CHAPTER 1 INTRODUCTION

1.1 Statement of the Problem

The nanoscale structures such as wires, rod and beams have a wide range of applications in physics, engineering, and several other fields. In engineering applications, nanobeams are commonly used in advanced technological devices such as sensors, actuators, transistors, and resonators in nanoelectromechanical systems (NEMSs), respectively. Since the mechanical properties of nanobeams are totally different from classical and macroscopic counterparts, therefore it is necessary to exactly characterize the mechanical properties of nanobeams for these applications. Nanobeams are size-independent physical properties and mechanical properties while their bending, buckling, and vibration behaviors are size-dependent, this is due to their large ratio of surface area to volume. Surface stress and nonlocal elasticity have been accepted as important effects that explain the behavior of nanobeams. Therefore, the main purpose of this study is to investigate the nanobeams behavior including of both surface stress and nonlocal elasticity effect on the model of nanobeams.

The main purpose of this work is to determine analytically the natural frequencies and corresponding mode shapes of nanobeams with boundary conditions of pinned-pinned, clamped-clamped, clamped-free, clamped-pinned, clamped-sliding, and sliding-pinned, respectively, and then verified numerically using the finite element method. The combined effects of surface stress and nonlocal elasticity on the natural frequencies, vibration mode shapes, and boundary conditions are also discussed. The equation of motion of nanobeams and the free vibration analysis results obtained in this work can be used to predict the vibration behaviors of nanobeams in NEMs technology applications.

1.2 Literature Review

For the solution of nanobeams obtained in literatures, it can be seen that most of the solutions of nanobeams were solved based on the consideration of surface stress, residual surface tension, as well as the surface elasticity. The surface stress theory was initiated by Gurtin and Murdoch (1978) and was included by many researchers in order to investigate the static and dynamic behaviors of micro and nanostructures. For examples, Wang and Feng (2007) presented the surface stress effect on the natural frequencies of simply-supported microbeams. The surface stress effect was also considered incorporated with nonlinear static and dynamic behaviors of nanobeams (Fu et al., 2010). In addition, Liu and Rajapzikse (2010) studied the surface stress effect on the bending, buckling, and the vibration of nanobeams and double nanobeams systems (Hui and Wang, 2011).

In case of nonlocal elasticity theory, the obtained literatures indicated that the nonlocal elasticity theory was initially proposed in 1983 by Eringen (1983, 2002). Later on, this theory was applied to the classical nanostructure of carbon nanotubes (CNTs). Based on this theory, the size scale is used to adjust the properties of elastic body continuum and it is assumed that the stress at a reference point depends on the strain at the same point and also at all other points in the continuum. In this way, the internal size scale could be considered in the constitutive equations, especially, the size scale of material parameters. The applications of nonlocal continuum theory on the nanotechnology works were initially addressed by Peddieson et al. (2003). This work used the simplified nonlocal model which presented by Eringen (1983) to solve the static displacement of nanostructures. In literatures, there were also found that applications of nonlocal elasticity for nanostructure analysis received much attention by many researchers. Reddy and Pang (2008) presented the analytical solutions of bending, vibration, and buckling of CNTs. Nonlocal elasticity for free vibration single-walled CNTs are also presented by author's previous work (Thongyothee and Chucheepsakul, 2013). This work presented the classical solutions and then compared with that of numerical solutions provided by finite element models. Lu et al. (2006) studied the dynamic properties of flexural beams using a nonlocal elasticity model. Frequency equations and model shape functions of nanobeams with general boundary conditions were derived based on a nonlocal Euler-beam model. Pandikar and Pradhan (2010) analyzed the bending, buckling, and the vibration of nanobeams and nanoplates by using the finite element solution. Moreover, a few researchers ((Eltaher et al., 2012) and (Eltaher et al., 2013)) were also applied the nonlocal elasticity with the free vibration behaviors of nanobeams using finite element method. Meanwhile, Ghannadpour et al. (2013) presented bending, buckling, and the vibration problems of nonlocal Euler-beams using Ritz method.

Focusing on the problem of nanobeams including the combined effects of surface stress and nonlocal elasticity, Mahmoud et al. (2012) presented the static displacement of simply support nanobeams by using finite element analysis while author's previous work (Juntarasaid et al., 2012) determined analytically the static displacement and buckling of nanowires with various boundary conditions and verified numerically using finite element method. Few studies included the surface stress effect on the frequency analysis of nanostructures using nonlocal elasticity beam theory was also found in works of Lee and Chang (2010). The first published paper presented the natural frequency of simply supported nanotubes using the nonlocal Timoshenko beam theory while the second work presented the natural frequency of a non-uniform cantilever nanobeam using the Raleigh-Ritz approximation solution method. In addition, the nonlinear frequency analysis of nonuniform cross section nanobeams was also investigated by Malekzadeh and Shojaee (2013) for both Timoshenko and Euler-Bernoulli beam theories.

1.3 Objectives

The objectives of this study are as follows:

- To study a formulation of nanobeams based on the surface stress and the nonlocal elasticity effects.
- To develop nanobeams model by considering both the surface stress and the nonlocal elasticity effects.
- To determine analytically the natural frequencies and corresponding mode shapes of nanobeams under various boundary conditions.
- To verify the results for natural frequencies of nanobeams by using finite-element method.
- To present the influences of surface stress and nonlocal elasticity on the natural frequencies of nanobeams.

1.4 Assumptions and Limitations

In order to limit the scope of the problem, the following assumptions are established.

- The nanobeams material is assumed to be homogeneous, isotropic and linearly elastic.
- The self weight of nanobeams is not considered.
- The boundary conditions of nanobeams are varied including pinned-pinned, clamped-clamped, clamped-free, clamped-pinned, clamped-sliding, and sliding-pinned.

1.5 Research Methodology

The first part of this research focuses on the theory and model formulation of nanobeams. The model of nanobeams with surface stress and nonlocal elasticity are described individually. Then, the governing differential equation considering both effects in order to compute the natural frequency of nanobeams is presented. For the method of solutions, the governing differential equation of nanobeams including the effects of surface stress and nonlocal elasticity is solved analytically to obtain the natural frequencies under various boundary conditions. Furthermore, