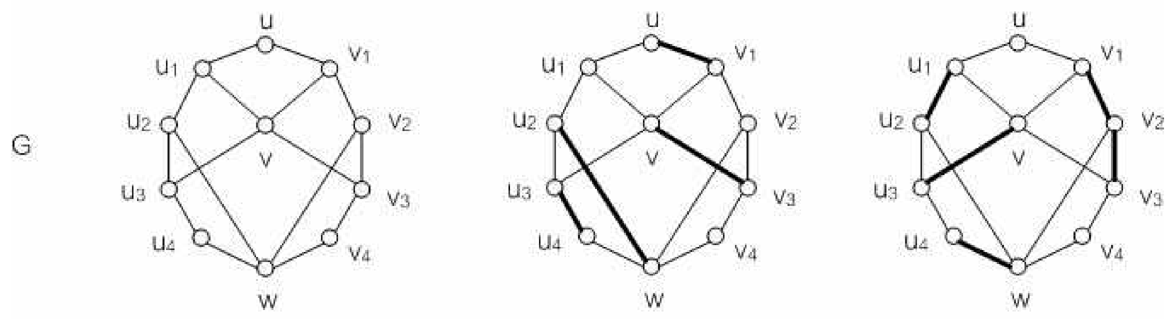

Figure 1.11: Two edge dominating sets in a graph G.

A set of edges in a graph is *independent* if no two edges in the set are adjacent. A subset M of the edge set E of a graph $G = (V, E)$ is an *independent edge set* or *matching* in G if no two distinct edges in M have a common vertex. A matching M is *maximum* in G if there is no matching M' of G with $|M'| > |M|$. The cardinality of a maximum matching of G is denoted by $\alpha_1(G)$ and is called the *matching number* of G.

For a real number x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

The next examples illustrate a change of $\kappa(G), \lambda(G), \gamma(G)$, and $\gamma'(G)$ of a graph G when an edge is deleted or added.

Example 1.1 Consider a graph $G \in \mathcal{CG}(15, 8)$.

Consider a given graph G in the Figure 1.12. We see that the sets $U_1 = \{u, w, y\}$ and $U_2 = \{s, v, y\}$ are both minimum vertex-cuts of G, $\kappa(G) = 3$. Let A be a set of vertices of G that have minimum degree, so $A = \{x, z\}$. Let e be any edge not incident with any vertex in A. We have $\kappa(G - e) = 3$. If e' is an edge that incident with any vertex in A, then $\kappa(G - e') = 2$. If we add an edge f' in G such that $\delta(G)$ increases 1, then $\kappa(G + f') = 4$. If we add any other edge f in G, then $\kappa(G + f) = 3$.

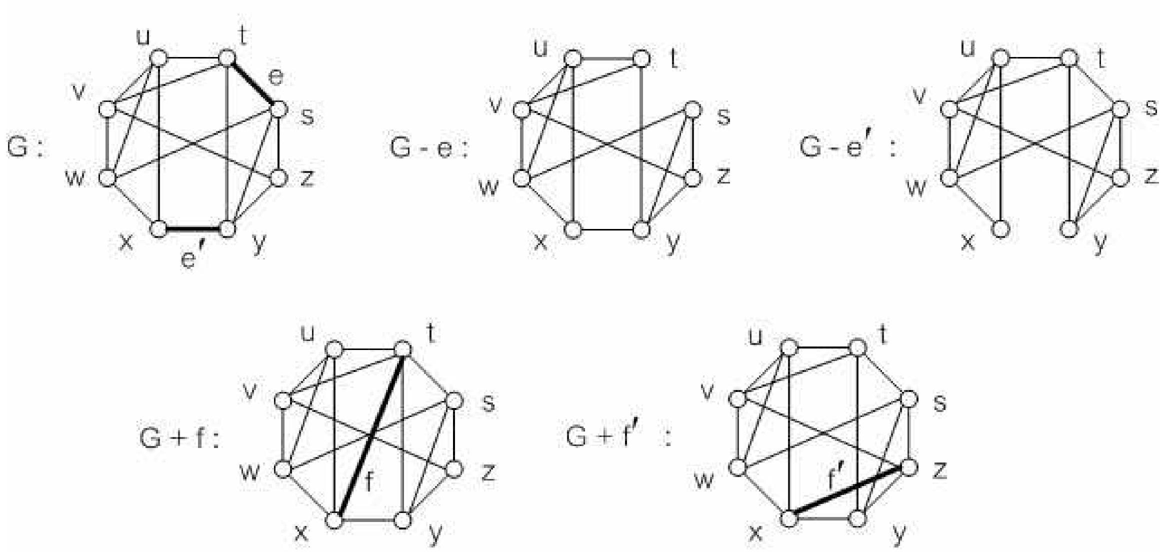

Figure 1.12: $G \in \mathcal{CG}(15, 8)$.

The sets $X_1 = \{ux, wx, xy\}$ and $X_2 = \{sz, vz, yz\}$ are both minimum edge-cuts of G, $\lambda(G) = 3$. Let e be any edge not incident with a vertex in A. We have $\lambda(G - e) = 3$. If e' is an edge that incident with a vertex in A, then $\lambda(G - e') = 2$. If we add an edge f' in G such that $\delta(G)$ increases 1, then $\lambda(G + f') = 4$. If we add an edge f in G, then $\lambda(G + f) = 3$.

Example 1.2 Consider a graph $G \in \mathcal{CG}(8, 9)$.

Consider a given graph G in the Figure 1.13. We see that the minimum dominating set of G is $S_3 = \{s, v, y\}$, $\gamma(G) = 3$. If e is an edge of G incident with a vertex not in S_3, then $\gamma(G - e) = 3$. If e' is an edge of G incident with a vertex in S_3, then $\gamma(G - e') = 4$. If we add an edge f in G, then $\gamma(G + f) = 3$.

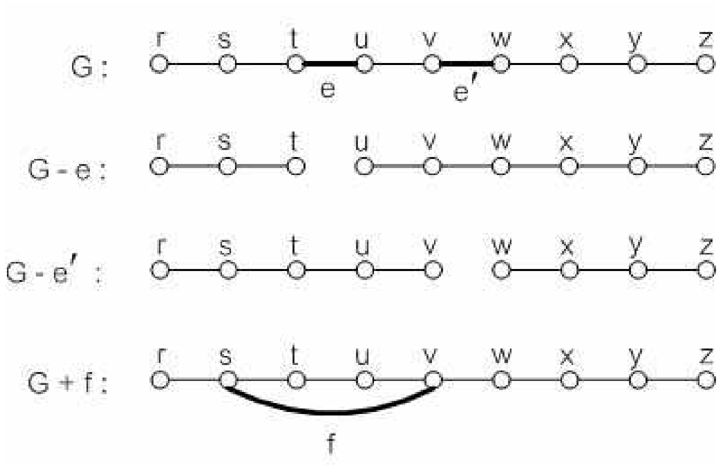

Figure 1.13: $G \in \mathcal{CG}(8, 9)$.

Example 1.3 Consider a graph $G \in \mathcal{CG}(6, 5)$.

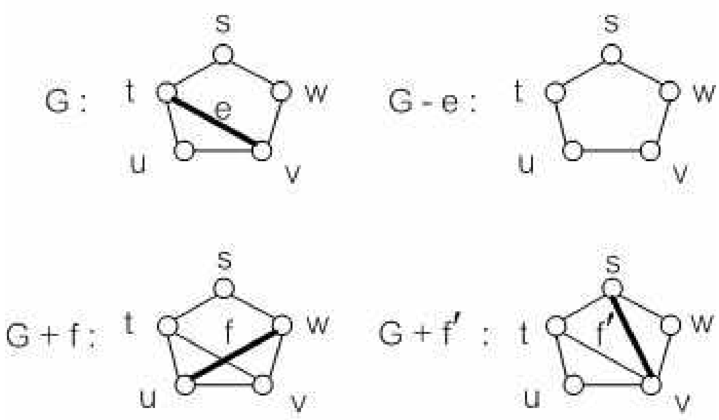

Figure 1.14: $G \in \mathcal{CG}(6, 5)$.

Consider a given graph G in the Figure 1.14. There are several minimum dominating sets of G, for example $S_4 = \{s, v\}$ and $S_5 = \{t, w\}$, $\gamma(G) = 2$. If we delete an edge e in G, then $\gamma(G - e) = 2$. If we add any edge f in G, then $\gamma(G + f) = 2$. If we add an edge f' in G, then $\gamma(G + f') = 1$.

Example 1.4 Consider a graph $G \in \mathcal{CG}(8,6)$.

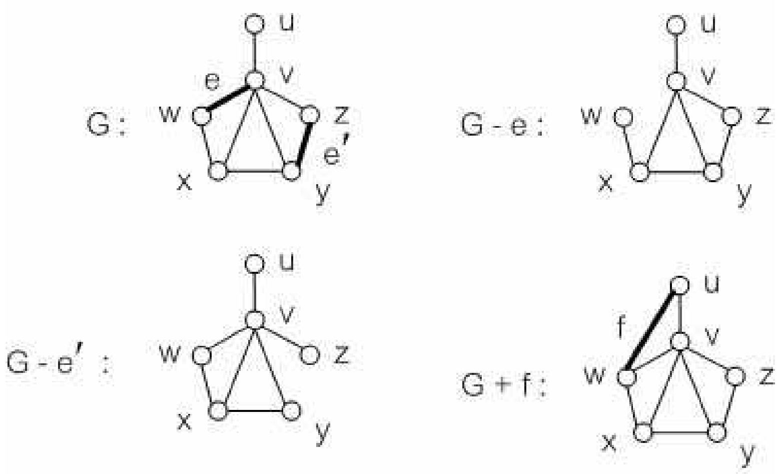

Figure 1.15: $G \in \mathcal{CG}(8,6)$.

Consider a given graph G in the Figure 1.15. There are several minimum edge dominating sets of G, for example $X_3 = \{vx, yz\}$ and $X_4 = \{uv, xy\}$, $\gamma'(G) = 2$. If we delete an edge e in G, then $\gamma'(G - e) = 2$. If we delete an edge e' in G, then $\gamma'(G - e') = 1$. If we add an edge f in G, then $\gamma'(G + f) = 2$.

Example 1.5 Consider a graph $G \in \mathcal{CG}(8,6)$.

Consider a given graph G in the Figure 1.16. There are several minimum edge dominating sets of G, for example $X_5 = \{vw, yz\}$ and $X_6 = \{vy, wz\}$, $\gamma'(G) = 2$. If we delete an edge e in G, then $\gamma'(G - e) = 2$. If we add an edge f in G, then $\gamma'(G + f) = 2$. If we add an edge f' in G, then $\gamma'(G + f') = 3$.

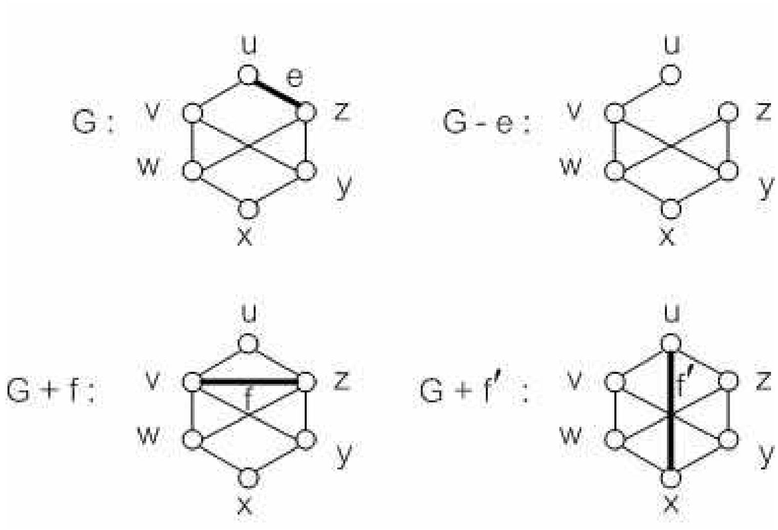

Figure 1.16: $G \in \mathcal{CG}(8, 6)$.

1.2 Basic Concepts

In this section we introduce the fundamental concepts of graph theory and the switching operation.

Let G be a graph of order n and let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. The sequence $(deg\ v_1, deg\ v_2, \ldots, deg\ v_n)$ is called a *degree sequence* of G, where $deg\ v_i$ is the degree of vertex v_i for $1 \le i \le n$. An r-regular graph G has degree sequence $d = r^n := (r, r, \ldots, r)$. A sequence $d = (d_1, d_2, \ldots, d_n)$ of non-negative integers is a *graphic degree sequence* if it is a degree sequence of some graph G. In this case, G is called a *realization* of d. The following theorem has been shown by Havel [16] and Hakimi [12].

Theorem 1.1 Let $d = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of non-negative integers and denote the sequence

$$d' = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$$

then d is graphic degree sequence if and only if d' is graphic degree sequence.

□

Let G be a graph, ab and cd are independent edges in G, and ac and bd are not edges in G. Define $G^{\sigma(a,b;c,d)}$ or G^{σ} to be the graph obtained from G by deleting the edges ab and cd and replacing the edges ac and bd (see the Figure 1.17). The operation $\sigma(a, b; c, d)$ is called a *switching operation*. It is easy to see that the graph obtained from G by a switching has the same degree sequence as G.

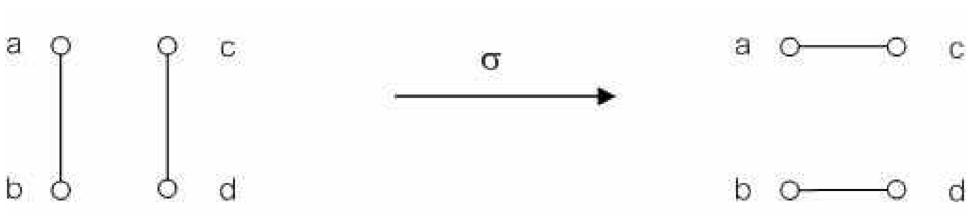

Figure 1.17: Switching Operation.

Theorem 1.2 Let $d = (d_1, d_2, \ldots, d_n)$ be a graphic degree sequence. If G_1 and G_2 are any two realizations of d, then G_2 can be obtained from G_1 by a finite sequence of switchings.

□

As a consequence of Theorem 1.2, Eggleton and Holton [9] defined *the graph* $\mathcal{R}(d)$ of realizations of d whose vertices are the graphs with degree sequence d; two vertices being adjacent in *the graph* $\mathcal{R}(d)$ if one can be obtained from the other by a switching. In particular, they obtained the following theorem.

Theorem 1.3 *The graph* $\mathcal{R}(d)$ is connected.

□

1.3 Interpolation Theorems

In this section we give the definition of a graph parameter. Then we shall detail the interpolation graph parameter.

Graph Parameters

Let $\mathcal{G}$ be the class of all graphs and $\mathbb{Z}$ be the set of integers. A function $f : \mathcal{G} \to \mathbb{Z}$ is called a *graph parameter* if $f(G) = f(H)$, whenever $G \cong H$.

We can see that κ, λ, γ and γ' are graph parameters for any graph.

Interpolation Theorems

Let $\mathcal{G}$ be the class of all graphs and $\mathcal{J} \subseteq \mathcal{G}$. If f is a graph parameter, f is called an *interpolation graph parameter with respect to* $\mathcal{J}$ if there exist integers a and b such that $\{f(G) : G \in \mathcal{J}\} = \{k \in \mathbb{Z} : a \leq k \leq b\}$.

In the study of interpolation of a graph parameter f with respect to $\mathcal{J}$, N. Punnim considers two parts. First, consider whether a given graph parameter f is an interpolation graph parameter with respect to $\mathcal{J}$. If it is, then develop techniques to find minimum and maximum values for the graph parameter.

In this thesis we study that the four graph parameters:

$\kappa(G)$, the vertex-connectivity of a graph G,

$\lambda(G)$, the edge-connectivity of a graph G,

$\gamma(G)$, the domination number of a graph G,

and $\gamma'(G)$, the edge domination number of a graph G,

are interpolation graph parameters with respect to the class of connected graphs with prescribed order and size.

For all $f \in \{\kappa, \lambda, \gamma, \gamma'\}$ of graphs, we also find their minimum values and maximum values.

We review the literature of the relevant works in chapter 2. Our results are proposed in the last chapter 3.

CHAPTER 2
LITERATURE REVIEW

This chapter we review some relevant works of interpolation theorems and extremal problems. Interpolation theorems have been studied in numerous researches by changing various kinds of graph parameters and class of graphs.

Harary and others [13, 14, 15, 23] defined transformations called fundamental exchange, neighbor exchange and end-line exchange, each of which is defined to transform a tree to another trees. Suppose T is a spanning tree of a graph G. An edge e of G is called a *chord* of T if $e \notin E(T)$. Let e be a chord of a spanning tree T of a graph G and let f be any edge which is in the cycle of $T+e$. Call the operation which transforms T to the spanning tree $T+e-f$ a *fundamental exchange*. The fundamental exchange $T \to T+e-f$ will be called a *neighbor exchange* if e and f are adjacent in G. Furthermore, if $T \to T+e-f$ is a neighbor exchange and f is a leaf of T, then the transformation is called an *end-line exchange*. Note that in this case e is a leaf of the resulting tree $T+e-f$.

Let G be a graph and $T(G)$ be the set of spanning trees of G. *The graph* $T(G)$ of spanning trees of G is defined as a graph whose vertices are the set $T(G)$; two vertices are adjacent in the graph if one can be obtained from the other by a single end-line exchange.

They answered the interpolation on various kinds of graph parameters with respect to the set of all spanning trees of a given graph. They used the edge exchanges to transform a spanning tree to another spanning tree of the same graph. In other words, let $T(G)$ be the set of all spanning trees of G and f be a graph parameter. If S and T are adjacent in *the graph* $T(G)$ and $|f(S)-f(T)| \leq 1$, then f is an interpolation graph parameter with respect to $T(G)$.

There are two sections in this chapter. In section 2.1, we give the definition of some graph parameters and introduce its bounds. In section 2.2, we describe the interpolation theorems on various kinds of graph parameters and classes of graphs.

2.1 Some Bounds of Graph Parameters

In this section we introduce each interesting graph parameter, then present a list of some related bounds for some graph parameters.

A *k-coloring* of a graph $G = (V, E)$ is a partition of its vertex set V as $V_1 \cup V_2 \cup \ldots \cup V_k$ such that no two vertices in V_i $(1 \leq i \leq k)$ are adjacent. The V_i's are called the *color classes.* A function $f : V \to \{1, 2, \ldots, k\}$ such that $f(v) = i$ for each $v \in V_i$ $(1 \leq i \leq k)$ is called a *color function.* If G has a k-coloring, it is said to be *k-colorable* and the minimum integer k for which G is k-colorable is called the *chromatic number* of G and is denoted by $\chi(G)$. If $\chi(G) = k$, we say that G is *k-chromatic.*

A maximal complete subgraph of a graph G is called a *clique* of G. The maximum order of clique of G is called the *clique number* of G and is denoted by $\omega(G)$.

A subset U of the vertex set V of a graph $G = (V, E)$ is said to be an *independent set* of G if the induced subgraph $G[U]$ of G is an empty graph. An independent set of G with maximum number of vertices is called a *maximum independent set* of G. The number of vertices of a maximum independent set of G is called the *independence number* of G and is denoted by $\alpha_0(G)$.

Theorem 2.1 ([6],p.272) For every graph G of order n,

$$\chi(G) \geq \omega(G) \quad \text{and} \quad \chi(G) \geq \tfrac{n}{\alpha_0(G)}.$$

□

Theorem 2.2 ([6],p.274) For every graph G, $\chi(G) \leq 1 + \Delta(G)$.

□

Theorem 2.3 (Brooks's Theorem) ([6],p.275) For every connected graph G that is not an odd cycle or a complete graph, $\chi(G) \leq \Delta(G)$.

□

Theorem 2.4 ([6],p.275) For every graph G,

$$\chi(G) \leq 1 + \max\{\delta(H)\},$$

where the maximum is taken over all induced subgraphs H of G.

□

Corollary 2.5 ([4],p.226) Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then

$$\chi(G) \leq \max_{1 \leq i \leq n} \{\min\, \{i, deg\ v_i + 1\}\}.$$

□

Corollary 2.6 ([4],p.226) For every graph G,

$$\chi(G) \leq 1 + \ell(G),$$

where $\ell(G)$ denotes the length of a longest path in G.

□

Corollary 2.7 ([4],p.229) If G is a chordal graph, then $\chi(G) = \omega(G)$.

□

Theorem 2.8 [26] Let G be a graph of order n and maximum degree Δ. Then $\omega(G) \geq \frac{n}{n-\Delta}$.

□

Corollary 2.9 [28] Let G be a graph with degree sequence $d = (d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$. Then

$$\omega(G) \geq \sum_{i=1}^{n} \frac{1}{n - d_i},$$

where $\omega(G)$ is the clique number of G.

□

Corollary 2.10 [28] Let G be a graph with degree sequence $d = (d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$. Then

$$\alpha_0(G) \geq \sum_{i=1}^{n} \frac{1}{d_i + 1},$$

where $\alpha_0(G)$ is the independence number of G.

□

Theorem 2.11 [35] Let G be a graph of order n with maximum degree Δ. Then $\alpha_0(G) \geq \frac{n}{\Delta+1}$.

□

Theorem 2.12 [35] Let G be a K_4-free graph of order n, $\Delta(G) = 3$. Then $\alpha_0(G) \geq \frac{n}{3}$.

□

Theorem 2.13 [35] Let G be a $K_{\Delta+1}$-free graph of order n with $\Delta \geq 4$. Then $\alpha_0(G) \geq \frac{n}{\Delta}$ or there exists an independent set S of G and $G - S$ is a K_Δ-free graph.

□

A vertex of a graph $G = (V, E)$ is said to cover the edges incident with it. A *vertex cover* of a graph G is a set of vertices covering all the edges of G. The minimum cardinality of a vertex cover of a graph G is called the *vertex covering number* of G and is denoted by $\beta_0(G)$.

Theorem 2.14 ([6],p.191) For every graph G of order n containing no isolated vertices,

$$\beta_0(G) + \alpha_0(G) = n.$$

□

Acyclic graph is a graph containing no cycle as its subgraph. An acyclic graph is called a forest. Therefore, each component of an acyclic graph is a tree. Since a tree is connected, every two vertices in a tree are connected by a unique path.

Let G be a graph and $F \subseteq V(G)$, F is called an *induced forest* of G, if the induced subgraph $G[F]$ of G contains no cycle. For a graph G, we define, $I(G)$ as:

$$I(G) := \max\{|F| : F \text{ is an induced forest in } G\}.$$

Theorem 2.15 [28] Let G be a graph with degree sequence $d = (d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$. Then

$$I(G) \geq 2 \sum_{i=1}^{n} \frac{1}{d_i + 1}.$$

□

Corollary 2.16 [28] If G is a graph of order n with maximum degree $\Delta = \Delta(G) \geq 1$, then $I(G) \geq \frac{2n}{\Delta+1}$.

□

An edge of a graph $G = (V, E)$ is said to cover the two vertices incident with it. An *edge cover* of a graph G is a set of edges covering all the vertices of G. The minimum cardinality of an edge cover of G is called the *edge covering number* of G and is denoted by $\beta_1(G)$.

Theorem 2.17 ([6],p.189) For every graph G of order n containing no isolated vertices,

$$\beta_1(G) + \alpha_1(G) = n.$$

□

Theorem 2.18 ([4],p.266) If G is a connected cubic graph of order n containing fewer than $3(k+1)$ bridges, then $\alpha_1(G) \geq \frac{n-2k}{2}$.

□

Theorem 2.19 ([4],p.266) If G is a connected cubic graph of order n all of whose bridges lie on r edge-disjoint paths of G, then

$$\alpha_1(G) \geq \frac{n}{2} - \left\lfloor \frac{2r}{3} \right\rfloor.$$

□

Theorem 2.20 ([4],p.269) Let G be a graph of order n without isolated vertices. Then

$$\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \alpha_1(G) \leq \left\lfloor \frac{n}{2} \right\rfloor .$$

Furthermore, these bounds are sharp.

□

Corollary 2.21 ([4],p.270) Let G be a graph of order n without isolated vertices. Then

$$\left\lceil \frac{n}{2} \right\rceil \leq \beta_1(G) \leq \left\lfloor \frac{n\cdot\Delta(G)}{1+\Delta(G)} \right\rfloor .$$

Furthermore, these bounds are sharp.

□

Next we turn our attention to sets of vertices in a graph G that are adjacent to all vertices of G and study minimum such sets and their cardinality.

An early result of Ore in [22] states that $\gamma(G) \leq \frac{n}{2}$ if G is a graph of order n with the minimum degree at least one. This result was improved as $\gamma(G) \leq \frac{2n}{5}$ by McCuaig and Shepherd in [20] for the connected graph G which has minimum degree at least two and is not one of seven exceptional graphs. In [34], Reed considered the case for the graphs with minimum degree at least three, and obtained that $\gamma(G) \leq \frac{3n}{8}$. From those results, an obvious conjecture (see [22]) seems to be that for any graph G with $\delta(G) \geq k, \gamma(G) \leq \frac{kn}{3k-1}$. However, for $\delta = \delta(G) \geq 7$, Caro and Roditty [39, 40] gave the following better bound.

Theorem 2.22 [39, 40] For any graph G of order n,

$$\gamma(G) \leq n[1 - \delta(\frac{1}{\delta+1})^{1+\frac{1}{\delta}}]$$

□

Theorem 2.23 ([6],p.364) If G is a graph of order n, then

$$\frac{n}{1+\Delta(G)} \leq \gamma(G) \leq n - \Delta(G).$$

□

Theorem 2.24 ([4],p.305) Let G be a graph of order n with $\delta = \delta(G) \geq 2$. Then

$$\gamma(G) \leq \frac{n(1+\ln(\delta+1))}{\delta+1}.$$

□

Theorem 2.25 ([4],p.306) If G is a graph of order n, then

$$\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \gamma(G) \leq n - \Delta(G).$$

□

Corollary 2.26 ([4],p.307) If G is a graph of order n, then $\gamma(G) \leq n - \kappa(G)$.

□

Theorem 2.27 ([4],p.307) If G is a graph of size m and order n for which $\gamma = \gamma(G) \geq 2$, then

$$m \leq \frac{(n-\gamma)(n-\gamma+2)}{2}.$$

□

Theorem 2.28 ([4],p.308) If G is a graph of size m and order n, then

$$n - m \leq \gamma(G) \leq n + 1 - \sqrt{1+2m}.$$

Furthermore, $\gamma(G) = n - m$ if and only if each component of G is a star or an isolated vertex.

□

Theorem 2.29 ([4],p.308) If G is a graph of order $n \geq 2$, then

1. $3 \leq \gamma(G) + \gamma(\overline{G}) \leq n + 1$,
2. $2 \leq \gamma(G) \cdot \gamma(\overline{G}) \leq n$.

□

Theorem 2.30 ([4],p.310) If G is a graph of order $n \geq 2$ such that neither G nor $\overline{G}$ has isolated vertices, then

$$\gamma(G) + \gamma(\overline{G}) \leq \frac{n+4}{2}.$$

□

L.A. Sanchis [37] proved the following results.

Theorem 2.31 [37] Let G be a graph with n vertices, domination number d where $3 \leq d \leq \frac{n}{2}$, and no isolated vertices. Then the number of edges of G is at most $\binom{n-d+1}{2}$. If G has exactly this number of edges then it must be of the following form:

1. An $(n-d)$-clique, together with an independent set of size d, such that each of the vertices in the $(n-d)$-clique is adjacent to exactly one of the vertices in the independent set, and such that each of these d vertices has at least one vertex adjacent to it.

2. For $d = 3$, G may consist of a clique of $n-5$ vertices, together with 5 vertices x_1, x_2, x_3, x_4, x_5, with edges x_1x_3, x_2x_4, x_2x_5, such that every vertex in the $(n-5)$-clique is adjacent to x_4 and x_5, and in addition adjacent to either x_1 or x_3. Moreover, at least one of these vertices is adjacent to x_1 and at least one to x_3.

□

In [1], S. Arumugam and S. Velammal used γ for the domination number and γ' for the edge domination number. A dominating set S is called an *independent dominating set* if no two vertices of S are adjacent. The *independent domination number* $\gamma_i(G)$ (or γ_i for short) of G is the minimum cardinality taken over all independent dominating sets of G.

The maximum order of a partition of E into edge dominating sets of G is called the *edge domatic number* of G and is denoted by $d'(G)$ (or d' for short). An edge dominating set X is called an *independent edge dominating set* if no two edges in X are adjacent. The *independent edge domination number* $\gamma_i'(G)$ (or γ_i' for short) of G is the minimum cardinality taken over all independent edge dominating sets of G. The *edge independence number* $\beta_i(G)$ (or β_i for short) is defined to be the number of edges in a maximum independent set of edges of G. The graph $S(G)$ obtained from G by subdividing each edge of G exactly once is called the *subdivision* of G.

Let G be a graph of order n and size m with edge domination number γ' and edge domatic number d'. S. Arumugam and S. Velammal characterized connected graphs for which $\gamma' = \frac{n}{2}$ and graphs for which $\gamma' + d' = m + 1$. They also characterized trees and unicyclic graphs for which $\gamma' = \lfloor \frac{n}{2} \rfloor$ and $\gamma' = m - \Delta'$, where Δ' denotes the maximum degree of an edge in G.

In [1], the authors used the following theorems to consider the problem of characterizing the class of graphs which they are interested.

Theorem 2.32 [5] For any graph G of order n and size m, $\gamma' \leq \lfloor \frac{n}{2} \rfloor$.

□

Theorem 2.33 [18] For any graph G of order n and size m, $\gamma' \leq m - \Delta'$ where Δ' denotes the maximum degree of an edge in G.

□

Theorem 2.34 [18] For any graph G of order n and size m, $\gamma' \leq m - \beta_i + m_0$ where m_0 is the number of isolated edges in G.

□

Theorem 2.35 [18] For any graph G of order n and size m, $\gamma' + d' \leq m + 1$.

□

S. Arumugam and S. Velammal [1] proved the following results.

Theorem 2.36 [1] For any connected graph G of even order n, $\gamma' = \frac{n}{2}$ if and only if G is isomorphic to K_n or $K_{n/2,n/2}$.

□

Theorem 2.37 [1] For any tree T of order $n \neq 2$, $\gamma' \leq \frac{n-1}{2}$; equality holds if and only if T is isomorphic to the subdivision of a star.

□

The authors use $C_{3,p}$ to denote the graph obtained from a C_3 and p (≥ 0) copies of K_2 by joining one end of each K_2 with a fixed vertex of C_3, and use $C_{4,p}$ to denote the graph obtained from C_4 by joining a vertex of C_4 with the center of $S(K_{1,p})$.

Theorem 2.38 [1] Let G be a connected unicyclic graph of order n. Then $\gamma' = \lfloor \frac{n}{2} \rfloor$ if and only if G is isomorphic to either $C_4, C_5, C_7, C_{3,p}$ or $C_{4,p}$ for some $p \geq 0$.

□

Theorem 2.39 [1] Let T be any tree with size m and let $e = uv$ be an edge of maximum degree Δ'. Then $\gamma' = m - \Delta'$ if and only if $\mathrm{diam}(T) \leq 4$ and $deg\ w \leq 2$ for every vertex $w \neq u, v$.

□

Theorem 2.40 [1] For any connected graph G with size m, $\gamma' = m - \beta_i$ if and only if G is isomorphic to C_4 or the subdivision graph of a star.

□

Theorem 2.41 [1] For any graph G of order n and size m, $\gamma' + d' = m + 1$ if and only if $G = C_3$ or $K_{1,n-1}$ or qK_2.

□

Chartrand and Harary [3] showed the following lower bound for $\kappa(G)$, where G is an arbitrary connected graph.

Theorem 2.42 [3] Let G be a connected graph of order n. If G is not a complete graph, then

$$\kappa(G) \geq 2\delta(G) - n + 2.$$

□

Sufficient conditions for a graph G to be maximally edge-connected ($\lambda(G) = \delta(G)$) were given by several authors, as for example, Chartrand [2], Lesniak [19], Plesník [24], Goldsmith and Entringer [10], Plesník and Znám [25], and Dankelmann and Volkmann [7]. We list as follows:

Theorem 2.43 [2] Let G be a connected graph of order n. If

$$n \leq 2\delta(G) + 1,$$

then $\lambda(G) = \delta(G)$.

□

Theorem 2.44 [19] Let G be a connected graph of order n. If $d(u)+d(v) \geq n-1$ for all pairs u, v of nonadjacent vertices, then $\lambda(G) = \delta(G)$.

□

Theorem 2.45 [24] If G is a connected graph of diameter $dm(G) \leq 2$, then $\lambda(G) = \delta(G)$.

□

Theorem 2.46 [10] Let G be a connected graph of order $n \geq 2$. If

$$\sum_{x \in N(u)} d(x) \geq \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor^2 - \left\lfloor \frac{n}{2} \right\rfloor & \text{for all even } n \text{ and for odd } n \leq 15, \\ \left\lfloor \frac{n}{2} \right\rfloor^2 - 7 & \text{for odd } n \geq 15 \end{cases}$$

for each vertex u of minimum degree, then $\lambda(G) = \delta(G)$.

□

Theorem 2.47 [25] Let G be a connected graph of order n. If there are no four vertices u_1, u_2, v_1, v_2 with

$$d(u_1, u_2), d(u_1, v_2), d(v_1, u_2), d(v_1, v_2) \geq 3,$$

then $\lambda(G) = \delta(G)$.

□

Theorem 2.48 [7] Let G be a connected graph of order n. If for all 3-distance maximal pairs of vertex sets $X, Y \subset V(G)$ there exists an isolated vertex in $G[X \cup Y]$, then $\lambda(G) = \delta(G)$.

□

Theorem 2.49 [7] Let G be a connected graph of order n with clique number $\omega(G) \leq p$. If

$$n \leq 2\left\lfloor \frac{p\delta(G)}{(p-1)} \right\rfloor - 1,$$

then $\lambda(G) = \delta(G)$.

□

2.2 Interpolation Theorems of Some Graph Parameters

N. Punnim has studied interpolation theorems on various kinds of graph parameters with respect to $\mathcal{R}(d)$. He used the switching operation (σ) to transform a realization of graphic degree sequence d to another realization of d. Let f be a graph parameter and d be a graphic degree sequence. It has been shown that *the graph* $\mathcal{R}(d)$ is connected. It follows that for a graph G of degree sequence d and a switching σ if $|f(G) - f(G^\sigma)| \leq 1$, then f is an interpolation graph parameter with respect to $\mathcal{R}(d)$.

N. Punnim showed in [27] the following results that the chromatic number is an interpolation graph parameter with respect to $\mathcal{R}(d)$ and also found its minimum and maximum values.

Theorem 2.50 [27] Let G be a graph and σ be a switching on G. Then

$$|\chi(G) - \chi(G^\sigma)| \leq 1$$

. □

Theorem 2.51 [27] If $r \geq 2$ and $n \geq 2r$, then

$$\min(\chi, r^n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

□

Theorem 2.52 [27] If $r \geq 2$, then

$$\min(\chi, r^{r+1}) = \max(\chi, r^{r+1}) = r + 1, \text{and}$$
$$\min(\chi, r^{r+2}) = \max(\chi, r^{r+2}) = \tfrac{r+2}{2}.$$

□

Theorem 2.53 [27] For any $r \geq 4$ and odd integer s such that $3 \leq s \leq r$, let q and t be integers satisfying $r + s = sq + t$, $0 \leq t < s$. Then

$$\min(\chi, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q + 1 & \text{if } 1 \leq t \leq s - 2, \\ q + 2 & \text{if } t = s - 1. \end{cases}$$

□

Theorem 2.54 [27] For any even integer $r \geq 6$ and any even number s such that $4 \leq s \leq r$, let q and t be integers satisfying $r + s = sq + t$, $0 \leq t < s$. Then

$$\min(\chi, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q + 1 & \text{if } t \geq 2. \end{cases}$$

□

Theorem 2.55 [27] Let $r \geq 2$. Then

1. $\max(\chi, r^{2r}) = r$,
2. $\max(\chi, r^{2r+1}) = \begin{cases} 3 & \text{if } r = 2, \\ r & \text{if } r \geq 4, \end{cases}$

3. $\max(\chi, r^n) = r + 1$ for $n \geq 2r + 2$.

□

Theorem 2.56 [27] For any r and s such that $3 \leq s \leq r - 1$, we have

1. $\max(\chi, r^{r+s}) \geq \frac{r+s}{2}$ if $r + s$ is even, and

2. $\max(\chi, r^{r+s}) \geq \frac{r+s-1}{2}$ if $r + s$ is odd.

□

N. Punnim showed in [26] the following results that the clique number is an interpolation graph parameter with respect to $\mathcal{R}(d)$ and also found its minimum and maximum values.

Theorem 2.57 [26] Let G be a graph and σ be a switching on G. Then

$$|\omega(G) - \omega(G^\sigma)| \leq 1$$

.

□

Theorem 2.58 [26] Let $d = r^n$ be a graphic degree sequence with $r + 2 \leq n \leq 2r + 1$. Then $\max(\omega, r^n) = \left\lfloor \frac{n}{2} \right\rfloor$.

□

Theorem 2.59 [26] For any $r \geq 6$ and odd integer s such that $5 \leq s < r$, let q and t be integers satisfying $r + s = sq + t$, $0 \leq t < s$. Then

$$\min(\omega, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q+1 & \text{if } 1 \leq t \leq s-2, \\ q+2 & \text{if } t = s-1. \end{cases}$$

□

Theorem 2.60 [26] For any even integer $r \geq 6$ and any even number s such that $4 \leq s \leq r$, let q and t be integers satisfying $r + s = sq + t$, $0 \leq t < s$. Then

$$\min(\omega, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q+1 & \text{if } t \geq 2. \end{cases}$$

□

It is clear that $\alpha_0(G)$ is a graph parameter and $\alpha_0(G) = \omega(\overline{G})$, for any graph G. Observe that for a graph G and a switching σ on G, $\overline{G}^{\sigma} = \overline{G^{\sigma}}$. Thus the interpolation result for α_0 in $\mathcal{R}(d)$ follows directly from the graph parameter ω.

N. Punnim showed in [30] the following results that the independence number is an interpolation graph parameter with respect to $\mathcal{R}(d)$.

Theorem 2.61 [30] Let G be a graph and σ be a switching on G. Then

$$|\alpha_0(G) - \alpha_0(G^{\sigma})| \leq 1$$

. □

In [35], R. Samanmoo found the minimum and maximum values of the independence number of regular graphs as follows.

Theorem 2.62 [35] Let $d = (d_1, d_2, \ldots, d_n), d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be a graphic degree sequence where $d_1 + 1 \leq n \leq 2d_1 + 1$. Then $\min(\alpha_0, d) = 1$ if and only if $\mathcal{R}(d) = \{K_n\}$.

□

Theorem 2.63 [35] For any $r \geq 3$, $n = r + j$ and $1 \leq j \leq r + 1$. Then

1. $\min(\alpha_0, r^n) = 1$ if and only if $n = r + 1$,
2. $\min(\alpha_0, r^n) = 2$ for all even integers n and $2 \leq j \leq r$,
3. $\min(\alpha_0, r^n) = 2$ for all odd integers n, $3 \leq j \leq r + 1$ and $n \geq f(j)$,
4. $\min(\alpha_0, r^n) = 3$ for all odd integers n, $3 \leq j \leq r + 1$ and $n < f(j)$,

where $f(j) = \frac{5}{2}(j - 1)$ if $j \equiv 3 \pmod 4$ and $f(j) = 1 + \frac{5}{2}(j - 1)$ if $j \equiv 1 \pmod 4$.

□

Theorem 2.64 [35] For $n \geq 2r+2$ and even integer $r \geq 4$, with $n = (r+1)q+t$, for some $q \geq 2$, and $0 \leq t \leq r$. Then

1. $\min(\alpha_0, r^n) = q$ if $t = 0$,

2. $\min(\alpha_0, r^n) = q + 1$ if $1 \leq t \leq r - 1$,

3. $\min(\alpha_0, r^n) = q + 2$ if $t = r$.

□

Theorem 2.65 [35] For any even integer $n \geq 2r + 2$ and any odd integer $r \geq 3$, with $n = (r + 1)q + t$, for some $q \geq 2$, and $0 \leq t < r$. Then

1. $\min(\alpha_0, r^n) = q$ if $t = 0$,

2. $\min(\alpha_0, r^n) = q + 1$ if $0 < t < r$.

□

Theorem 2.66 [35] For any integer $r \geq 3$ and any integer s such that $3 \leq s < r$, let q and t be integers satisfying $n = r + s = sq + t, q \geq 2$ and $0 \leq t < s$. Then $\max(\alpha_0, r^n) = s$.

□

Theorem 2.67 [35] Let $d = r^n$ be a graphic degree sequence with $n \geq 2r$. Then $\max(\alpha_0, r^n) = \lfloor \frac{n}{2} \rfloor$.

□

In [35], R. Samanmoo also found the minimum and maximum values of the independence number of connected regular graphs as follows:

Theorem 2.68 [35] For $r \geq 3, n \geq 2r + 2$ and a graphic degree sequence r^n. Then $\max(\alpha_0, r^n) = \lfloor \frac{n}{2} \rfloor$ if $n \geq 2r$.

□

Theorem 2.69 [35] $\min(\alpha_0, r^n) \geq \frac{n}{r}$, for all integer $r \geq 3$.

□

Theorem 2.70 [35] For $n \geq 2r + 2$ and even integer $r \geq 4$, write $n = rq + t$ where $0 \leq t < r$. Then

1. $\min(\alpha_0, r^n) = q$ if $t = 0$,

2. $\min(\alpha_0, r^n) = q + 1$ if $1 \le t < r$.

□

Theorem 2.71 [35] For even integer $n \ge 2r + 2$ and odd integer $r \ge 3$, write $n = rq + t$ where $0 \le t < r$. Then

1. $\min(\alpha_0, r^n) = q$ if $t = 0$,

2. $\min(\alpha_0, r^n) = q + 1$ if $1 \le t < r$.

□

N. Punnim showed in [29] the following results that $I(G)$ is an interpolation graph parameter with respect to $\mathcal{R}(d)$ and also found its maximum.

Theorem 2.72 [29] If σ is any switching on G, then $|I(G) - I(G^\sigma)| \le 1$.

□

Theorem 2.73 [29]

$$\max(I, r^n) = \begin{cases} n - r + 1 & \text{if } r + 1 \le n \le 2r - 1, \\ \left\lfloor \frac{nr-2}{2(r-1)} \right\rfloor & \text{if } n \ge 2r. \end{cases}$$

□

Theorem 2.74 [28] Let $d = (d_1, d_2, \ldots, d_n), d_1 \ge d_2 \ge \ldots \ge d_n \ge 1$ be a graphic degree sequence and $d_1 + 1 \le n \le 2d_1 + 1$. Then

1. $\min(I, d) = 2$ if and only if $d_1 = d_2 = d_3 = \ldots = d_n$ and $n = d_1 + 1$ and

2. if d does not have a complete graph as its realization, then $\min(I, d)$= 3 if and only if $\overline{d}$ has a union of stars as its realization.

□

The values of $\min(I, r^n)$, for all r and n, were obtained by N. Punnim in [33] in terms of the graph parameter ϕ as stated in the following theorems. Note that $\min(I, r^n) = \max(\phi, r^n)$.

Theorem 2.75 [33] For $r \geq 3$, and $n = r + j$, $1 \leq j \leq r + 1$

1. $\min(I, r^n) = 2$ if and only if $n = r + 1$,

2. $\min(I, r^n) = 3$ if and only if $n = r + 2$,

3. $\min(I, r^n) = 4$ for all even integers n, $r + 3 \leq n$,

4. $\min(I, r^n) = 4$ for all odd integers n, $r + 3 \leq n$ and $n \geq f(j)$,

5. $\min(I, r^n) = 5$ for all odd integers n, $r + 3 \leq n$ and $n < f(j)$,

where $f(j) = \frac{5}{2}(j-1)$ if $j \equiv 3 \pmod 4$, and $f(j) = 1 + \frac{5}{2}(j-1)$ if $j \equiv 1 \pmod 4$.

□

Theorem 2.76 [33] For $n \geq 2r + 2$ and $r \geq 3$, write $n = (r+1)q + t, q \geq 2$ and $0 \leq t \leq r$.

1. $\min(I, r^n) = 2q$ if $t = 0$,

2. $\min(I, r^n) = 2q + 1$ if $t = 1$,

3. $\min(I, r^n) = 2q + 2$ if $2 \leq t \leq r - 1$,

4. $\min(I, r^n) = 2q + 3$ if $t = r$.

□

Let Δ be a nonnegative integer and n be a positive integer such that $n \geq \Delta+1$. Let $\mathbb{G}(\Delta, n)$ be the class of all graphs of order n and of maximum degree Δ.
The (Δ, n)-graph is a graph having $\mathbb{G}(\Delta, n)$ as its vertex set and two such graphs being adjacent if one can be obtained from the other by either adding or deleting an edge.

N. Punnim showed in [28] the following results that $I(G)$ is an interpolation graph parameter with respect to $\mathbb{G}(\Delta, n)$ and also found its minimum.

Theorem 2.77 [28] *The* (Δ, n)-*graph* is connected.

□

Theorem 2.78 [28] If G_1 and G_2 are adjacent in $\mathbb{G}(\Delta, n)$, then

$$|I(G_1) - I(G_2)| \leq 1.$$

□

Theorem 2.79 [28] Let $n = (\Delta + 1)q + t$, $0 \leq t \leq \Delta$. Then

1. $\min(I, \mathbb{G}(\Delta, n)) = 2q$ if $t = 0$,
2. $\min(I, \mathbb{G}(\Delta, n)) = 2q + 1$ if $t = 1$, and
3. $\min(I, \mathbb{G}(\Delta, n)) = 2q + 2$ if $2 \leq t \leq \Delta$.

□

For a graph G, the minimum number of vertices whose removal eliminates all cycles in a graph G is the *decycling number* of G, and is denoted by $\phi(G)$. It is easy to see that for a graph G of order n, $\phi(G) + I(G) = n$. Thus the interpolation result for ϕ in $\mathcal{R}(d)$ is easily obtained.

Theorem 2.80 [33] If σ is a switching on G, then $|\phi(G) - \phi(G^\sigma)| \leq 1$.

□

In [33], N. Punnim showed the minimum and maximum values of the decycling number of regular graphs.

Theorem 2.81 [33]

$$\min(\phi, r^n) = \begin{cases} r - 1 & \text{if } r + 1 \leq n \leq 2r - 1, \\ \left\lceil \frac{nr-2n+2}{2(r-1)} \right\rceil & \text{if } n \geq 2r. \end{cases}$$

□

Theorem 2.82 [33] Let $d = (d_1, d_2, \ldots, d_n), d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be a graphic degree sequence and $d_1 + 1 \leq n \leq 2d_1 + 1$. Then

1. $\max(\phi, d) = n - 2$ if and only if $\mathcal{R}(d) = \{K_n\}$ and

2. if $K_n \notin \mathcal{R}(d)$, then $\max(\phi, d) = n - 3$ if and only if there exists a union of stars as a realization of $\overline{d}$, where $\overline{d} = (n - d_n, n - d_{n-1}, \ldots, n - d_1)$.

□

Theorem 2.83 [33] For $r \geq 3$, and $r + 1 \leq n \leq 2r + 1$,

1. $\max(\phi, r^n) = n - 2$ if and only if $n = r + 1$,

2. $\max(\phi, r^n) = n - 3$ if and only if $n = r + 2$,

3. $\max(\phi, r^n) = n - 4$ for all even integers n, $r + 3 \leq n$,

4. $\max(\phi, r^n) = n - 4$ for all odd integers n, $r + 3 \leq n$ and $n \geq f(j)$,

5. $\max(\phi, r^n) = n - 5$ for all odd integers n, $r + 3 \leq n$ and $n < f(j)$,

where $f(j) = \frac{5}{2}(j - 1)$ if $j \equiv 3 \pmod 4$, and $f(j) = 1 + \frac{5}{2}(j - 1)$ if $j \equiv 1 \pmod 4$.

□

Theorem 2.84 [33] For $n \geq 2r + 2$ and $r \geq 3$, write $n = (r + 1)q + t, q \geq 2$ and $0 \leq t \leq r$. Then

1. $\max(\phi, r^n) = n - 2q$ if $t = 0$,

2. $\max(\phi, r^n) = n - 2q - 1$ if $t = 1$,

3. $\max(\phi, r^n) = n - 2q - 2$ if $2 \leq t \leq r - 1$,

4. $\max(\phi, r^n) = n - 2q - 3$ if $t = r$.

□

N. Punnim also showed in [33] the following results that the decycling number is an interpolation graph parameter with respect to $\mathbb{G}(\Delta, n)$ and also found its maximum.

Theorem 2.85 [33] If G_1 and G_2 are adjacent in $\mathbb{G}(\Delta, n)$, then

$$|\phi(G_1) - \phi(G_2)| \leq 1.$$

□

Theorem 2.86 [33] Let $n = (\Delta+1)q + t, 0 \leq t \leq \Delta$. Then

1. $\max(\phi, \mathbb{G}(\Delta, n)) = n - 2q$ if $t = 0$,
2. $\max(\phi, \mathbb{G}(\Delta, n)) = n - 2q - 1$ if $t = 1$, and
3. $\max(\phi, \mathbb{G}(\Delta, n)) = n - 2q - 2$ if $2 \leq t \leq \Delta$.

□

Moreover, in [33], N. Punnim showed the minimum and maximum values of the decycling number of cubic graphs.

Theorem 2.87 [33] For any integer $n \geq 2$,

$$\min(\phi, 3^{2n}) = \left\lceil \frac{n+1}{2} \right\rceil, \text{ and}$$

$$\max(\phi, 3^{2n}) = \begin{cases} n & \text{if } n \text{ is even,} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

□

N. Punnim showed in [31] the following results that the matching number is an interpolation graph parameter with respect to $\mathcal{R}(d)$.

Theorem 2.88 [31] If σ is a switching on G, then $|\alpha_1(G) - \alpha_1(G^\sigma)| \leq 1$.

□

In [32], N. Punnim investigated the values of $\min(\alpha_1, r^n)$ and $\max(\alpha_1, r^n)$ for all r and n. It is easy to see that $\min(\alpha_1, 0^n) = \max(\alpha_1, 0^n) = 0$ and $\min(\alpha_1, 1^{2n}) = \max(\alpha_1, 1^{2n}) = n$. Because of this fact, we will consider $r \geq 2$ and $n \geq r+1$.

Theorem 2.89 [32] For $r \geq 2, n \geq r+1$ and $nr \equiv 0 \pmod 2$, there exists an r-regular hamiltonian graph of order n. In particular, $\max(\alpha_1, r^n) = \left\lfloor \frac{n}{2} \right\rfloor$.

□

Let $F(r, d)$ be the minimum order of an r-regular graph G with $\alpha_1(G) = \frac{1}{2}(|V(G)| - d)$. It is clear that $|V(G)| \equiv d \pmod 2$.

Theorem 2.90 [32] Let r be an even integer, $r \geq 2$. Then $F(r,d) = d(r+1)$.

□

Theorem 2.91 [32] Let r be an even integer, $r \geq 2$. If $n = (r+1)d+e, 0 \leq e \leq r$, then $\min(\alpha_1, r^n) = \frac{dr}{2} + \lfloor \frac{1+e}{2} \rfloor$.

□

Theorem 2.92 [32] For an odd integer $r \geq 3$. Then

1. $F(r, 2q) = (r+2)(1+2q)+1$ for $q = 1, 2, \ldots, \frac{r-1}{2}$,
2. if $q = \frac{r-1}{2}s + t, 0 \leq t < \frac{r-1}{2}$, then $F(r,2q) = sF(r, r-1) + F(r, 2t)$ where $F(r, 0) = 0$.

□

Corollary 2.93 [32] Let r be an odd integer, $r \geq 3$.
If $F(r, 2q) \leq n < F(r, 2(q+1))$, then $\min(\alpha_1, r^n) = \frac{1}{2}(n-2q)$.

□

N. Punnim showed in [31] that the domination number is an interpolation graph parameter with respect to $\mathcal{R}(d)$.

Theorem 2.94 [31] If σ is a switching on G, then $|\gamma(G) - \gamma(G^\sigma)| \leq 1$.

□

In [38], Topp and Vestergaard considered the interpolating character of a number of domination related parameters on the set of all spanning trees of connected graphs.

For a connected graph G, let $\mathcal{T}(G)$ be the set of all spanning trees of G. Let T be a spanning tree of G and let e be an edge of G which is not in T. If f is an edge which belongs to the unique cycle of $T+e$, then $T+e-f$ is a spanning tree of G and the transformation of T into $T+e-f$ is called a *simple edge-exchange*. If e and f are adjacent edges of G, then the transformation of T into $T+e-f$ is called an *adjacent edge-exchange*. An adjacent edge-exchange of T into $T+e-f$

is called an *end edge-exchange* if e and f are incident with a common end vertex of T (and then also of $T + e - f$).

An integer-valued graph function π is said to *interpolate over (the spanning trees of) a connected graph* G if the set $\pi(\mathcal{T}(G)) = \{\pi(T) : T \in \mathcal{T}(G)\}$ consists of consecutive integers, i.e. $\pi(\mathcal{T}(G))$ is an integer interval. We shall call π an *interpolating function* if π interpolates over each connected graph.

They consider vertex, edge and mixed (here called total) versions of independent, dominating and irredundant sets in a graph. More precisely, a subset I of $V(G)$ $(E(G), V(G) \cup E(G)$, resp.) is said to be *vertex (edge, total,* resp.) *distance k-independent* (shortly $VDkI$ $(EDkI, TDkI$, resp.)) in G if $N^k_G(I) \cap I = \emptyset$ $(N^k_{L(G)}(I) \cap I = \emptyset$, $N^k_{T(G)}(I) \cap I = \emptyset$, resp.) A subset D of $V(G)$ $(E(G), V(G) \cup E(G)$, resp.) is said to be *vertex (edge, total,* resp.) *distance k-dominating* (shortly $VDkD$ $(EDkD, TDkD$, resp.)) in G if $N^k_G[D] = V(G)$ $(N^k_{L(G)}[D] = E(G)$, $N^k_{T(G)}[D] = V(G) \cup E(G)$, resp.). A subset X of $V(G)$ $(E(G), V(G) \cup E(G)$, resp.) is said to be *vertex (edge, total,* resp.) *distance k-irredundant* (shortly $VDkIr$ $(EDkIr, TDkIr$, resp.)) in G if $I^k_G[x, X] \neq \emptyset$ $(I^k_{L(G)}[x, X] \neq \emptyset$, $I^k_{T(G)}[x, X] \neq \emptyset$, resp.) for every $x \in X$. A vertex distance 1-independent (1-dominating, 1-irredundant, resp.) set in a graph G is shortly said to be independent (dominating, irredundant, resp.) in G. The *lower vertex (edge, total,* resp.) *distance k-independence number* $i_k(G)$ $(i'_k(G), i''_k(G)$, resp.) of a graph G is defined to be the cardinality of a minimum maximal $VDkI$ $(EDkI, TDkI$, resp.) set of G. The *upper vertex (edge, total,* resp.) *distance k-independence number* $\alpha_k(G)$ $(\alpha'_k(G), \alpha''_k(G)$, resp.) of G is the cardinality of a maximum $VDkI$ $(EDkI, TDkI$, resp.) set of G. The *lower vertex (edge, total,* resp.) *distance k-domination number* $\gamma_k(G)$ $(\gamma'_k(G), \gamma''_k(G)$, resp.) of G is the cardinality of a minimum $VDkD$ $(EDkD, TDkD$, resp.) set of G. The *upper vertex (edge, total,* resp.) *distance k-domination number* $\Gamma_k(G)$ $(\Gamma'_k(G), \Gamma''_k(G)$, resp.) of G is the cardinality of a maximum $VDkD$ $(EDkD, TDkD$, resp.) set of G. The *lower vertex (edge, total,* resp.) *distance k-irredundance number* $ir_k(G)$ $(ir'_k(G), ir''_k(G)$, resp.) of G is the cardinality of a minimum maximal $VDkIr$ $(EDkIr, TDkIr$, resp.) set of G. The *upper vertex (edge, total,* resp.) *distance k-irredundance num-*

ber $IR_k(G)$ ($IR'_k(G), IR''_k(G)$, resp.) of G is the cardinality of a maximum $VDkIr$ ($EDkIr, TDkIr$, resp.) set of G.

It is clear from the above definitions that if π_k is one of the six vertex parameters ir_k, γ_k, i_k, α_k, Γ_k, IR_k and if π'_k and π''_k are, respectively, the edge and total versions of the parameter π_k, then for any graph G,

$$\pi'_k(G) = \pi_k(L(G)) \quad \text{and} \quad \pi''_k(G) = \pi_k(T(G)). \tag{2.1}$$

It is also easy to observe that a set S of vertices of a graph G is a $VDkI$ ($VDkD, VDkIr$, resp.) set of G if and only if S is a $VD1I$ ($VD1D, VD1Ir$, resp.) set of G^k. Consequently, for any graph G,

$$\pi_k(G) = \pi_1(G^k), \quad \pi'_k(G) = \pi_1((L(G))^k) \quad \text{and} \quad \pi''_k(G) = \pi_1((T(G))^k), \tag{2.2}$$

where $\pi_k \in \{i_k, \alpha_k, \gamma_k, \Gamma_k, ir_k, IR_k\}$ and again π'_k and π''_k are respectively the edge and total versions of π_k. The parameters $i_1, \alpha_1, \gamma_1, \Gamma_1, ir_1$ and IR_1 are well known and it is clear (see [17]) that for any graph H,

$$ir_1(H) \le \gamma_1(H) \le i_1(H) \le \alpha_1(H) \le \Gamma_1(H) \le IR_1(H). \tag{2.3}$$

Now it is clear from (2.1)-(2.3) that for any graph G,

$$\begin{aligned} ir_k(G) \le \gamma_k(G) \le i_k(G) \le \alpha_k(G) \le \Gamma_k(G) \le IR_k(G). \\ ir'_k(G) \le \gamma'_k(G) \le i'_k(G) \le \alpha'_k(G) \le \Gamma'_k(G) \le IR'_k(G). \\ ir''_k(G) \le \gamma''_k(G) \le i''_k(G) \le \alpha''_k(G) \le \Gamma''_k(G) \le IR''_k(G). \end{aligned} \tag{2.4}$$

Topp and Vestergaard [38] proved the following results.

Theorem 2.95 [38] For any positive integer k, the upper distance k-independence numbers α_k, α'_k and α''_k are interpolating functions.

□

Corollary 2.96 [38] For any positive integer k, the upper distance k-domination numbers Γ_k, Γ'_k and Γ''_k and the upper distance k-irredundance numbers IR_k, IR'_k and IR''_k are interpolating functions.

□

Theorem 2.97 [38] For any positive integer k, the lower distance k-domination numbers γ_k, γ'_k and γ''_k are interpolating functions.

□

Theorem 2.98 [38] The lower vertex distance 1- and 2-independence numbers i_1 and i_2 interpolate over every 2-connected graph.

□

Corollary 2.99 [38] The lower edge distance 1-independence number i'_1 is an interpolating function.

□

Theorem 2.100 [38] The lower edge distance 2-independence number i'_2 interpolates over every 2-connected graph.

□

Theorem 2.101 [38] The lower total distance 1- and 2-independence numbers i''_1 and i''_2 interpolate over any 2-connected graph.

□

The n-domination and n-dependence numbers of a graph were introduced by Fink and Jacobson in [11]. Let n be a positive integer. An *n-dominating set* of a graph G is a subset D of $V(G)$ such that $|N_G(v) \cap D| \geq n$ for every $v \in V(G) - D$. The *n-domination number* of G, denoted by $\gamma_{(n)}(G)$, is the minimum cardinality of an n-dominating set of G. An *n-depending set* of a graph G is a set $I \subseteq V(G)$ such that $|N_G(v) \cap I| < n$ for every $v \in I$. The *n-dependence number* of G, denoted by $\alpha_{(n)}(G)$, is the maximum cardinality of an n-depending set of G. Certainly, $\gamma_{(1)}(G) = \gamma_1(G)$ and $\alpha_{(1)}(G) = \alpha_1(G)$.

Theorem 2.102 [38] For any positive integer n, the n-domination number $\gamma_{(n)}$ and the n-dependence number $\alpha_{(n)}$ are interpolating functions.

□

A set S of vertices of a graph G is said to be a *global dominating set* of G if S is a dominating set both of G and of its complement $\overline{G}$. The *global domination* *number* of G, denoted by $\gamma_g(G)$, is the minimum cardinality of a global dominating set of G. The global domination number was introduced by Sampathkumar in [36].

Theorem 2.103 [38] The global domination number γ_g is an interpolating function.

□

A set D of vertices in a graph G is a *total dominating set* if each vertex of G is adjacent to a vertex in D. Total dominating sets were first defined and studied by Cockayne et al. [8]. The cardinality of a minimum total dominating set in a graph G is called the *total domination number* of G and is denoted by $\gamma_t(G)$.

Theorem 2.104 [38] The total domination number γ_t interpolates over any 2-connected graph.

□

CHAPTER 3

INTERPOLATION THEOREMS FOR THE CONNECTIVITY AND THE DOMINATION NUMBERS OF CONNECTED GRAPHS

To establish the interpolation property of the graph parameters, we use a technique to transform graphs in $\mathcal{CG}(m,n)$. We introduce the transformation and define *the graph* $\mathcal{CG}(m,n)$ as follows:

Let $G = (V, E)$ be a graph, $e \in E(G)$ and $f \in E(\overline{G})$. We define the graph $G^{t(e,f)}$ is a graph with $V(G^{t(e,f)}) = V(G)$ and $E(G^{t(e,f)}) = (E(G) - \{e\}) \cup \{f\}$. The transformation $t(e,f)$ described above is called a *jumping transformation.* Let $\mathcal{CG}(m,n)$ be the class of all nonisomorphic graphs of size m and order n. *The graph* $\mathcal{CG}(m,n)$ is a graph having $\mathcal{CG}(m,n)$ as its vertex set; two vertices are being adjacent if one can be obtained from the other by a single jumping transformation. Note that for any graph G and a jumping transformation $t(e,f)$ is well defined on G if and only if a jumping transformation $t(f,e)$ is well defined on $G^{t(e,f)}$. Thus *the graph* $\mathcal{CG}(m,n)$ is simple.

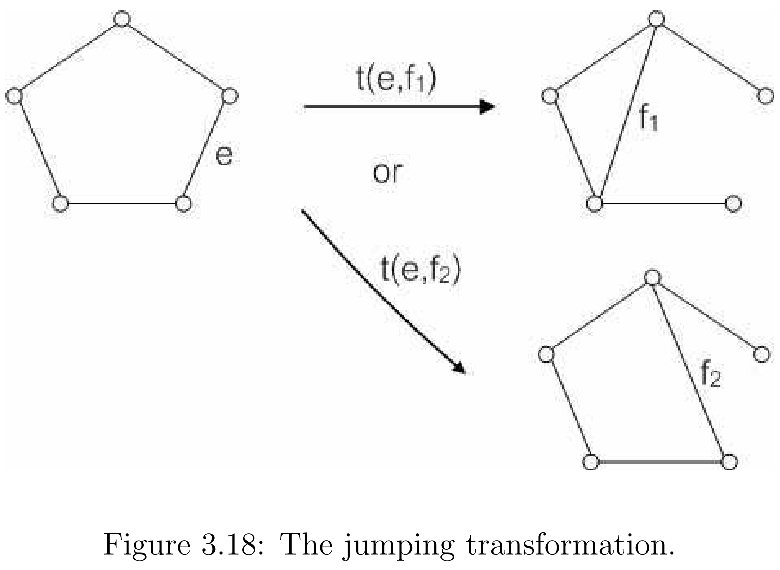

Figure 3.18: The jumping transformation.

The next example illustrates *the graph* $\mathcal{CG}(6,5)$. We can see that *the graph* $\mathcal{CG}(6,5)$ is connected.

Example 3.1.1 Consider the class of connected graphs of size 6 and order 5 ($\mathcal{CG}(6,5)$), there are 5 nonisomorphic graphs G_1, G_2, G_3, G_4, G_5 in $\mathcal{CG}(6,5)$ and it can be illustrated as in Figure 3.19.

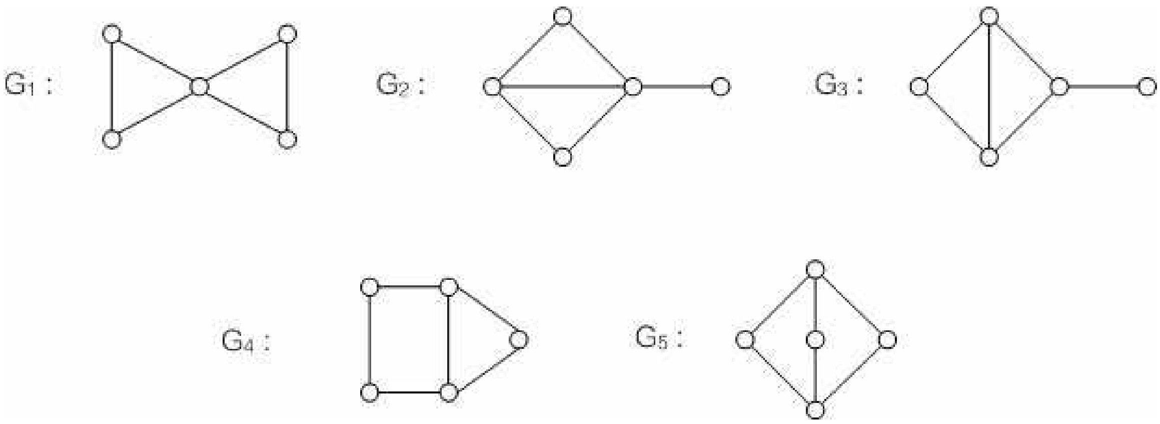

Figure 3.19: Graphs $G_1, G_2, G_3, G_4, G_5 \in \mathcal{CG}(6,5)$.

To illustrate that *the graph* $\mathcal{CG}(6,5)$ is connected, let's use the nonisomorphic graphs G_1, G_2, G_3, G_4, G_5 to represent each vertex of $\mathcal{CG}(6,5)$ as shown in Figure 3.20.

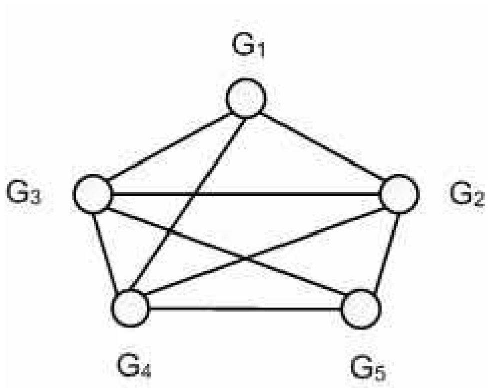

Figure 3.20: *The graph* $\mathcal{CG}(6,5)$.

This chapter consists of two sections. In section 3.1, we show the interpolation property of the graph parameters κ, λ, γ and γ' with respect to $\mathcal{CG}(m,n)$.

In section 3.2, we determine the extreme values of all those graph parameters for connected graphs of order n and size m.

3.1 Interpolation Theorems

To show the interpolation theorems of our interesting graph parameters, we need the following results.

Let $G, H \in \mathcal{CG}(m,n)$ and $G \ncong H$. We have $|E(G)-E(H)| = |E(H)-E(G)|$. Let $E(G)-E(H) = \{e_1, e_2, \ldots, e_k\}$ and $E(H)-E(G) = \{f_1, f_2, \ldots, f_k\}$. Observe that for $e_i \in E(G)-E(H)$ and $f_j \in E(H)-E(G)$, $G^{t(e_i,f_j)}$ may be disconnected. We will show the following lemma.

Lemma 3.1 Let $G, H \in \mathcal{CG}(m,n)$ and $G \ncong H$. For every edge $e \in E(G)-E(H)$ there exists an edge $f \in E(H)-E(G)$ such that a graph $G^{t(e,f)}$ is connected.

Proof Let $e \in E(G)-E(H)$. If a graph $G-e$ is connected, then a graph $G-e+f$ is connected for any edge $f \in E(H)-E(G)$. We may assume that a graph $G-e$ is disconnected. Thus the graph $G-e$ has exactly two components, say $G-e = G_1 \cup G_2$. Since the graph H is connected, there exists a spanning tree T of H. It follows that $E(T)-E(G-e) \neq \emptyset$. That is there exists an edge $f \in E(T)-E(G-e)$ such that $f = uv$ where $u \in E(G_1)$ and $v \in E(G_2)$. Therefore a graph $G-e+f = G^{t(e,f)}$ is connected.

□

Let $G \in \mathcal{CG}(m,n)$. By Lemma 3.1, there is a jumping transformation $t(e_1, f_1)$ on G such that $G_1 = G^{t(e_1,f_1)} \in \mathcal{CG}(m,n)$. By applying Lemma 3.1 again there is a jumping transformation $t(e_2, f_2)$ on G_1 such that $G_2 = G_1^{t(e_2,f_2)} \in \mathcal{CG}(m,n)$ and so on.

Theorem 3.2 Let $G, H \in \mathcal{CG}(m,n)$. Then $G \cong H$ or H can be obtained from G by a finite sequence of jumping transformations.

Proof Assume that $G \ncong H$. Let $|E(G)-E(H)| = |E(H)-E(G)| = k \geq 1$.

We prove by induction on k. By Lemma 3.1, the statement is true when $k = 1$. Suppose that $k > 1$. Then there exists a jumping transformation $t(e_1, f_1)$ such that $G_1 = G^{t(e_1,f_1)}$ and $|E(G_1) - E(H)| \leq k - 1$. By induction hypothesis, H can be obtained from G_1 by a finite sequence of jumping transformations. It is clear that G_1 can be obtained from G by a jumping transformation $t(e_1, f_1)$. Thus H can be obtained from G by a finite sequence of jumping transformations.

□

As a consequence of Theorem 3.2, we have the following corollary.

Corollary 3.3 *The graph* $\mathcal{CG}(m, n)$ is connected.

□

Let $G \in \mathcal{CG}(m, n)$ and let t be a jumping transformation on G. As the definition of *the graph* $\mathcal{CG}(m, n)$, it is clear that graphs G and G^t are adjacent in *the graph* $\mathcal{CG}(m, n)$. Let f be a graph parameter. Since *the graph* $\mathcal{CG}(m, n)$ is connected, so we have the following theorems.

Theorem 3.4 For a graph $G \in \mathcal{CG}(m, n)$ and a jumping transformation t, if $|f(G) - f(G^t)| \leq 1$, then f is an interpolation graph parameter with respect to $\mathcal{CG}(m, n)$.

□

Theorem 3.5 Let $f \in \{\kappa, \lambda, \gamma, \gamma'\}$. Then there exist integers $a(f) = \min\{f(G) \mid G \in \mathcal{CG}(m, n)\}$ and $b(f) = \max\{f(G) \mid G \in \mathcal{CG}(m, n)\}$ such that there is a graph $G \in \mathcal{CG}(m, n)$ with $f(G) = c$ if and only if c is an integer satisfying $a(f) \leq c \leq b(f)$.

□

We now prove our results on interpolation theorems of the graph parameters κ, λ, γ, and γ', respectively, as follows :

Theorem 3.6 Let $G \in \mathcal{CG}(m, n)$ and let t be a jumping transformation on G. Then $\mid \kappa(G) - \kappa(G^t) \mid \leq 1$.

Proof Let $e = uv \in E(G)$ where $G \in \mathcal{CG}(m, n)$ and let S be a minimum vertex-cut of $G-e$ with cardinality $\kappa(G-e)$. Then $S \cup \{u\}$ or $S \cup \{v\}$ is also a vertex-cut of G, that is, $\kappa(G) \leq \kappa(G-e)+1$. Since $G-e$ is a subgraph of G, so $\kappa(G-e) \leq \kappa(G)$. Thus

$$\kappa(G) - 1 \leq \kappa(G - e) \leq \kappa(G). \tag{3.6.1}$$

Let $f = xy \in E(\overline{G})$. Since $G - e$ is a subgraph of $G - e + f$, so $\kappa(G - e) \leq \kappa(G - e + f)$. Then $S \cup \{x\}$ or $S \cup \{y\}$ is also a vertex-cut of $G - e + f$, that is $\kappa(G - e + f) \leq \kappa(G - e) + 1$. Thus

$$\kappa(G - e) \leq \kappa(G - e + f) \leq \kappa(G - e) + 1. \tag{3.6.2}$$

Combine (3.6.1) and (3.6.2), we have $\kappa(G) - 1 \leq \kappa(G - e + f) \leq \kappa(G) + 1$. That is $-1 \leq \kappa(G) - \kappa(G - e + f) \leq 1$. Thus $\mid \kappa(G) - \kappa(G^t) \mid \leq 1$.

□

Theorem 3.7 Let $G \in \mathcal{CG}(m, n)$ and let t be a jumping transformation on G. Then $\mid \lambda(G) - \lambda(G^t) \mid \leq 1$.

Proof Let $e \in E(G)$ where $G \in \mathcal{CG}(m, n)$ and let X be a minimum edge-cut of $G - e$ with cardinality $\lambda(G - e)$. Then $X \cup \{e\}$ is also an edge-cut of G, that is, $\lambda(G) \leq \lambda(G - e) + 1$. Since $G - e$ is a subgraph of G, so $\lambda(G - e) \leq \lambda(G)$. Thus

$$\lambda(G) - 1 \leq \lambda(G - e) \leq \lambda(G). \tag{3.7.1}$$

Let $f \in E(\overline{G})$. Since $G - e$ is a subgraph of $G - e + f$, so $\lambda(G - e) \leq \lambda(G - e + f)$. Then $X \cup \{f\}$ is also an edge-cut of $G - e + f$, that is $\lambda(G - e + f) \leq \lambda(G - e) + 1$. Thus

$$\lambda(G - e) \leq \lambda(G - e + f) \leq \lambda(G - e) + 1. \tag{3.7.2}$$

Combine (3.7.1) and (3.7.2), we have $\lambda(G) - 1 \leq \lambda(G - e + f) \leq \lambda(G) + 1$. That is $-1 \leq \lambda(G) - \lambda(G - e + f) \leq 1$. Thus $\mid \lambda(G) - \lambda(G^t) \mid \leq 1$.

□

Theorem 3.8 Let $G \in \mathcal{CG}(m, n)$ and let t be a jumping transformation on G. Then $\mid \gamma(G) - \gamma(G^t) \mid \leq 1$.

Proof Let $e = uv \in E(G)$ where $G \in \mathcal{CG}(m,n)$. Since every dominating set of $G - e$ is also a dominating set of G, it follows that $\gamma(G) \leq \gamma(G-e)$. Let D be a minimum dominating set of G. If either both $u, v \in D$ or both $u, v \notin D$, then D is a dominating set of $G - e$. That is $\gamma(G-e) \leq \gamma(G)$. If u or v is an element of D, say u, then $D \cup \{v\}$ is a dominating set of $G - e$. That is $\gamma(G-e) \leq \gamma(G) + 1$. Thus

$$\gamma(G) \leq \gamma(G-e) \leq \gamma(G) + 1. \qquad (3.8.1)$$

Let $f = xy \in E(\overline{G})$. Since every dominating set of $G - e$ is also a dominating set of $G - e + f$, it follows that $\gamma(G-e+f) \leq \gamma(G-e)$. Let D' be a minimum dominating set of $G - e + f$. If either both $x, y \in D'$ or both $x, y \notin D'$, then D' is a dominating set of $G - e$. That is $\gamma(G-e) - 1 \leq \gamma(G-e) \leq \gamma(G-e+f)$. If x or y is an element of D', say x, then $D' \cup \{y\}$ is a dominating set of $G - e$. That is $\gamma(G-e) \leq \gamma(G-e+f) + 1$. Thus

$$\gamma(G-e) - 1 \leq \gamma(G-e+f) \leq \gamma(G-e). \qquad (3.8.2)$$

Combine (3.8.1) and (3.8.2), we have $\gamma(G) - 1 \leq \gamma(G-e+f) \leq \gamma(G) + 1$. That is $-1 \leq \gamma(G) - \gamma(G-e+f) \leq 1$. Thus $\mid \gamma(G) - \gamma(G^t) \mid \leq 1$.

□

Theorem 3.9 Let $G \in \mathcal{CG}(m,n)$ and let t be a jumping transformation on G. Then $\mid \gamma'(G) - \gamma'(G^t) \mid \leq 1$.

Proof Let $e \in E(G)$ where $G \in \mathcal{CG}(m,n)$. Since every edge dominating set of G is also edge dominating set of $G - e$, so $\gamma'(G-e) \leq \gamma'(G)$. Let X be an edge dominating set of $G - e$. If e is adjacent to any edge in X, then X is an edge dominating set of G. That is $\gamma'(G) \leq \gamma'(G-e)$. If e is not adjacent to an edge in X, then $X \cup \{e\}$ is an edge dominating set of G. That is $\gamma'(G) \leq \gamma'(G-e) + 1$. Thus

$$\gamma'(G) - 1 \leq \gamma'(G-e) \leq \gamma'(G). \qquad (3.9.1)$$

Let $f \in E(\overline{G})$. Since every edge dominating set of $G-e+f$ is also edge dominating set of $G - e$, so $\gamma'(G-e) \leq \gamma'(G-e+f)$. If f is adjacent to any edge in X, then X is an edge dominating set of $G-e+f$. That is $\gamma'(G-e+f) \leq \gamma'(G-e)$. If f is not adjacent to an edge in X, then $X \cup \{f\}$ is an edge dominating set of $G - e + f$. That is $\gamma'(G - e + f) \leq \gamma'(G - e) + 1$. Thus

$$\gamma'(G - e) \leq \gamma'(G - e + f) \leq \gamma'(G - e) + 1. \tag{3.9.2}$$

Combine (3.9.1) and (3.9.2), we have $\gamma'(G) - 1 \leq \gamma'(G - e + f) \leq \gamma'(G) + 1$. That is $-1 \leq \gamma'(G) - \gamma'(G - e + f) \leq 1$. Thus $\mid \gamma'(G) - \gamma'(G^t) \mid \leq 1$.

□

3.2 Extremal Problems

In this section we show the second part of the interpolation theorems for all graph parameters $f \in \{\kappa, \lambda, \gamma, \gamma'\}$ with respect to $\mathcal{CG}(m, n)$. We shall determine $\min\{f(G) \mid G \in \mathcal{CG}(m, n)\}$ and $\max\{f(G) \mid G \in \mathcal{CG}(m, n)\}$, simply written $\min(f; m, n)$ and $\max(f; m, n)$, respectively, for all $f \in \{\kappa, \lambda, \gamma, \gamma'\}$.

To achieve the minimum and maximum values of a graph parameter f for graphs in $\mathcal{CG}(m, n)$, first, we find its lower bound and upper bound that are the best possible. We show each $\min(f; m, n) = k$ or $\max(f; m, n) = \ell$ in the sense that for every two integers m and n with $1 \leq n - 1 \leq m \leq \binom{n}{2}$, there exist a graph G and a graph H of size m and order n such that $f(G) = k$ and $f(H) = \ell$; $f \in \{\kappa, \lambda, \gamma, \gamma'\}$. Otherwise, we will prove that it cannot construct any graph G with $f(G)$ equal to those bounds then construct a graph G whose $f(G)$ is almost equal to the bounds. This means that we obtain the minimum or maximum values of that graph parameter.

3.2.1 Vertex - Connectivity (κ)

The vertex-connectivity $\kappa(G)$ of G is simply called the connectivity. For a nontrivial graph G, $\kappa(G) = 0$ if and only if G is disconnected and $\kappa(G) = 1$ if and only if $G \cong K_2$. If $G \cong K_n$ for a positive integer n, then $\kappa(G) = n - 1$. If G is a graph of size m and order n such that $\kappa(G) \geq 1$, then, necessarily, $m \geq n - 1$.

Thus, we consider $\mathcal{CG}(m,n)$ for $n-1 \leq m < \binom{n}{2}$ and $n \geq 3$. We shall determine $\min\{\kappa(G) \mid G \in \mathcal{CG}(m,n)\}$ and $\max\{\kappa(G) \mid G \in \mathcal{CG}(m,n)\}$, written $\min(\kappa; m, n)$ and $\max(\kappa; m, n)$, respectively.

The following theorem due to Hassler Whitney in 1932 is useful for our proof.

Theorem 3.10 ([6],p.117) For every graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

□

We now determine the minimum values of $\kappa(G)$ for all $G \in \mathcal{CG}(m,n)$.

Proposition 3.11 If $n-1 \leq m \leq \binom{n-1}{2}+1$, then $\min(\kappa; m, n) = 1$.

Proof It is clear that any connected graph G of order n with size $m = n-1$ is a tree. Then $\kappa(G) = 1$. Consider $m = \binom{n-1}{2}+1$. Let H be a graph obtained from K_{n-1} by adding a new vertex v to K_{n-1} and then joining any one vertex of K_{n-1} to the vertex v. It is clear that $H \in \mathcal{CG}(m,n)$. Since H has exactly one vertex v of degree one, so $\kappa(H) = 1$. By the fact that for any connected graph G, $\kappa(G) \geq 1$ and $\min(\kappa; k, n) \leq \min(\kappa; k+1, n)$ for a fix integer n and $n-1 \leq k \leq \binom{n-1}{2}$, we can conclude that $\min(\kappa; m, n) = 1$ for $n-1 \leq m \leq \binom{n-1}{2}+1$.

□

Note that for positive integers p and q, $(1-p)(1-q) = 1-p-q+pq \geq 0$. Thus $1+pq \geq p+q$.

Proposition 3.12 Let $G \in \mathcal{CG}(m,n)$. If $\binom{n-1}{2}+1 < m < \binom{n}{2}$, then $\kappa(G) \geq m - \binom{n-1}{2}$.

Proof Let U be a minimum vertex-cut of G with cardinality k. If the vertex set $V(G)$ is partitioned into $U, V_1(G)$, and $V_2(G)$ where $|V_1(G)| = p$ and $|V_2(G)| = q$, then $n = k+p+q$. To achieve the best lower bound of $\kappa(G)$ we have to consider the maximum possible number of edges in G. That is $m \leq \binom{n}{2} - pq$. Observe that

$$\begin{aligned} m &\leq \binom{n}{2} - pq \\ &= \binom{n-1}{2} + (n-1) - pq \\ &\leq \binom{n-1}{2} + n - (p+q) \\ &= \binom{n-1}{2} + k. \end{aligned}$$

Hence $\kappa(G) = k \geq m - \binom{n-1}{2}$.

□

Next, we show that the lower bound in Proposition 3.12 is sharp.

Proposition 3.13 If $\binom{n-1}{2} + 1 < m < \binom{n}{2}$, then $\min(\kappa; m, n) = m - \binom{n-1}{2}$.

Proof Let $G \in \mathcal{CG}(m, n)$. From Proposition 3.12, we have $\kappa(G) = k \geq m - \binom{n-1}{2}$. We can see that if $m > \binom{n-1}{2} + 1$, then $k > 1$. Consider $k = m - \binom{n-1}{2}$. Let H be a graph obtained from K_{n-1} by adding a new vertex u to K_{n-1} and then joining k vertices of K_{n-1} to u. It is clear that the graph $H \in \mathcal{CG}(m, n)$ where $m = \binom{n-1}{2} + k$. Since $k = m - \binom{n-1}{2} < \binom{n}{2} - \binom{n-1}{2}$, so $k \leq n - 2$. Hence $\kappa(H) = k$. Thus $\min(\kappa; m, n) = k$. Figure 3.21 illustrates the graph H.

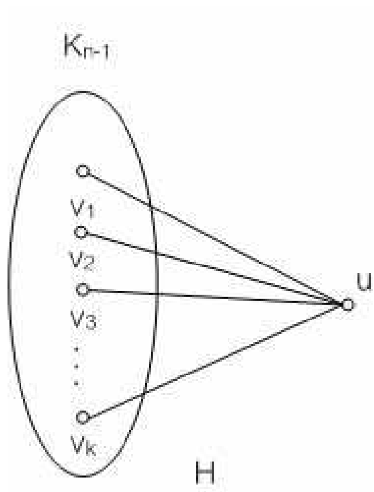

Figure 3.21: The graph H.

□

Theorem 3.14 For an integer $n \geq 3$,

$$\min(\kappa; m, n) = \begin{cases} 1 & \text{if } n-1 \leq m \leq \binom{n-1}{2}+1, \\ m - \binom{n-1}{2} & \text{if } \binom{n-1}{2}+1 < m < \binom{n}{2}. \end{cases}$$

The proof of Theorem 3.14 follows from Propositions 3.11 and 3.13.

□

If $m \geq n-1$, then there is a sharp upper bound for $\kappa(G)$, the proof is presented in [6] p.120-122. The bound is given in the following theorem.

Theorem 3.15 ([6],p.120) If G is a graph of order n and size $m \geq n-1$, then $\kappa(G) \leq \lfloor \frac{2m}{n} \rfloor$.

□

Theorem 3.16 ([6],p.120) For every two integers r and n with $2 \leq r < n$, $\kappa(H_{r,n}) = r$.

□

The Harary graphs play an important role to show that the bound given in Theorem 3.15 is sharp.

In [6], the authors discussed that for every two integers m and n such that $1 \leq n-1 \leq m \leq \binom{n}{2}$, there exists a graph G of size m and order n such that $\kappa(G) = \lfloor \frac{2m}{n} \rfloor$. If $m = n-1$, then every tree T of order n has the desired property as $\kappa(T) = \lfloor \frac{2m}{n} \rfloor = \lfloor \frac{2n-2}{n} \rfloor = 1$. Hence we may assume that $3 \leq n \leq m \leq \binom{n}{2}$. Then there exists a unique positive integer r such that $r \leq \frac{2m}{n} < r+1$. Consequently, $\frac{rn}{2} \leq \lceil \frac{rn}{2} \rceil \leq m < \frac{n(r+1)}{2}$. Thus the smallest size of a graph G of order n such that $\kappa(G) = r$ is $\lceil \frac{rn}{2} \rceil$, which, as we have seen, is the size of Harary graph $H_{r,n}$. Since $\lceil \frac{rn}{2} \rceil \leq m < \frac{n(r+1)}{2}$, it follows that $m = \lceil \frac{rn}{2} \rceil + t$, where $0 \leq t < \frac{n}{2}$ if n is even, $0 \leq t < \frac{n-1}{2}$ if r is even and n is odd, and $0 \leq t < \frac{n-3}{2}$ if r and n are both odd. Adding t edges to the graph $H_{r,n}$ produces a graph H' of size m. Since $H_{r,n}$ is a spanning subgraph of H', it follows that $r = \kappa(H_{r,n}) \leq \kappa(H')$.

Necessarily, H' contains one or more vertices of degree r and so $\delta(H') = r$. It then follows by Theorem 3.10 that $\kappa(H') \leq r$. Therefore, $\kappa(H') = r$. Hence the equality in Theorem 3.15 holds. Then we can conclude with the following theorem.

Theorem 3.17 For an integer $n \geq 3$ and $n-1 \leq m \leq \binom{n}{2}$, $\max(\kappa; m, n) = \left\lfloor \frac{2m}{n} \right\rfloor$.

□

Next example illustrates Theorems 3.14 and 3.17 and shows some graphs whose connectivity is greater than $\min(\kappa; m, n)$ and lower than $\max(\kappa; m, n)$.

Example 3.2.1 Let $n = 8$.

$$\text{Then } \min(\kappa; m, 8) = \begin{cases} 1 & \text{if } 7 \leq m \leq 22, \\ m - 21 & \text{if } 22 < m \leq 28 \end{cases}$$

and $\max(\kappa; m, 8) = \left\lfloor \frac{2m}{8} \right\rfloor$.

First, we consider $m = 19$. Then $\min(\kappa; 19, 8) = 1$ and $\max(\kappa; 19, 8) = \left\lfloor \frac{2\times 19}{8} \right\rfloor = 4$. We construct the graph G_1 with $\kappa(G_1) = 1$ and the graph G_2 as a Harary graph $H_{4,8}$ with $\kappa(G_2) = 4$ where $G_1, G_2 \in \mathcal{CG}(19, 8)$ as shown in Figure 3.22. The minimum vertex-cuts of G_1, G_2 are indicated by solid vertices.

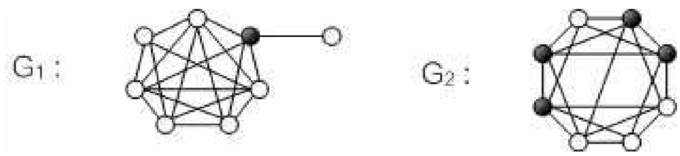

Figure 3.22: $G_1, G_2 \in \mathcal{CG}(19, 8)$.

By the interpolation property of $\kappa(G)$, we know that there exists a graph $G \in \mathcal{CG}(19, 8)$ with $\kappa(G) = c$ where $c = 2, 3$. The graph $G_3 \in \mathcal{CG}(19, 8)$ with $\kappa(G_3) = 2$ and graph $G_4 \in \mathcal{CG}(19, 8)$ with $\kappa(G_4) = 3$ are shown in Figure 3.23 and the minimum vertex-cuts of G_3 and G_4 are indicated by solid vertices.

Figure 3.23: $G_3, G_4 \in \mathcal{CG}(19, 8)$.

Next, we consider $m = 24$. Then $\min(\kappa; 24, 8) = 24 - 21 = 3$ and $\max(\kappa; 24, 8) = \left\lfloor \frac{2\times 24}{8} \right\rfloor = 6$. We construct the graph H_1 with $\kappa(H_1) = 3$ by Proposition 3.13 and the graph H_2 as a Harary graph $H_{6,8}$ with $\kappa(H_2) = 6$ where $H_1, H_2 \in \mathcal{CG}(24, 8)$ as shown in Figure 3.24. The minimum vertex-cuts of H_1, H_2 are indicated by solid vertices.

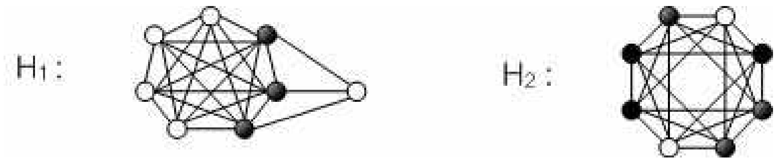

Figure 3.24: $H_1, H_2 \in \mathcal{CG}(24, 8)$.

By the interpolation property of $\kappa(G)$, we know that there exists a graph $G \in \mathcal{CG}(24, 8)$ with $\kappa(G) = c$ where $c = 4, 5$. The graph $H_3 \in \mathcal{CG}(24, 8)$ with $\kappa(H_3) = 4$ and graph $H_4 \in \mathcal{CG}(24, 8)$ with $\kappa(H_4) = 5$ are shown in Figure 3.25 and the minimum vertex-cuts of H_3 and H_4 are indicated by solid vertices.

Figure 3.25: $H_3, H_4 \in \mathcal{CG}(24, 8)$.

3.2.2 Edge - Connectivity (λ)

Note that $\lambda(G) = 0$ if and only if G is disconnected or G is trivial, while $\lambda(G) = 1$ if and only if G is connected and contains a bridge. If $G \cong K_n$ for a positive integer n, then $\lambda(G) = n-1$. If G is a graph of size m and order n such that $\lambda(G) \geq 1$, then, necessarily, $m \geq n-1$. Thus, we consider $\mathcal{CG}(m,n)$ where $n-1 \leq m < \binom{n}{2}$ and $n \geq 3$. We shall determine $\min\{\lambda(G) \mid G \in \mathcal{CG}(m,n)\}$ and $\max\{\lambda(G) \mid G \in \mathcal{CG}(m,n)\}$, written $\min(\lambda; m,n)$ and $\max(\lambda; m,n)$, respectively.

Proposition 3.18 If $n-1 \leq m \leq \binom{n-1}{2}+1$, then $\min(\lambda; m,n) = 1$.

Proof It is clear that any connected graph G of order n with size $n-1$ is a tree. Then $\lambda(G) = 1$. Consider $m = \binom{n-1}{2}+1$. Let H be a graph obtained from K_{n-1} by adding a new vertex v to K_{n-1} and then joining any one vertex of K_{n-1} to the vertex v. It is clear that $H \in \mathcal{CG}(m,n)$. Since H has exactly one vertex v of degree one, so $\lambda(H) = 1$. By the fact that for any connected graph G, $\lambda(G) \geq 1$ and $\min(\lambda; k,n) \leq \min(\lambda; k+1,n)$ for a fix integer n and $n-1 \leq k \leq \binom{n-1}{2}$, we can conclude that $\min(\lambda; m,n) = 1$ for $n-1 \leq m \leq \binom{n-1}{2}+1$.

□

Proposition 3.19 If $\binom{n-1}{2}+1 < m < \binom{n}{2}$, then $\min(\lambda; m,n) = m - \binom{n-1}{2}$.

Proof Let $G \in \mathcal{CG}(m,n)$. From Proposition 3.12, we have $\kappa(G) \geq m - \binom{n-1}{2}$. By Theorem 3.10, we have $\lambda(G) \geq m - \binom{n-1}{2}$. Let $k = m - \binom{n-1}{2}$. We can see that if $m > \binom{n-1}{2}+1$, then $k > 1$. Let H be a graph obtained from K_{n-1} by adding a new vertex u to K_{n-1} and then joining k vertices of K_{n-1} to u. It is clear that the graph $H \in \mathcal{CG}(m,n)$ where $m = \binom{n-1}{2}+k$. Since $k = m - \binom{n-1}{2} < \binom{n}{2} - \binom{n-1}{2}$, so $k \leq n-2$. Hence $\lambda(H) = k$. Thus $\min(\lambda; m,n) = k$. Figure 3.26 illustrates the graph H.

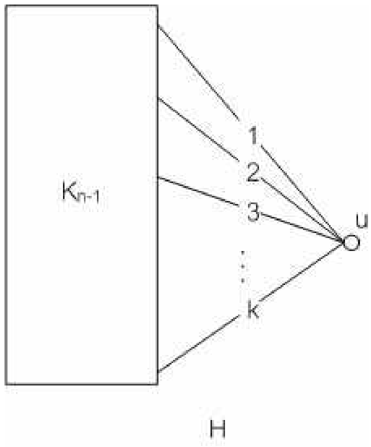

Figure 3.26: The graph H.

□

Theorem 3.20 For an integer $n \geq 3$,

$$\min(\lambda; m, n) = \begin{cases} 1 & \text{if } n-1 \leq m \leq \binom{n-1}{2} + 1, \\ m - \binom{n-1}{2} & \text{if } \binom{n-1}{2} + 1 < m < \binom{n}{2}. \end{cases}$$

The proof of Theorem 3.20 follows from Propositions 3.18 and 3.19.

□

Theorem 3.21 For an integer $n \geq 3$ and $n-1 \leq m \leq \binom{n}{2}$, $\max(\lambda; m, n) = \left\lfloor \frac{2m}{n} \right\rfloor$.

Proof Let $G \in \mathcal{CG}(m, n)$. Since the sum of the degrees of the vertices of G is $2m$, the average degree of the vertices of G is $\frac{2m}{n}$ and so $\delta(G) \leq \frac{2m}{n}$. By Theorem 3.10, $\lambda(G) \leq \left\lfloor \frac{2m}{n} \right\rfloor$. We use the process (above Theorem 3.17) described to obtain a graph H' of size m and order n from $H_{r,n}$ where $r = \left\lfloor \frac{2m}{n} \right\rfloor$. It is clear that $\kappa(H') = \left\lfloor \frac{2m}{n} \right\rfloor$ and $\delta(H') = \left\lfloor \frac{2m}{n} \right\rfloor$. Therefore, by Theorem 3.10, we have that $\lambda(H') = \left\lfloor \frac{2m}{n} \right\rfloor$ and $H' \in \mathcal{CG}(m, n)$ as desired.

□

Next example illustrates Theorems 3.20 and 3.21 and shows some graphs whose edge-connectivity is greater than $\min(\lambda; m, n)$ and lower than $\max(\lambda; m, n)$.

Example 3.2.2 Let $n = 6$.

Then $\min(\lambda; m, 6) = \begin{cases} 1 & \text{if } 5 \le m \le 11, \\ m - 10 & \text{if } 11 < m \le 15 \end{cases}$

and $\max(\lambda; m, 6) = \lfloor \frac{2m}{6} \rfloor$.

First, we consider $m = 10$. Then $\min(\lambda; 10, 6) = 1$ and $\max(\lambda; 10, 6) = \lfloor \frac{2\times 10}{6} \rfloor = 3$. We construct the graph G_1 with $\lambda(G_1) = 1$ and the graph G_2 as a Harary graph $H_{3,6}$ with $\lambda(G_2) = 3$ where $G_1, G_2 \in \mathcal{CG}(10, 6)$, as shown in Figure 3.27. By the interpolation property of $\lambda(G)$, we know that there exists a graph $G \in \mathcal{CG}(10, 6)$ with $\lambda(G) = 2$. The graph $G_3 \in \mathcal{CG}(10, 6)$ with $\lambda(G_3) = 2$ is also shown in Figure 3.27 and the minimum edge-cuts of G_1, G_2 and G_3 are indicated by solid edges.

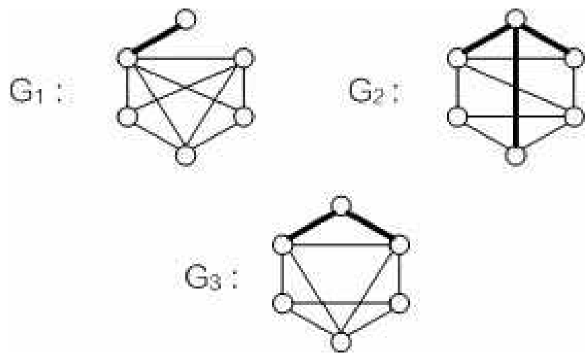

Figure 3.27: G_1, G_2 and $G_3 \in \mathcal{CG}(10, 6)$.

Next, we consider $m = 13$. Then $\min(\lambda; 13, 6) = 13 - 10 = 3$ and $\max(\lambda; 13, 6) = \lfloor \frac{2\times 13}{6} \rfloor = 4$. We construct the graph H_1 with $\lambda(H_1) = 3$ by Proposition 3.19 and the graph H_2 as a Harary graph $H_{4,6}$ with $\lambda(H_2) = 4$ where $H_1, H_2 \in \mathcal{CG}(13, 6)$ as shown in Figure 3.28. The minimum edge-cuts of H_1, H_2 are indicated by solid edges.

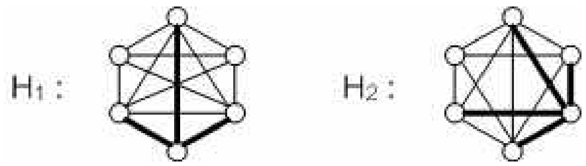

Figure 3.28: $H_1, H_2 \in \mathcal{CG}(13, 6)$.

3.2.3 Domination Number (γ)

Since the vertex set of a graph is always a dominating set, the domination number is defined for every graph. If G is a graph of order n, then $1 \leq \gamma(G) \leq n$ where n is a positive integer. A graph G of order n has domination number 1 if and only if G contains a vertex v of degree $n-1$, in which case $\{v\}$ is a minimum dominating set; while $\gamma(G) = n$ if and only if $G \cong \overline{K}_n$, in which case $V(G)$ is the unique minimum dominating set. If G is a cycle of order $n \geq 3$, then $\gamma(G) = \lceil \frac{n}{3} \rceil$. It is easy to see that any connected graph G of order $n = 2, 3$, $\gamma(G) = 1$. We shall determine $\min\{\gamma(G) \mid G \in \mathcal{CG}(m,n)\}$ and $\max\{\gamma(G) \mid G \in \mathcal{CG}(m,n)\}$, written $\min(\gamma; m, n)$ and $\max(\gamma; m, n)$, respectively. We consider the minimum and maximum values of the domination number where $4 \leq n \leq 7$ and $n - 1 \leq m \leq \binom{n}{2}$. We observe that:

For $4 \leq n \leq 7$, $\min(\gamma; m, n) = 1$ for all m.

For $n = 4$, $\max(\gamma; m, n) = \begin{cases} 2 & \text{if } m = 3 \text{ and } 4, \\ 1 & \text{if } m = 5 \text{ and } 6. \end{cases}$

For $n = 5$, $\max(\gamma; m, n) = \begin{cases} 2 & \text{if } 4 \leq m \leq 7, \\ 1 & \text{if } 8 \leq m \leq 10. \end{cases}$

For $n = 6$, $\max(\gamma; m, n) = \begin{cases} 3 & \text{if } m = 5 \text{ and } 6, \\ 2 & \text{if } 7 \leq m \leq 12, \\ 1 & \text{if } 13 \leq m \leq 15. \end{cases}$

For $n = 7$, $\max(\gamma; m, n) = \begin{cases} 3 & \text{if } 7 \leq m \leq 10, \\ 2 & \text{if } 11 \leq m \leq 17, \\ 1 & \text{if } 18 \leq m \leq 21. \end{cases}$

We now determine the minimum and maximum values of $\gamma(G)$ for all $G \in \mathcal{CG}(m, n)$ where $n \geq 8$ and $n - 1 \leq m \leq \binom{n}{2}$.

Theorem 3.22 For an integer $n \geq 8$ and $n - 1 \leq m \leq \binom{n}{2}$, $\min(\gamma; m, n) = 1$.

Proof It is clear that for $m = n - 1$, we have $\gamma(K_{1,n-1}) = 1$ and for $m = \binom{n}{2}$, $\gamma(K_n) = 1$. By the fact that $\min(\gamma; k, n) \geq \min(\gamma; k + 1, n)$ for a fix integer n and $n - 1 \leq k \leq \binom{n}{2} - 1$, therefore the domination number for any graph with size $n - 1 < m < \binom{n}{2}$ is also one. Since any graph G of order n, $\gamma(G) \geq 1$, that is $\min(\gamma; m, n) = 1$ for all $G \in \mathcal{CG}(m, n)$.

□

To determine $\max(\gamma; m, n)$, we need the following theorems.

Theorem 3.23 ([6],p.367) Let G be a graph without isolated vertices. If S is a minimal dominating set of G, then $V(G) - S$ is a dominating set of G.

□

Theorem 3.24 ([6],p.367) If G is a graph of order n without isolated vertices, then $\gamma(G) \leq \frac{n}{2}$.

□

Note that if n is even, the $(n - 2)$-regular graph of order n has $\frac{n(n-2)}{2}$ edges. Then a graph of order n with size $m = \frac{n(n-2)}{2} + 1$ contains a vertex of degree $n - 1$. If n is odd, a graph having $\frac{n(n-2)+1}{2}$ edges contains exactly one vertex of degree $n - 1$ and $n - 1$ vertices of degree $n - 2$.

Proposition 3.25 If $\binom{n}{2} - \left\lfloor \frac{n-1}{2} \right\rfloor \leq m \leq \binom{n}{2}$, then $\max(\gamma; m, n) = 1$.

Proof Consider $m = \binom{n}{2} - \left\lfloor \frac{n-1}{2} \right\rfloor$. Let $G \in \mathcal{CG}(m, n)$. If n is odd, then $m = \binom{n}{2} - \frac{n-1}{2} = \frac{(n-1)^2}{2} = \frac{n(n-2)+1}{2}$. Hence the graph G contains a vertex of degree $n - 1$, that is $\gamma(G) = 1$. If n is even, then $m = \binom{n}{2} - \frac{n-2}{2} = \frac{n(n-2)}{2} + 1$. Hence the graph G contains a vertex of degree $n-1$, that is $\gamma(G)=1$. Thus $\gamma(G)=1$ if $m=\binom{n}{2}-\lfloor\frac{n-1}{2}\rfloor$. We know that $|E(K_n)|=\binom{n}{2}$ and every vertex of K_n has degree $n-1$. Thus $\gamma(K_n)=1$. By the fact that $\max(\gamma;k,n)\geq\max(\gamma;k+1,n)$ for a fix integer n and $\binom{n}{2}-\lfloor\frac{n-1}{2}\rfloor\leq k\leq\binom{n}{2}-1$, therefore $\max(\gamma;m,n)=1$ for $\binom{n}{2}-\lfloor\frac{n-1}{2}\rfloor\leq m\leq\binom{n}{2}$.

□

Proposition 3.26 If $\binom{n-2}{2}<m<\binom{n}{2}-\lfloor\frac{n-1}{2}\rfloor$, then $\max(\gamma;m,n)=2$.

Proof Let $G\in\mathcal{CG}(m,n)$. By Theorem 2.31, we have $m\leq\binom{n-d+1}{2}$ if $\gamma(G)=d$ where $3\leq d\leq\frac{n}{2}$. We can say that if $m>\binom{n-2}{2}$, then $d<3$ or $d\leq 2$. That is $\gamma(G)\leq 2$.

Consider $m=\binom{n-2}{2}+1$. Let $\{v_1,v_2,\ldots,v_{n-2}\}$ be a vertex set of a graph K_{n-2} and let H be a graph obtained from K_{n-2} by adding two new vertices u_1 and u_2 and adding edges u_1v_1 and u_2v_2. Thus $\{v_1,v_2\}$ is a dominating set of H, that is $\gamma(H)\leq 2$. Observe that the graph H contains no vertices of degree $n-1$. Hence $\gamma(H)\geq 2$. Thus $\gamma(H)=2$ and $|E(H)|=\binom{n-2}{2}+2$. Then a graph H' of size $m=\binom{n-2}{2}+1$ can be obtained from H by deleting one edge in K_{n-2}. It is clear that $H'\in\mathcal{CG}(m,n)$. Since every dominating set of H' is also a dominating set of H, so $\gamma(H')\geq\gamma(H)=2$. We have that $\{v_1,v_2\}$ is also a dominating set of H', that is $\gamma(H')\leq 2$. Thus $\gamma(H')=2$. Then we obtain the graph $H'\in\mathcal{CG}(m,n)$ where $m=\binom{n-2}{2}+1$ with $\gamma(H')=2$.

For $m=\binom{n}{2}-\lfloor\frac{n-1}{2}\rfloor-1$, we consider two cases according to whether n is even or odd.

Case 1 Assume that n is even. Then $m=\binom{n}{2}-\frac{n-2}{2}-1=\frac{n(n-2)}{2}$. Let H be the $(n-2)$-regular graph of order n. Then $H\in\mathcal{CG}(m,n)$ and $|E(H)|=\frac{n(n-2)}{2}$. It is clear that every vertex of H has degree $n-2$. That is $\gamma(H)\geq 2$. By Theorem 2.23, we have $\gamma(H)\leq n-(n-2)=2$. Thus $\gamma(H)=2$.

Case 2 Assume that n is odd. Then $m=\binom{n}{2}-\frac{n-1}{2}-1=\frac{n^2-2n-1}{2}$. Let $s:$ $n-3,n-2,n-2,\ldots,n-2$ be a sequence of n nonnegative integers consisting of $n-1$ terms of $n-2$. Observe that a sequence $s':2,1,1,\ldots,1$ of n integers is a graphic degree sequence. Let H be a realization of s'. Then $\overline{H}$ is a graph with degree sequence s since the sum of each corresponding terms in s' and s equal $n-1$. Thus s is a graphic degree sequence. Let $G = \overline{H}$. Observe that $|E(G)| = \frac{1}{2}[(n-3)+(n-1)(n-2)] = \frac{n^2-2n-1}{2} = m$. Since the graph G contains no vertices of degree $n-1$, so $\gamma(G) \neq 1$. That is $\gamma(G) \geq 2$. By Theorem 2.23, we have $\gamma(G) \leq n-(n-2) = 2$. Thus $\gamma(G) = 2$.

Note that $\max(\gamma; k, n) \geq \max(\gamma; k+1, n)$ for a fix integer n and $\binom{n-2}{2} < k < \binom{n}{2} - \lfloor \frac{n-1}{2} \rfloor - 1$. That is $\max(\gamma; m, n) = 2$ for $\binom{n-2}{2} < m < \binom{n}{2} - \lfloor \frac{n-1}{2} \rfloor$.

□

Recall Theorem 2.31 established by L.A. Sanchis [37]. Let G be a graph with n vertices, the domination number d where $3 \leq d \leq \frac{n}{2}$, and no isolated vertices. Then the number of edges of G is at most $\binom{n-d+1}{2}$. This means that if $d=3$, then $|E(G)| \leq \binom{n-2}{2}$ and if $d = \lfloor \frac{n}{2} \rfloor$, then $|E(G)| \leq \binom{\lceil n/2 \rceil + 1}{2}$. To consider $\max(\gamma; m, n)$ where $m \leq \binom{n-2}{2}$, we can establish that there exists a connected graph G of order n and size $m = \binom{n-2}{2}$ with $\gamma(G) = 3$ as follows:

Let $m = \binom{n-2}{2}$ and let $\{u_1, u_2, \ldots, u_{n-3}\}$ be a vertex set of a graph K_{n-3}. Let G be a graph obtained from K_{n-3} by adding three new vertices v_1, v_2 and v_3 and adding edges u_1v_1, u_2v_2 and u_iv_3 for $3 \leq i \leq n-3$. Then $|E(G)| = \binom{n-3}{2} + (n-3) = \binom{n-2}{2}$. It is clear that $\{v_1, v_2, v_3\}$ is a minimum dominating set of G. Thus $\gamma(G) = 3$.

Proposition 3.27 If $\binom{\lceil n/2 \rceil + 1}{2} < m \leq \binom{n-2}{2}$, then $\max(\gamma; m, n) = \left\lfloor \frac{(2n+1)-\sqrt{8m+1}}{2} \right\rfloor$.

Proof Let S be a minimum dominating set of $G \in \mathcal{CG}(m,n)$ with cardinality $k \geq 3$. By Theorem 2.31, we have $m \leq \binom{n-k+1}{2}$. That is $k^2 - (2n+1)k + (n^2+n-2m) \geq 0$. Using the quadratic formula, $k \leq \frac{(2n+1)-\sqrt{8m+1}}{2}$. Thus $\gamma(G) = |S| = k \leq \left\lfloor \frac{(2n+1)-\sqrt{8m+1}}{2} \right\rfloor$. We claim that this bound is sharp. Consider $k = \left\lfloor \frac{(2n+1)-\sqrt{8m+1}}{2} \right\rfloor$. We now construct a graph $G \in \mathcal{CG}(m,n)$ with $\gamma(G) = k$ as follows. Let $\{v_1, v_2, \ldots, v_{n-k}\}$ be the vertex set of K_{n-k}. Adding k new vertices $u_1, u_2, \ldots, u_k$ to K_{n-k} and joining each pair of vertices u_i, v_i for $1 \leq i \leq k$ yields a connected graph H of order n and size $\binom{n-k}{2} + k$. Thus $\{v_1, v_2, \ldots, v_k\}$ is a dominating set of H, by the fact that $k \leq n-k$, that is $\gamma(H) \leq k$. Since each vertex u_i is adjacent to exactly one of vertices v_i's for $1 \leq i \leq k$ which are distinct, so $\gamma(H) \geq k$. That is $\gamma(H) = k$. To construct the connected graph $G \in \mathcal{CG}(m,n)$

where $\binom{\lceil n/2\rceil+1}{2} < m \le \binom{n-2}{2}$ with $\gamma(G) = k$, we consider two cases according to m.

Case 1 Assume that $\binom{\lceil n/2\rceil+1}{2} < m < \binom{n-k}{2} + k$. Let T be a spanning tree of K_{n-k}. Since $\binom{n-k}{2} - (n-k-1) > \binom{n-k}{2} + k - m$, so the connected graph G with size m can be obtained from the graph H by deleting $\binom{n-k}{2} + k - m$ edges from $E(K_{n-k}) - E(T)$. We have that $\{v_1, v_2, \ldots, v_k\}$ is also a dominating set of G, that is $\gamma(G) \le k$. Since every dominating set of G is also a dominating set of H, so $\gamma(G) \ge \gamma(H) = k$. That is $\gamma(G) = k$.

Case 2 Assume that $\binom{n-k}{2} + k \le m \le \binom{n-2}{2}$. Now we have the graph H with size $\binom{n-k}{2} + k$ and $\gamma(H) = k$. Consider $\binom{n-k}{2} + k < m \le \binom{n-2}{2}$. Note that for a graph with size m and the domination number k, $m \le \binom{n-k+1}{2}$. We have $m - \binom{n-k}{2} \le n-k$. Then the graph G with size m can be obtained from the graph H by joining u_k to v_j for $k+1 \le j \le m - \binom{n-k}{2}$. Hence $G \in \mathcal{CG}(m, n)$. We have that $\{v_1, v_2, \ldots, v_k\}$ is also a dominating set of G, that is $\gamma(G) \le k$. Since each vertex u_i is adjacent to exactly one of vertex v_i's for $1 \le i \le k-1$ and u_k adjacent to v_j for $k+1 \le j \le m - \binom{n-k}{2}$ which are distinct, $\gamma(G) \ge k$. That is $\gamma(G) = k$.

□

Proposition 3.28 If $n - 1 \le m \le \binom{\lceil n/2\rceil+1}{2}$, then $\max(\gamma; m, n) = \lfloor \frac{n}{2} \rfloor$.

Proof Let $G \in \mathcal{CG}(m, n)$. By Theorem 3.24, $\gamma(G) \le \lfloor \frac{n}{2} \rfloor$. To construct a connected graph of order n and size m where $n - 1 \le m \le \binom{\lceil n/2\rceil+1}{2}$ with the domination number $\lfloor \frac{n}{2} \rfloor$, we divide into two cases.

Case 1 Suppose that n is even, $n = 2k$. Let $V(K_k) = \{v_1, v_2, \ldots, v_k\}$ and let T be a spanning tree of K_k. Let G be a graph obtained from K_k by adding k new vertices u_i; $1 \le i \le k$ and k new edges u_iv_i for $1 \le i \le k$. Then G is a connected graph of order n and size $\binom{k}{2} + k$. Thus $\{v_1, v_2, \ldots, v_k\}$ is a dominating set of G, that is $\gamma(G) \le k$. Since each vertex u_i is adjacent to exactly one of vertices v_i's for $1 \le i \le k$ which are distinct, so $\gamma(G) \ge k$. That is $\gamma(G) = k$. Let $n - 1 \le m \le \binom{\lceil n/2\rceil+1}{2}$. Since $\binom{k}{2} - (k-1) \ge \binom{k}{2} + k - m$, so a graph $H \in \mathcal{CG}(m, n)$ with $\gamma(H) = k$ can be obtained from the graph G by deleting $\binom{k}{2} + k - m$ edges from $K_k - E(T)$. It is clear that the graph $H \in \mathcal{CG}(m, n)$. We have that $\{v_1, v_2, \ldots, v_k\}$ is a dominating set of H, that is $\gamma(H) \le k$. Since every dominating set of H is also a dominating set of G, so $\gamma(H) \geq \gamma(G) = k$. That is $\gamma(H) = k$.

Case 2 Suppose that n is odd, $n = 2k + 1$. Let $V(K_{k+1}) = \{v_1, v_2, \ldots, v_{k+1}\}$ and let T' be a spanning tree of K_{k+1}. Let G' be a graph obtained from K_{k+1} by adding k new vertices u_i; $1 \leq i \leq k$ and k new edges u_iv_i for $1 \leq i \leq k$. Then G' is a connected graph of order n and size $\binom{k+1}{2} + k$. Thus $\{v_1, v_2, \ldots, v_k\}$ is a dominating set of G', that is $\gamma(G') \leq k$. Since each vertex u_i is adjacent to exactly one of vertices v_i's for $1 \leq i \leq k$ which are distinct, so $\gamma(G') \geq k$. That is $\gamma(G') = k$. Let $n - 1 \leq m \leq \binom{\lceil n/2 \rceil + 1}{2}$. Since $\binom{k+1}{2} - k \geq \binom{k+1}{2} + k - m$, so a graph $H' \in \mathcal{CG}(m, n)$ with $\gamma(H') = k$ can be obtained from the graph G' by deleting $\binom{k+1}{2} + k - m$ edges from $K_{k+1} - E(T')$. It is clear that the graph $H' \in \mathcal{CG}(m, n)$. We have that $\{v_1, v_2, \ldots, v_k\}$ is also a dominating set of H', that is $\gamma(H') \leq k$. Since every dominating set of H' is also a dominating set of G', $\gamma(H') \geq \gamma(G') = k$. That is $\gamma(H') = k$.

□

Theorem 3.29 For an integer $n \geq 8$ and $n - 1 \leq m \leq \binom{n}{2}$,

1. $\max(\gamma; m, n) = \lfloor \frac{n}{2} \rfloor$ if $n - 1 \leq m \leq \binom{\lceil n/2 \rceil + 1}{2}$,
2. $\max(\gamma; m, n) = \left\lfloor \frac{(2n+1) - \sqrt{8m+1}}{2} \right\rfloor$ if $\binom{\lceil n/2 \rceil + 1}{2} < m \leq \binom{n-2}{2}$,
3. $\max(\gamma; m, n) = 2$ if $\binom{n-2}{2} < m < \binom{n}{2} - \lfloor \frac{n-1}{2} \rfloor$,
4. $\max(\gamma; m, n) = 1$ if $\binom{n}{2} - \lfloor \frac{n-1}{2} \rfloor \leq m \leq \binom{n}{2}$.

□

Next example illustrates Theorems 3.22 and 3.29 and shows all graphs of every value of the domination number between those minimum and maximum values.

Example 3.2.3 Let $n = 10$.

Then $\min(\gamma; m, 10) = 1$

$$\text{and } \max(\gamma; m, 10) = \begin{cases} 5 & \text{if } 9 \le m \le 15, \\ \left\lfloor \frac{21-\sqrt{8m+1}}{2} \right\rfloor & \text{if } 15 < m \le 28, \\ 2 & \text{if } 28 < m < 41, \\ 1 & \text{if } 41 \le m \le 45. \end{cases}$$

For example, firstly, we consider $m = 14$. Then $\min(\gamma; 14, 10) = 1$ and $\max(\gamma; 14, 10) = 5$. We construct the graph G_1 with $\gamma(G_1) = 1$ and the graph G_2 by Proposition 3.28 with $\gamma(G_2) = 5$ where $G_1, G_2 \in \mathcal{CG}(14, 10)$ as shown in Figure 3.29. The minimum dominating sets of G_1, G_2 are indicated by solid vertices.

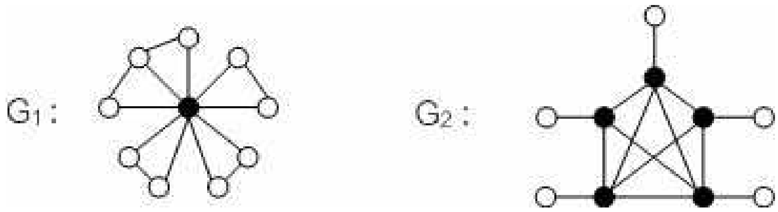

Figure 3.29: $G_1, G_2 \in \mathcal{CG}(14, 10)$.

By the interpolation property of $\gamma(G)$, we know that there exists a graph $G \in \mathcal{CG}(14, 10)$ with $\gamma(G) = c$ where $c = 2, 3, 4$. The graph $G_3 \in \mathcal{CG}(14, 10)$ with $\gamma(G_3) = 2$, graph $G_4 \in \mathcal{CG}(14, 10)$ with $\gamma(G_4) = 3$ and graph $G_5 \in \mathcal{CG}(14, 10)$ with $\gamma(G_5) = 4$ are shown in Figure 3.30 and the minimum dominating sets of G_3, G_4 and G_5 are indicated by solid vertices.

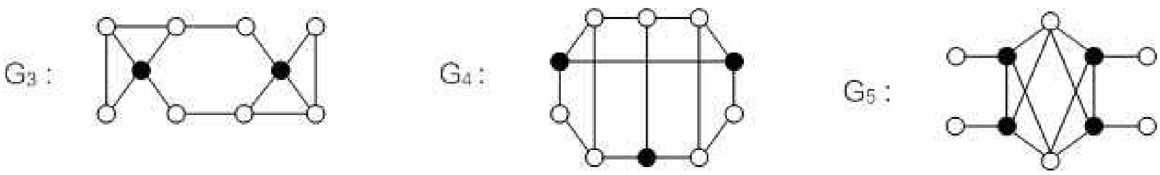

Figure 3.30: G_3, G_4 and $G_5 \in \mathcal{CG}(14, 10)$.

Secondly, we consider $m = 17$. Then $\min(\gamma; 17, 10) = 1$ and $\max(\gamma; 17, 10) = 4$. We construct the graph H_1 with $\gamma(H_1) = 1$ and the graph H_2 by Proposition 3.27

with $\gamma(H_2) = 4$ where $H_1, H_2 \in \mathcal{CG}(17, 10)$ as shown in Figure 3.31. The minimum dominating sets of H_1, H_2 are indicated by solid vertices.

Figure 3.31: $H_1, H_2 \in \mathcal{CG}(17, 10)$.

By the interpolation property of $\gamma(G)$, we know that there exists a graph $G \in \mathcal{CG}(17, 10)$ with $\gamma(G) = c$ where $c = 2, 3$. The graph $H_3 \in \mathcal{CG}(17, 10)$ with $\gamma(H_3) = 2$ and graph $H_4 \in \mathcal{CG}(17, 10)$ with $\gamma(H_4) = 3$ are shown in Figure 3.32 and the minimum dominating sets of H_3 and H_4 are indicated by solid vertices.

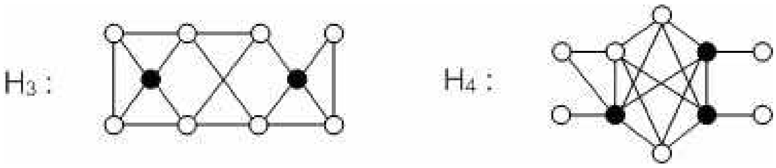

Figure 3.32: $H_3, H_4 \in \mathcal{CG}(17, 10)$.

Thirdly, we consider $m = 30$. Then $\min(\gamma; 30, 10) = 1$ and $\max(\gamma; 30, 10) = 2$. We construct the graph F_1 with $\gamma(F_1) = 1$ and the graph F_2 with $\gamma(F_2) = 2$ where $F_1, F_2 \in \mathcal{CG}(30, 10)$ as shown in Figure 3.33. The minimum dominating sets of F_1, F_2 are indicated by solid vertices.

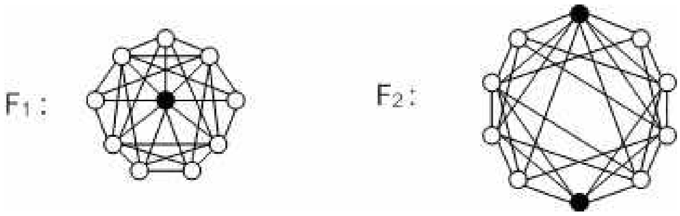

Figure 3.33: $F_1, F_2 \in \mathcal{CG}(30, 10)$.

Finally, we consider $m = 41$. Then $\min(\gamma; 41, 10) = \max(\gamma; 41, 10) = 1$. We construct the graph G with $\gamma(G) = 1$ where $G \in \mathcal{CG}(41, 10)$ as shown in Figure 3.34. A minimum dominating set of G is indicated by solid vertices.

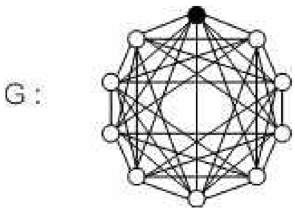

Figure 3.34: $G \in \mathcal{CG}(41, 10)$.

Example 3.2.4 Let $n = 11$.
Then $\min(\gamma; m, 11) = 1$

$$\text{and } \max(\gamma; m, 11) = \begin{cases} 5 & \text{if } 10 \leq m \leq 21, \\ \left\lfloor \frac{23-\sqrt{8m+1}}{2} \right\rfloor & \text{if } 21 < m \leq 36, \\ 2 & \text{if } 36 < m < 50, \\ 1 & \text{if } 50 \leq m \leq 55. \end{cases}$$

For example, firstly, we consider $m = 15$. Then $\min(\gamma; 15, 11) = 1$ and $\max(\gamma; 15, 11) = 5$. We construct the graph G_1 with $\gamma(G_1) = 1$ and the graph G_2 by Proposition 3.28 with $\gamma(G_2) = 5$ where $G_1, G_2 \in \mathcal{CG}(15, 11)$ as shown in Figure 3.35. The minimum dominating sets of G_1, G_2 are indicated by solid vertices.

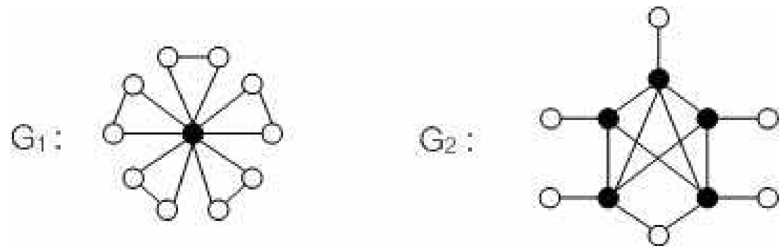

Figure 3.35: $G_1, G_2 \in \mathcal{CG}(15, 11)$.

By the interpolation property of $\gamma(G)$, we know that there exists a graph $G \in \mathcal{CG}(15, 11)$ with $\gamma(G) = c$ where $c = 2, 3, 4$. The graph $G_3 \in \mathcal{CG}(15, 11)$ with $\gamma(G_3) = 2$, graph $G_4 \in \mathcal{CG}(15, 11)$ with $\gamma(G_4) = 3$ and graph $G_5 \in \mathcal{CG}(15, 11)$ with $\gamma(G_5) = 4$ are shown in Figure 3.36 and the minimum dominating sets of G_3, G_4 and G_5 are indicated by solid vertices.

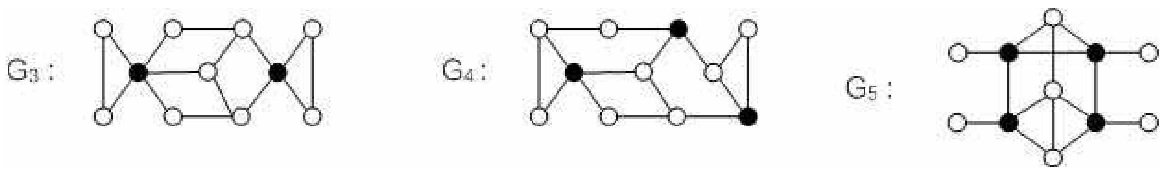

Figure 3.36: G_3, G_4 and $G_5 \in \mathcal{CG}(15, 11)$.

Secondly, we consider $m = 27$. Then $\min(\gamma; 27, 11) = 1$ and $\max(\gamma; 27, 11) = 4$. We construct the graph H_1 with $\gamma(H_1) = 1$ and the graph H_2 by Proposition 3.27 with $\gamma(H_2) = 4$ where $H_1, H_2 \in \mathcal{CG}(27, 11)$ as shown in Figure 3.37. The minimum dominating sets of H_1, H_2 are indicated by solid vertices.

Figure 3.37: $H_1, H_2 \in \mathcal{CG}(27, 11)$.

By the interpolation property of $\gamma(G)$, we know that there exists a graph $G \in \mathcal{CG}(27, 11)$ with $\gamma(G) = c$ where $c = 2, 3$. The graph $H_3 \in \mathcal{CG}(27, 11)$ with $\gamma(H_3) = 2$ and graph $H_4 \in \mathcal{CG}(27, 11)$ with $\gamma(H_4) = 3$ are shown in Figure 3.38 and the minimum dominating sets of H_3 and H_4 are indicated by solid vertices.

Figure 3.38: H_3 and $H_4 \in \mathcal{CG}(27, 11)$.

Thirdly, we consider $m = 40$. Then $\min(\gamma; 40, 11) = 1$ and $\max(\gamma; 40, 11) = 2$. We construct the graph F_1 with $\gamma(F_1) = 1$ and the graph F_2 with $\gamma(F_2) = 2$ where $F_1, F_2 \in \mathcal{CG}(40, 11)$ as shown in Figure 3.39. The minimum dominating sets of F_1, F_2 are indicated by solid vertices.

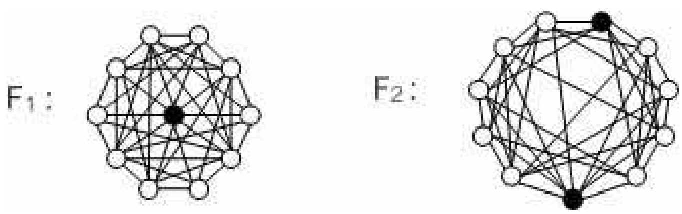

Figure 3.39: $F_1, F_2 \in \mathcal{CG}(40, 11)$.

Finally, we consider $m = 50$. Then $\min(\gamma; 50, 11) = \max(\gamma; 50, 11) = 1$. We construct the graph G with $\gamma(G) = 1$ where $G \in \mathcal{CG}(50, 11)$ as shown in Figure 3.40. A minimum dominating set of G is indicated by solid vertices.

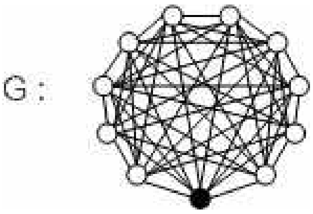

Figure 3.40: $G \in \mathcal{CG}(50, 11)$.

3.2.4 Edge Domination Number (γ')

For any graph G of size m and order n, $\gamma'(G) \leq \lfloor \frac{n}{2} \rfloor$. Let G be a connected graph of even order n. Then $\gamma'(G) = \frac{n}{2}$ if and only if G is isomorphic to K_n or $K_{n/2,n/2}$. In other words, $\gamma'(G) = \frac{n}{2}$ if and only if G has size $m = \binom{n}{2}$ and $m = \frac{n^2}{4}$. For any tree T of order $n \neq 2$, $\gamma'(T) \leq \frac{n-1}{2}$; equality holds if and only if T is isomorphic to the subdivision of a star. It is easy to see that any connected graph G of order $n = 2, 3$, $\gamma'(G) = 1$. We now consider $\mathcal{CG}(m, n)$ where $n \geq 4$ and $n - 1 \leq m \leq \binom{n}{2}$. We shall determine $\min\{\gamma'(G) \mid G \in \mathcal{CG}(m, n)\}$ and $\max\{\gamma'(G) \mid G \in \mathcal{CG}(m, n)\}$, written $\min(\gamma'; m, n)$ and $\max(\gamma'; m, n)$, respectively.

Proposition 3.30 If $n - 1 \leq m \leq 2n - 3$, then $\min(\gamma'; m, n) = 1$.

Proof It is clear that if $m = n - 1$, then $\gamma'(K_{1,m}) = 1$. Let $V(K_{1,n-1}) = \{u_1, u_2, \ldots, u_n\}$ and $\deg u_1 = n - 1$. Then we consider $m = 2n - 3 = (n - 1) + (n - 2)$. Let H be a graph obtained from $K_{1,n-1}$ by joining u_2 to u_{i+2} for $1 \leq i \leq n - 2$. We can see that the edge u_1u_2 is adjacent to every other edge of H. Thus $\gamma'(H) = 1$. By the fact that for any connected graph G, $\gamma'(G) \geq 1$ and $\min(\gamma'; k, n) \leq \min(\gamma'; k + 1, n)$ for a fix integer n and $n - 1 \leq k \leq 2n - 4$, we can conclude that $\min(\gamma'; m, n) = 1$ for $n - 1 \leq m \leq 2n - 3$.

□

The next lemma shows that for any graph $G \in \mathcal{CG}(m, n)$ we can choose an edge dominating set which is an independent edge set.

Lemma 3.31 Let $G \in \mathcal{CG}(m,n)$ and $\gamma'(G) = k$. Then there exists an edge dominating set of G with cardinality k and it is independent.

Proof Let $X = \{e_1, e_2, \ldots, e_k\}$ be an edge dominating set of G. Suppose that there exist $e_i, e_j, e_k \in X$ such that $\{e_i, e_j, e_k\}$ induced a path of 4 vertices. Then $X - \{e_j\}$ is an edge dominating set of G with cardinality $k-1$. That is $\gamma'(G) = k-1$. It is contradiction. Next we consider two cases.

Case 1 Suppose that there exist $e_i, e_j \in X$ such that $\{e_i, e_j\}$ induced a path of 3 vertices. Then there is an edge $f \notin X$ adjacent to the edge e_j. We can obtained a new edge dominating set X' with $|X'| = k$ by adding f in X and deleting e_j from X such that e_i and f are not adjacent in X'.

Case 2 Suppose that there exist $e_i, e_j, e_k \in X$ such that $\{e_i, e_j, e_k\}$ induced a star of 4 vertices. Then there is an edge $f' \notin X$ adjacent to the edge e_j. We can obtained a new edge dominating set X'' with $|X''| = k$ by adding f' in X and deleting e_j from X such that e_i, e_k are not adjacent to f in X''. Then the remaining e_i, e_k such that $\{e_i, e_k\}$ induced a path of 3 vertices, is the Case 1.

Combining Case 1 and Case 2 we can conclude that there exists an edge dominating set of G with cardinality k and it is independent.

□

Proposition 3.32 If $2n-3 < m \leq \binom{n}{2}$, then $\min(\gamma'; m, n) = \left\lceil \frac{(2n-1)-\sqrt{(2n-1)^2-8m}}{4} \right\rceil$.

Proof Let $G \in \mathcal{CG}(m,n)$ and let X be a minimum edge dominating set of G with cardinality k. By Lemma 3.31, there exists an edge dominating set of G with cardinality k and it is independent. Let S be a set of vertices of G incident with edges of the edge dominating set. Then $|S| = 2k$. Since $\gamma'(G) = k$, so the induced subgraph $G[V(G)-S]$ of G has no edge and each vertex in $V(G)-S$ is joined to some vertices in S. Hence $m \leq 2k(n-2k)+\binom{2k}{2} = 2kn-k(2k+1)$. Thus $-2k^2+(2n-1)k-m \geq 0$. Using the quadratic formula, $k \geq \frac{(2n-1)-\sqrt{(2n-1)^2-8m}}{4}$. Since $n \geq 4$ and $2n-3 < m \leq \binom{n}{2}$, so $(2n-1)^2 - 8m > 0$ and $(2n-1) - \sqrt{(2n-1)^2-8m} \geq 5$. Thus $\gamma'(G) = k \geq \left\lceil \frac{(2n-1)-\sqrt{(2n-1)^2-8m}}{4} \right\rceil$. For $n \geq 4$ and $m = (2n-3)+1$, we can check that $\left\lceil \frac{(2n-1)-\sqrt{(2n-1)^2-8m}}{4} \right\rceil > 1$. That is $\gamma'(G) \geq 2$.

Consider $k = \left\lceil \frac{(2n-1)-\sqrt{(2n-1)^2-8m}}{4} \right\rceil$. Let G be a graph of order n with $V(G) = \{u_1, u_2, \ldots, u_n\}$. We add edges $u_i u_{j+i}$ for $1 \leq i \leq 2k-2$, $1 \leq j \leq n-i$, and join u_{2k-1} to u_{2k}. Now we have a graph G with $|E(G)| = 1 + \sum_{i=1}^{2k-2}(n-i)$ and $\deg_G u_{2k-1} = \deg_G u_{2k} = 2k-1$. Observe that $\deg_{\overline{G}} u_{2k-1} = \deg_{\overline{G}} u_{2k} = (n-1)-(2k-1) = n-2k$. We have that $\{u_1u_2, u_3u_4, \ldots, u_{2k-1}u_{2k}\}$ is an edge dominating set of G. Hence $\gamma'(G) \leq k$. Since $\{u_1, u_2, \ldots, u_{2k}\}$ is the set of vertices incident with all edges in G, so $\gamma'(G) \geq k$. Thus $\gamma'(G) = k$. Since $m \leq 2kn - k(2k+1)$, so $m - 1 - \sum_{i=1}^{2k-2}(n-i) \leq 2n - 4k$. Then a graph G' can be obtained from the graph G by adding $m - 1 - \sum_{i=1}^{2k-2}(n-i)$ edges joining u_{2k-1} or u_{2k} to any vertex of degree less than $n-1$. We have that $\{u_1u_2, u_3u_4, \ldots, u_{2k-1}u_{2k}\}$ is an edge dominating set of G'. Hence $\gamma'(G') \leq k$. Since $\{u_1, u_2, \ldots, u_{2k}\}$ is the set of vertices incident with all edges in G, so $\gamma'(G') \geq \gamma'(G) = k$. Hence $\gamma'(G') = k$. We obtain the graph $G' \in \mathcal{CG}(m, n)$, $2n - 3 < m \leq \binom{n}{2}$ with $\gamma'(G') = k$ as desired.

□

Theorem 3.33 For an integer $n \geq 4$,

$$\min(\gamma'; m, n) = \begin{cases} 1 & \text{if } n-1 \leq m \leq 2n-3, \\ \left\lceil \frac{(2n-1)-\sqrt{(2n-1)^2-8m}}{4} \right\rceil & \text{if } 2n-3 < m \leq \binom{n}{2}. \end{cases}$$

The proof of Theorem 3.33 follows from the Propositions 3.30 and 3.32.

□

From Theorem 2.36, we can say that for any connected graph G of even order n with size $m = \binom{n}{2}$ or $\frac{n^2}{4}$, $\max(\gamma'; m, n) = \frac{n}{2}$. Hence we shall determine $\max(\gamma'; m, n)$ if n is odd and $\max(\gamma'; m, n)$ if n is even and $m \neq \binom{n}{2}$ and $m \neq \frac{n^2}{4}$.

Proposition 3.34 Let n be an odd integer and $n - 1 \leq m \leq \binom{n}{2}$. Then $\max(\gamma'; m, n) = \frac{n-1}{2}$.

Proof Let $G \in \mathcal{CG}(m, n)$. From Theorem 2.32, we have that $\gamma'(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$.
Let $p = \frac{n-1}{2}$ and let $V(K_{1,p}) = \{u, u_1, u_2, \ldots, u_p\}$ with $\deg u = p$. Let H be a graph obtained from $K_{1,p}$ by adding p new vertices $v_1, v_2, \ldots, v_p$ to $K_{1,p}$ and joining each pair of vertices $u_i, v_i; 1 \leq i \leq p$. Hence $|E(H)| = n-1$ and $|V(H)| = n$.

Then $X = \{u_1v_1, u_2v_2, \ldots, u_pv_p\}$ is an edge dominating set of H. Hence $\gamma'(H) \leq p$. Since each edge u_iv_i $(1 \leq i \leq p)$ is adjacent to exactly one corresponding edge uu_i $(1 \leq i \leq p)$, so $\gamma'(H) \geq p$. That is $\gamma'(H) = p$.

If $n - 1 < m \leq 3p$, then a graph H' can be obtained from the graph H by adding the edges uv_i for $1 \leq i \leq m - (n - 1)$. We can see that X is also an edge dominating set of H'. That is $\gamma'(H') \leq p$. Let $X_1 = \{uu_1, uv_1, u_1v_1\}$ and consider the subgraph $\langle X_1 \rangle$ in H'. Since each edge of $\langle X_1 \rangle$ dominates 3 edges and $|X_1| = 3$, at least one edge from X_1 are needed to dominated the edges of X_1. Let $X_2 = \{uu_p, u_pu_p\}$ and consider the subgraph $\langle X_2 \rangle$ in H'. Since each edge of $\langle X_2 \rangle$ dominates two edges and $|X_2| = 2$, at least one edge from X_2 are needed to dominated the edges of X_2. Applying this argument to the other subgraphs isomorphic to $\langle X_1 \rangle$ and $\langle X_2 \rangle$ in H', we see that $\gamma'(H') \geq p$. Thus $\gamma'(H') = p$.

If $3p < m \leq \binom{n}{2}$, then we consider two cases according to whether p is odd or even.

Case 1 Suppose that p is odd. Then we divide into two subcases.

Subcase 1.1 For m even, let $V(K_{1,p-2}) = \{x, x_1, x_2, \ldots, x_{p-2}\}$ where $\deg x = p-2$ and let $V(K_4) = \{z_1, z_2, z_3, z_4\}$. Then a graph F can be obtained from $K_{1,p-2}$ by adding $p - 2$ new vertices $y_1, y_2, \ldots, y_{p-2}$ to $K_{1,p-2}$ and joining each pair of vertices $x_i, y_i; 1 \leq i \leq p - 2$ and adding the edge xz_1. Hence $|E(F)| = 2p + 3$ and $|V(F)| = n$. Then $X' = \{z_1z_2, z_3z_4, x_1y_1, x_2y_2, \ldots, x_{p-2}y_{p-2}\}$ is an edge dominating set of F. Hence $\gamma'(F) \leq p$. Since each edge x_iy_i $(1 \leq i \leq p - 2)$ is adjacent to exactly one corresponding edge xx_i $(1 \leq i \leq p - 2)$ and $\gamma'(K_4) = 2$, so $\gamma'(F) \geq p$. That is $\gamma'(H) = p$. Then a graph F' can be obtained from a graph F by adding the edges xy_i for $1 \leq i \leq p - 2$. Then $|E(F')| = 3p + 1$ and X' is also an edge dominating set of F'. Hence $\gamma'(F') \leq p$. Let $Y_1 = \{z_1z_2, z_1z_3, z_1z_4, z_2z_3, z_2z_4, z_3z_4\}$ and consider the subgraph $\langle Y_1 \rangle$ in F'. Since each edge of $\langle Y_1 \rangle$ dominates 5 edges and $|Y_1| = 6$, at least two edges from Y_1 are needed to dominated the edges of Y_1. Let $Y_2 = \{xx_1, xy_1, x_1y_1\}$ and consider the subgraph $\langle Y_2 \rangle$ in F'. Since each edge of $\langle Y_2 \rangle$ dominates 3 edges and $|Y_2| = 3$, at least one edge from Y_2 are needed to dominated the edges of Y_2. Applying this argument to the other subgraphs isomorphic to $\langle Y_1 \rangle$

and $\langle Y_2 \rangle$ in F', we see that $\gamma'(F') \geq p$. Thus $\gamma'(F') = p$.

If $3p + 1 < m \leq 2\binom{p-2}{2} + 3p + 1 = p^2 - 2p + 7$, then a graph F_1 can be obtained from the graph F' by adding $m - (3p + 1)$ edges which are the pairs of edges $x_i y_j$ and $x_j y_i$ for $1 \leq i < j \leq p - 2$. Hence X' is also an edge dominating set of F_1. That is $\gamma'(F_1) \leq p$. Let $Y_3 = \{z_1z_2, z_1z_3, z_1z_4, z_2z_3, z_2z_4, z_3z_4\}$ and consider the subgraph $\langle Y_3 \rangle$ in F_1. Since each edge of $\langle Y_3 \rangle$ dominates 5 edges and $|Y_3| = 6$, at least two edges from Y_3 are needed to dominated the edges of Y_3. Let $Y_4 = \{xx_1, xx_2, xy_1, xy_2, x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$ and consider the subgraph $\langle Y_4 \rangle$ in F_1. Since every edge of $\langle Y_4 \rangle$ dominates at most 6 edges and $|Y_4| = 8$, at least two edges from Y_4 are needed to dominated the edges of Y_4. Let $Y_5 = \{xx_{p-2}, xy_{p-2}, x_{p-2}y_{p-2}\}$ and consider the subgraph $\langle Y_5 \rangle$ in F_1. Since each edge of $\langle Y_5 \rangle$ dominates 3 edges and $|Y_5| = 3$, at least one edge from Y_5 are needed to dominated the edges of Y_5. Applying this argument to the other subgraphs isomorphic to $\langle Y_3 \rangle, \langle Y_4 \rangle$ and $\langle Y_5 \rangle$ in F_1, we see that $\gamma'(F_1) \geq p$. Thus $\gamma'(F_1) = p$.

If $p^2 - 2p + 7 < m \leq 4\binom{p-2}{2} + 3p + 1 = 2p^2 - 7p + 13$, then a graph F_2 can be obtained from the graph F' by adding $2\binom{p-2}{2}$ edges which are the pairs of edges $x_i y_j$ and $x_j y_i$ for $1 \leq i < j \leq p - 2$ and adding $m - (p^2 - 2p + 7)$ edges which are the pairs of edges $x_i x_j$ and $y_i y_j$ for $1 \leq i < j \leq p - 2$. Hence X' is also an edge dominating set of F_2. That is $\gamma'(F_2) \leq p$. Let $Y_6 = \{z_1z_2, z_1z_3, z_1z_4, z_2z_3, z_2z_4, z_3z_4\}$ and consider the subgraph $\langle Y_6 \rangle$ in F_2. Since each edge of $\langle Y_6 \rangle$ dominates 5 edges and $|Y_6| = 6$, at least two edges from Y_6 are needed to dominated the edges of Y_6. Let $Y_7 = \{xx_1, xx_2, xy_1, xy_2, x_1x_2, x_1y_1, x_1y_2, x_2y_1, x_2y_2, y_1y_2\}$ and consider the subgraph $\langle Y_7 \rangle$ in F_2. Since every edge of $\langle Y_7 \rangle$ dominates at most 7 edges and $|Y_7| = 10$, at least two edges from Y_7 are needed to dominated the edges of Y_7. Let $Y_8 = \{xx_{p-2}, xy_{p-2}, x_{p-2}y_{p-2}\}$ and consider the subgraph $\langle Y_8 \rangle$ in F_2. Since each edge of $\langle Y_8 \rangle$ dominates three edges and $|Y_8| = 3$, at least one edge from Y_8 are needed to dominated the edges of Y_8. Applying this argument to the other subgraphs isomorphic to $\langle Y_6 \rangle, \langle Y_7 \rangle$ and $\langle Y_8 \rangle$ in F_2, we see that $\gamma'(F_2) \geq p$. Thus $\gamma'(F_2) = p$.

If $2p^2 - 7p + 13 < m \leq \binom{n}{2}$, then a graph F_3 can be obtained from the graph F' by adding $2\binom{p-2}{2}$ edges which are the pairs of edges x_iy_j and x_jy_i for $1 \le i < j \le p-2$ and adding $2\binom{p-2}{2}$ edges which are the pairs of edges x_ix_j and y_iy_j for $1 \le i < j \le p-2$ and adding $m-(2p^2-7p+13)$ new edges. Hence X' is also an edge dominating set of F_3. That is $\gamma'(F_3) \le p$. Since F_3 contains $K_{2(p-2)+1}$ and K_4, so $\gamma'(F_3) \ge p-2+2 = p$. Thus $\gamma'(F_3) = p$. Figure 3.41 illustrates the graph F', F_1, F_2 and F_3.

Subcase 1.2 For m odd, if $3p < m \le 2\binom{p}{2} + 3p = p^2 + 2p$, then a graph J_1 can be obtained from the graph H by adding the edges uv_i for $1 \le i \le p$ and adding $m-3p$ edges which are the pairs of edges u_iv_j and u_jv_i for $1 \le i < j \le p$. Hence X is also an edge dominating set of J_1. That is $\gamma'(J_1) \le p$. Let $Y_1' = \{uu_1, uu_2, uv_1, uv_2, u_1v_1, u_1v_2, u_2v_1, u_2v_2\}$ and consider the subgraph $\langle Y_1' \rangle$ in J_1. Since every edge of $\langle Y_1' \rangle$ dominates at most 6 edges and $|Y_1'| = 8$, at least two edges from Y_1' are needed to dominated the edges of Y_1'. Let $Y_2' = \{uu_p, uv_p, u_pv_p\}$ and consider the subgraph $\langle Y_2' \rangle$ in J_1. Since each edge of $\langle Y_2' \rangle$ dominates three edges and $|Y_2'| = 3$, at least one edge from Y_2' are needed to dominated the edges of Y_2'. Applying this argument to the other subgraphs isomorphic to $\langle Y_1' \rangle$ and $\langle Y_2' \rangle$ in J_1, we see that $\gamma'(J_1) \ge p$. Thus $\gamma'(J_1) = p$.

If $p^2 + 2p < m \le 2p^2 + p = \binom{n}{2}$, then a graph J_2 can be obtained from the graph H by adding the edges uv_i for $1 \le i \le p$ and adding $2\binom{p}{2}$ edges which are the pairs of edges u_iv_j and u_jv_i for $1 \le i < j \le p$ and adding $m - (p^2 + 2p)$ edges which are the pairs of edges u_iu_j and v_iv_j for $1 \le i < j \le p$. Hence X is also an edge dominating set of J_2. That is $\gamma'(J_2) \le p$. Let $Y_3' = \{uu_1, uu_2, uv_1, uv_2, u_1u_2, u_1v_1, u_1v_2, u_2v_1, u_2v_2, v_1v_2\}$ and consider the subgraph $\langle Y_3' \rangle$ in J_2. Since every edge of $\langle Y_3' \rangle$ dominates at most 7 edges and $|Y_3'| = 10$, at least two edges from Y_3' are needed to dominated the edges of Y_3'. Let $Y_4' = \{uu_p, uv_p, u_pv_p\}$ and consider the subgraph $\langle Y_4' \rangle$ in J_2. Since each edge of $\langle Y_4' \rangle$ dominates three edges and $|Y_4'| = 3$, at least one edge from Y_4' are needed to dominated the edges of Y_4'. Applying this argument to the other subgraphs isomorphic to $\langle Y_3' \rangle$ and $\langle Y_4' \rangle$ in J_2, we see that $\gamma'(J_2) \ge p$. Thus $\gamma'(J_2) = p$. Figure 3.42 illustrates the graph H', J_1 and J_2.

Case 2 Suppose that p is even. Then we divide into two subcases.

Subcase 2.1 For m odd, the construction is similar to Subcase 1.1.

Subcase 2.2 For m even, the construction is similar to Subcase 1.2.

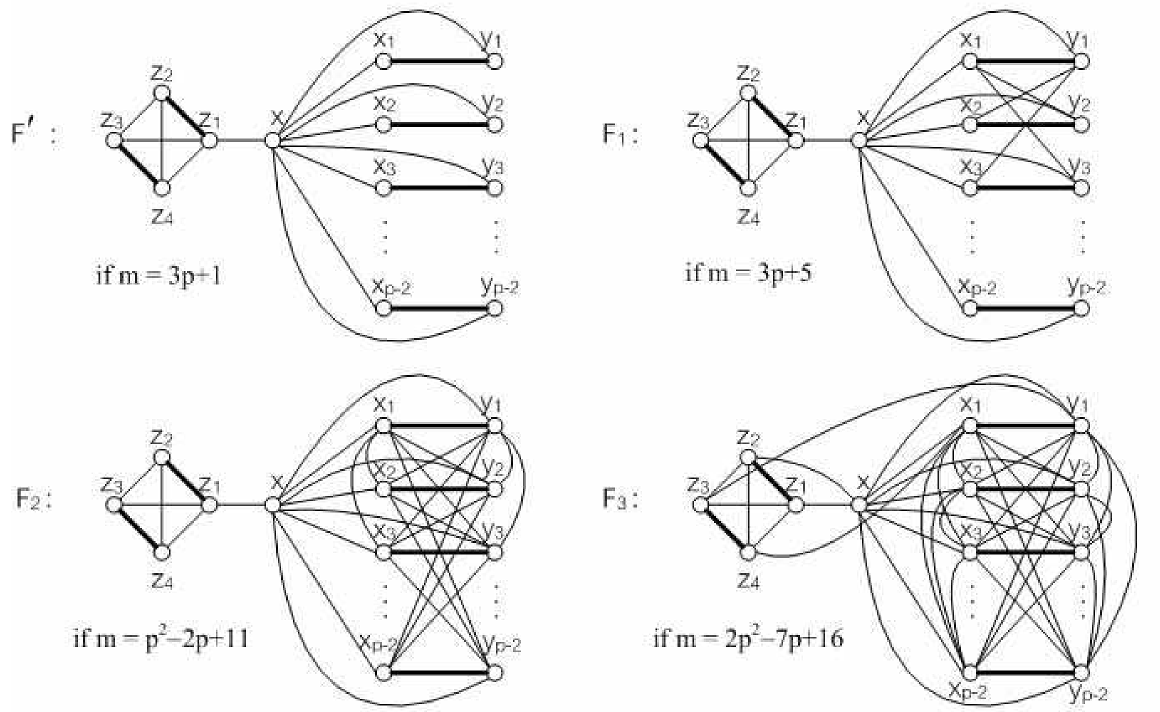

Figure 3.41: The graph F', F_1, F_2 and F_3.

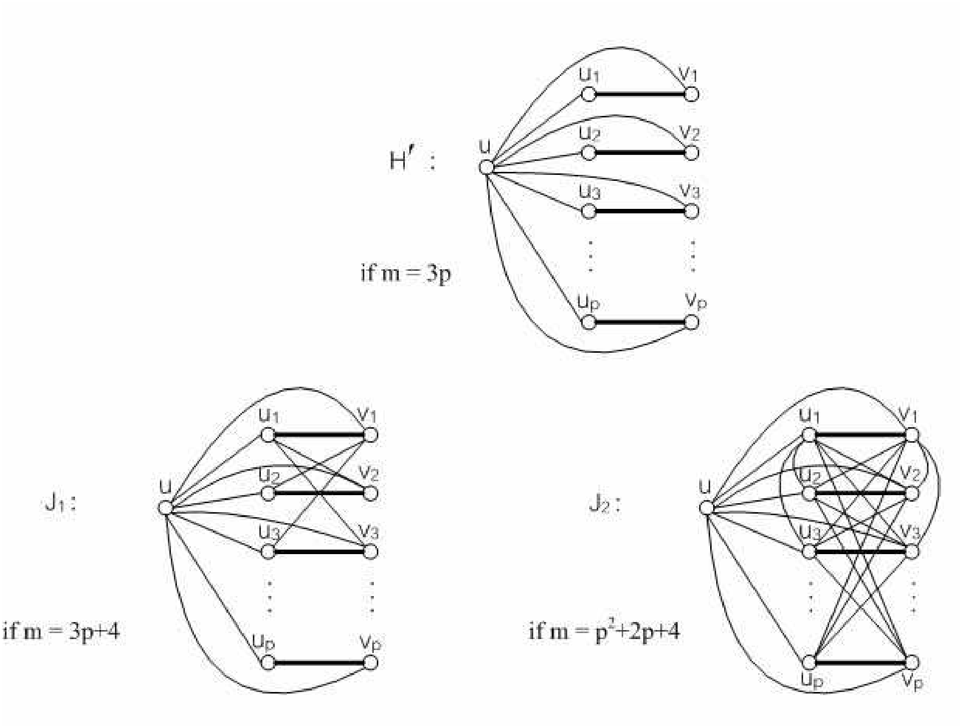

Figure 3.42: The graph H', J_1 and J_2.

□

Proposition 3.35 Let n be an even integer and $n-1 \leq m < \binom{n}{2}$. If $m \notin \left\{\binom{n}{2}, \frac{n^2}{4}\right\}$, then $\max(\gamma'; m, n) = \frac{n}{2} - 1$.

Proof Let $G \in \mathcal{CG}(m,n)$, and let $m \notin \left\{\binom{n}{2}, \frac{n^2}{4}\right\}$. We can say Theorem 2.36 in the other words that if $m \neq \binom{n}{2}$ and $m \neq \frac{n^2}{4}$, then $\gamma'(G) \neq \frac{n}{2}$. By Theorem 2.32, that is $\gamma'(G) \leq \frac{n}{2} - 1$.

Let $p = \frac{n}{2}$ and let $V(K_{1,p}) = \{u, u_1, u_2, \ldots, u_p\}$ with $\deg u = p$. Let H_1 be a graph obtained from $K_{1,p}$ by adding $p-1$ new vertices $v_1, v_2, \ldots, v_{p-1}$ to $K_{1,p}$ and joining each pair of vertices $u_i, v_i; 1 \leq i \leq p-1$. Hence $|E(H_1)| = n-1$ and $|V(H_1)| = n$. Then $X = \{uu_1, u_2v_2, u_3v_3, \ldots, u_{p-1}v_{p-1}\}$ is an edge dominating set of H_1. Hence $\gamma'(H_1) \leq p-1$. Since each edge u_iv_i $(1 \leq i \leq p-1)$ is adjacent to exactly one edge uu_i $(1 \leq i \leq p-1)$, so $\gamma'(H_1) \geq p-1$. That is $\gamma'(H_1) = p-1$. If $n-1 < m \leq 3(p-1)+1 = 3p-2$, then a graph H_2 can be obtained from the graph H_1 by adding the edges uv_i for $1 \leq i \leq m-(n-1)$. Hence X is also an edge dominating set of H_2. That is $\gamma'(H_2) \leq p-1$. Let $Y_1 = \{uu_1, uv_1, uu_p, u_1v_1\}$ and consider the subgraph $\langle Y_1 \rangle$ in H_2. Since every edge of $\langle Y_1 \rangle$ dominates at most 4 edges and $|Y_1| = 4$, at least one edge from Y_1 are needed to dominated the edges of Y_1. Let $Y_2 = \{uu_{p-1}, u_{p-1}u_{p-1}\}$ and consider the subgraph $\langle Y_2 \rangle$ in H_2. Since each edge of $\langle Y_2 \rangle$ dominates two edges and $|Y_2| = 2$, at least one edge from Y_2 are needed to dominated the edges of Y_2. Applying this argument to the other subgraphs isomorphic to $\langle Y_1 \rangle$ and $\langle Y_2 \rangle$ in H_2, we see that $\gamma'(H_2) \geq p-1$. Thus $\gamma'(H_2) = p-1$.

If $3p-2 < m \leq 4(p-1)+1 = 4p-3$, then a graph H_3 can be obtained from the graph H_1 by adding the edges uv_i for $1 \leq i \leq p-1$ and adding the edges u_iu_p for $1 \leq i \leq m-(3p-2)$. Hence X is also an edge dominating set of H_3. That is $\gamma'(H_3) \leq p-1$. Let $Y_3 = \{uu_1, uv_1, uu_p, u_1v_1, u_1u_p\}$ and consider the subgraph $\langle Y_3 \rangle$ in H_3. Since every edge of $\langle Y_3 \rangle$ dominates at most 5 edges and $|Y_3| = 5$, at least one edge from Y_3 are needed to dominated the edges of Y_3. Let $Y_4 = \{uu_{p-1}, uv_{p-1}, u_{p-1}u_{p-1}\}$ and consider the subgraph $\langle Y_4 \rangle$ in H_3. Since each edge of $\langle Y_4 \rangle$ dominates 3 edges and $|Y_4| = 3$, at least one edge from Y_4 are needed to dominated the edges of Y_4. Applying this argument to the other subgraphs isomorphic to $\langle Y_3 \rangle$ and $\langle Y_4 \rangle$ in H_3, we see that $\gamma'(H_3) \geq p-1$. Thus $\gamma'(H_3) = p-1$.

If $4p-3 < m \leq 5(p-1) = 5p-5$, then a graph H_4 can be obtained from the graph H_1 by adding the edges uv_i for $1 \leq i \leq p-1$ and adding the edges u_iu_p for $1 \leq i \leq p-1$ and adding the edges u_pv_i for $2 \leq i \leq m-(4p-3)$. Hence X is also an edge dominating set of H_4. That is $\gamma'(H_4) \leq p-1$. Let $Y_5 = \{uu_2, uv_2, uu_p, u_2v_2, u_2u_p, u_pv_2\}$ and consider the subgraph $\langle Y_5 \rangle$ in H_4. Since every edge of $\langle Y_5 \rangle$ dominates at most 5 edges and $|Y_5| = 6$, at least two edges from Y_5 are needed to dominated the edges of Y_5. Let $Y_6 = \{uu_{p-1}, uv_{p-1}, u_{p-1}u_{p-1}\}$ and consider the subgraph $\langle Y_6 \rangle$ in H_4. Since each edge of $\langle Y_6 \rangle$ dominates 3 edges and $|Y_6| = 3$, at least one edge from Y_6 are needed to dominated the edges of Y_6. Applying this argument to the other subgraphs isomorphic to $\langle Y_5 \rangle$ and $\langle Y_6 \rangle$ in H_4, we see that $\gamma'(H_4) \geq p-1$. Thus $\gamma'(H_4) = p-1$.

If $5p-5 < m \leq \binom{n}{2}-1$, then a graph H_5 can be obtained from the graph

H_1 by adding the edges uv_i for $1 \le i \le p-1$ and adding the edges u_iu_p for $1 \le i \le p-1$ and adding the edges u_pv_i for $2 \le i \le p-1$ and adding $m-(5p-5)$ new edges. Hence X is also an edge dominating set of H_5. That is $\gamma'(H_5) \le p-1$. Let $Y_7 = \{uu_2, uu_3, uu_p, u_2u_p, u_2v_2, u_3u_p, u_3v_3, u_pv_2, u_pv_3\}$ and consider the subgraph $\langle Y_7 \rangle$ in H_5. Since every edge of $\langle Y_7 \rangle$ dominates at most 7 edges and $|Y_7| = 9$, at least two edges from Y_7 are needed to dominated the edges of Y_7. Let $Y_8 = \{uu_{p-1}, uv_{p-1}, u_{p-1}u_{p-1}\}$ and consider the subgraph $\langle Y_8 \rangle$ in H_5. Since each edge of $\langle Y_8 \rangle$ dominates 3 edges and $|Y_8| = 3$, at least one edge from Y_8 are needed to dominated the edges of Y_8. Applying this argument to the other subgraphs isomorphic to $\langle Y_7 \rangle$ and $\langle Y_8 \rangle$ in H_5, we see that $\gamma'(H_5) \ge p-1$. Thus $\gamma'(H_5) = p-1$. Figure 3.43 illustrates the graph H_2, H_3, H_4 and H_5.

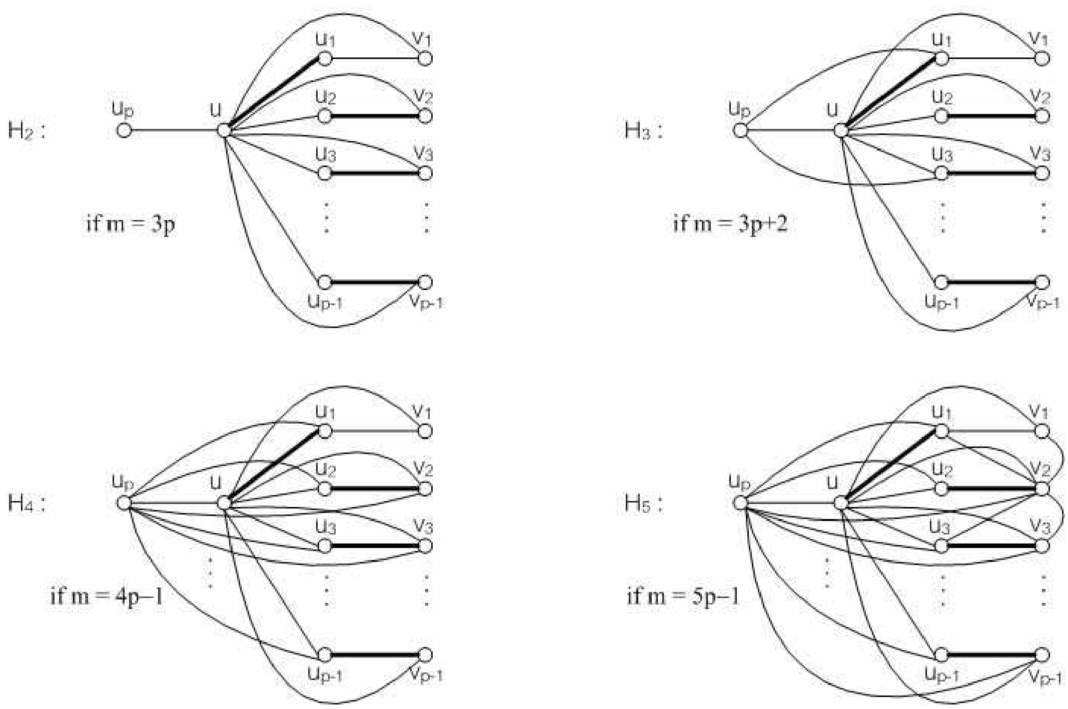

Figure 3.43: The graph H_2, H_3, H_4 and H_5.

□

Theorem 3.36 For an integer $n \geq 4$ and $n-1 \leq m \leq \binom{n}{2}$,

1. $\max(\gamma'; m, n) = \frac{n-1}{2}$ if n is odd,

2. $\max(\gamma'; m, n) = \frac{n}{2}$ if n is even and $m = \binom{n}{2}$ or $m = \frac{n^2}{4}$,

3. $\max(\gamma'; m, n) = \frac{n}{2} - 1$ otherwise.

□

Next example illustrates Theorems 3.33 and 3.36 and shows all graphs of every value of the edge domination number between those minimum and maximum values.

Example 3.2.5 Let $n = 7$.

$$\text{Then } \min(\gamma'; m, 7) = \begin{cases} 1 & \text{if } 6 \leq m \leq 11, \\ \left\lceil \frac{13-\sqrt{169-8m}}{4} \right\rceil & \text{if } 11 < m \leq 21 \end{cases}$$

and $\max(\gamma'; m, 7) = \frac{7-1}{2} = 3$.

For example, we consider $m = 11$. Then $\min(\gamma'; 11, 7) = 1$ and $\max(\gamma'; 11, 7) = 3$. We construct the graph $G_1 \in \mathcal{CG}(11, 7)$ by Proposition 3.30 with $\gamma'(G_1) = 1$. Since $p = 3$ is odd, $m = 11$ is odd and $3p < m \leq p^2 + 2p$, so the construction of the graph $G_2 \in \mathcal{CG}(11, 7)$ with $\gamma'(G_2) = 3$ follows the graph J_1 in Proposition 3.34 Subcase 1.2. By the interpolation property of $\gamma'(G)$, we know that there exists a graph $G \in \mathcal{CG}(11, 7)$ with $\gamma'(G) = c$ where $c = 2$. The graph $G_3 \in \mathcal{CG}(11, 7)$ with $\gamma'(G_3) = 2$ is shown in Figure 3.44 and the minimum edge dominating sets of G_1, G_2 and G_3 are indicated by solid edges.

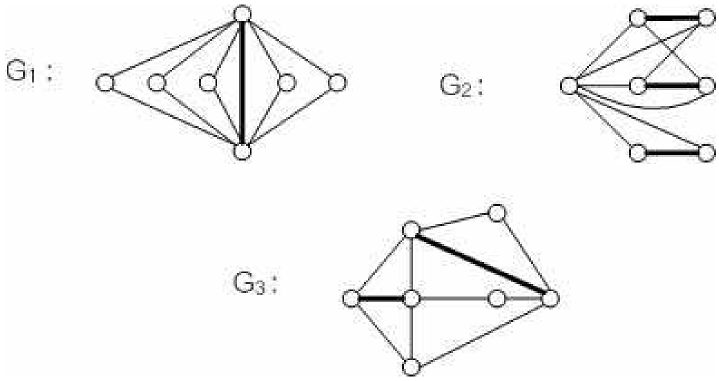

Figure 3.44: G_1, G_2 and $G_3 \in \mathcal{CG}(11, 7)$.

Next, we consider $m = 16$. Then $\min(\gamma'; 16, 7) = 2$ and $\max(\gamma'; 16, 7) = 3$. We construct the graph H_1 by Proposition 3.32 with $\gamma'(H_1) = 2$. Since $p = 3$ is odd, $m = 16$ is even and $2p^2 - 7p + 13 < m \leq \binom{n}{2}$, so the construction of the graph $H_2 \in \mathcal{CG}(16, 7)$ with $\gamma'(H_2) = 3$ follows the graph F_3 in Proposition 3.34 Subcase 1.1. The minimum edge dominating sets of H_1 and H_2 are indicated by solid edges in Figure 3.45.

Figure 3.45: $H_1, H_2 \in \mathcal{CG}(16, 7)$.

Example 3.2.6 Let $n = 8$.

$$\text{Then } \min(\gamma'; m, 8) = \begin{cases} 1 & \text{if } 7 \leq m \leq 13, \\ \left\lceil \frac{15-\sqrt{225-8m}}{4} \right\rceil & \text{if } 13 < m \leq 28 \end{cases}$$

$$\text{and } \max(\gamma'; m, 8) = \begin{cases} 4 & \text{if } m = 16 \text{ and } m = 28, \\ 3 & \text{if } 7 \leq m < 16 \text{ and } 16 < m < 28. \end{cases}$$

For example, we consider $m = 10$. Then $\min(\gamma'; 10, 8) = 1$ and $\max(\gamma'; 10, 8) = 3$. We construct the graph G_1 by Proposition 3.30 with $\gamma'(G_1) = 1$. Since $p = 4, m = 10$ and $n - 1 < m \leq 3p - 2$, so the construction of the graph $G_2 \in \mathcal{CG}(10, 8)$ with $\gamma'(G_2) = 3$ follows the graph H_2 in Proposition 3.35. By the interpolation property of $\gamma'(G)$, we know that there exists a graph $G \in \mathcal{CG}(10, 8)$ with $\gamma'(G) = c$ where $c = 2$. The graph $G_3 \in \mathcal{CG}(10, 8)$ with $\gamma'(G_3) = 2$ is shown in Figure 3.46 and the minimum edge dominating sets of G_1, G_2 and G_3 are indicated by solid edges.

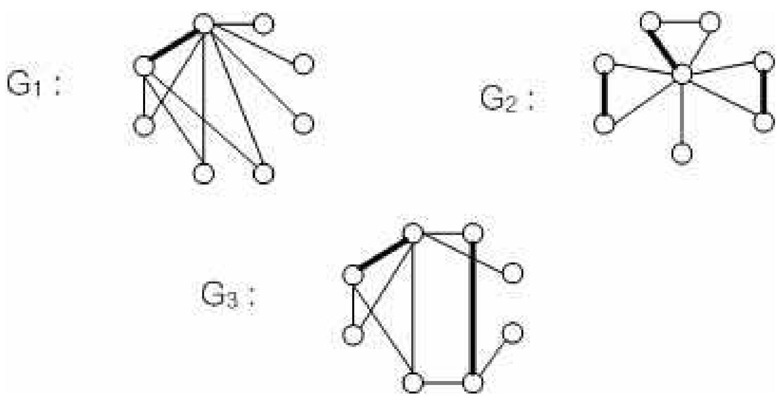

Figure 3.46: G_1, G_2 and $G_3 \in \mathcal{CG}(10, 8)$.

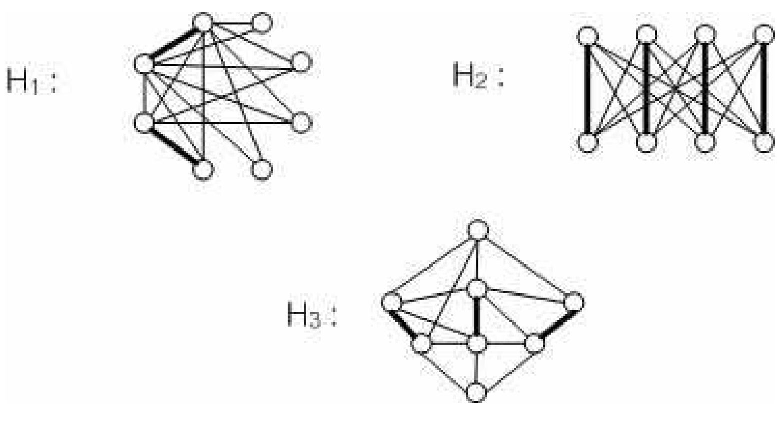

Figure 3.47: H_1, H_2 and $H_3 \in \mathcal{CG}(16, 8)$.

Next, we consider $m = 16$. Then $\min(\gamma'; 16, 8) = 2$ and $\max(\gamma'; 16, 8) = 4$. We construct the graph H_1 by Proposition 3.32 with $\gamma'(H_1) = 2$ and the graph H_2 by Theorem 2.36 with $\gamma'(H_2) = 4$ where $H_1, H_2 \in \mathcal{CG}(16, 8)$. By the interpolation property of $\gamma'(G)$, we know that there exists a graph $G \in \mathcal{CG}(16, 8)$ with $\gamma'(G) = c$ where $c = 3$. The graph $H_3 \in \mathcal{CG}(16, 8)$ with $\gamma'(H_3) = 3$ is shown in Figure 3.47 and the minimum edge dominating sets of H_1, H_2 and H_3 are indicated by solid edges.

REFERENCES

[1] S. Arumugam and S. Velammal, *Edge domination in graphs*, Taiwanese Journal of Mathematics **2** (1998), no. 2, 173–179.

[2] G. Chartrand, *A graph-theoretic approach to a communications problem*, SIAM J. Appl. Math. **14** (1966), 778–781.

[3] G. Chartrand and F. Harary, *Graphs with prescribed connectivities*, Proceedings of the Colloquium Held at Tihany, Hungary. Budapest, 1968, pp. 61–63.

[4] G. Chartrand and L. Lesniak, *Graphs and digraphs*, third ed., Chapman and Hall, 1996.

[5] G. Chartrand and S. Schuster, *On the independence number of complementary graphs*, Trans. New York Acad. Sci.,Series II **36** (1974), no. 3, 247–251.

[6] G. Chartrand and P. Zhang, *Introduction to Graph Theory*, international ed., McGraw-Hill, 2005.

[7] P. Dankelmann and L. Volkmann, *New suffcient conditions for equality of minimum degree and edge-connectivity*, Ars Combin. **40** (1995), 270–278.

[8] E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi, *Total domination in graphs*, Networks **10**.

[9] R. B. Eggleton and D. A. Holton, *Graphic sequences*, Combinatorial mathematics **6** (1979), 1–10, (Proc. Sixth Austral. Conf., Univ. New England, Armidale, 1978) Lecture Note in Math, 748.

[10] R.C. Entringer and D.L. Goldsmith, *A suffcient condition for equality of edge connectivity and minimum degree of a graph*, J. Graph Theory **3** (1979), 251–255.

[11] J.F. Fink and M.S. Jacobson, *On n-domination, n-dependence and forbidden subgraphs*, Proc. 5th Quadrennial International Conf., 1984, pp. 283–300.

[12] S. Hakimi, *On the realizability of a set of integers as the degree of the vertices of a graph*, SIAM J.Appl. Math. **10** (1962), 496–506.

[13] F. Harary and M. Plantholt, *Classification of interpolation theorems for spanning trees and other families of spanning subgraphs*, Graph Theory **13** (1989), no. 6, 703–712.

[14] F. Harary and S. Schuster, *Interpolation theorems for the invariants of spanning trees of a given graph: edge-covering*, Congr. Number. **59** (1978), 107–114, Eighteenth Southeastern International conference on Combinatorics, Graph Theory, and Computing (Boca Raton, Fla, 1987).

[15] ______, *Interpolation theorems for the independence and domination numbers of spanning trees*, Ann. Discrete Math. **41** (1989), 221–227, Graph theory in memory of G.A. Dirac (Sandbjerg, 1985).

[16] M. Havel, *A remark on the existence of finite graphs (in hungarian)*, Casopis Pest. Mat. **80** (1955), 477–480.

[17] E.J. Cockayne, S.T. Hedetniemi and D.J. Miller, *Properties of hereditary hypergraphs and middle graphs*, Canad. Math. Bull. **21**.

[18] S. R. Jayaram, *Line domination in graphs*, Graphs Combin. **3** (1987), 357–363.

[19] L. Lesniak, *Results on the edge-connectivity of graphs*, Discrete Math. **8** (1974), 351–354.

[20] W. McCuaig and B. Shepherd, *Domination in graphs with minimum degree two*, J. Graph Theory **13** (1989), 749–762.

[21] S. Mitchell and S. T. Hedetniemi, *Edge domination in trees*, Congr. Numer. **19** (1977), 489–509.

[22] O. Ore, *Theory of graphs*, Amer. Math. Soc. Colloq. Publ. **38** (1962).

[23] F. Harary, M. Plantholt and J. Robert, *Interpolation theorems for diameters of spanning trees*, IEEE Trans. Circuits and systems **30** (1983), no. 7, 429–432.

[24] J. Plesník, *Critical graphs of given diameter*, Acta Fac. Rerum Natur. Univ. Commenian Math. **30** (1975), 71–93.

[25] J. Plesník and S. Znám, *On equality of edge-connectivity and minimum degree of a graph*, Arch. Match. (Brno) **25** (1989), 19–25.

[26] N. Punnim, *The clique number of regular graphs*, Graphs Combinatorics **18** (2002), no. 4, 781–785.

[27] ______, *Degree sequences and chromatic number of graphs*, Graphs and Combinatorics **18** (2002), no. 3, 597–603.

[28] ______, *Forests in random graphs*, SEAMS Bull. Math. **27** (2003), 333–339.

[29] ______, *On maximal induced forest in graphs*, SEAMS Bull. Math. **27** (2003), 667–673.

[30] ______, *Spectrum of graph parameters*, East-West Journal of Mathematics **5** (2003), no. 1, 28–34.

[31] ______, *Interpolation theorems on graph parameters*, SEAMS Bull. Math. **28** (2004), 533–538.

[32] ______, *The matching number of regular graphs*, Thai Journal of Mathematics **2** (2004), 133–140.

[33] ______, *Decycling regular graphs*, Australasian J. Combinatorics **32** (2005), 147–162.

[34] B.A. Reed, *Paths, stars and the number three*, Combin. Probab. Comput. **5** (1996), 277–295.

[35] R. Samanmoo, *Extremal theorems for the independence number of connected regular graphs*, Master's thesis, Department of Mathematics, Science and Technology, Thammasat University, 2005.

[36] E. Sampathkumar, *The global domination number of a graph*, J. Math. Phys. Sci. **23**.

[37] L.A. Sanchis, *Maximum number of edges in connected graphs with a given domination number*, Discrete Math. **87** (1991), 65–72.

[38] J. Topp and P.D. Vestergaard, *Interpolation theorems for domination numbers of a graph*, Discrete Mathematics **191** (1998), 207–221.

[39] Roditty Y. Caro, *On the vertex-independence number and star decomposition of graphs*, Ars Combin **20** (1985), 167–180.

[40] Roditty Y.Caro, *A note on the k-domination number of a graph*, Internat. J. Math. Sci. **13** (1990), 205–206.

VITA

Name	Miss Aunyarat Bunyawat
Date of Birth	January 22, 1982
Education	2003 : B.S. in Physics,
	Naresuan University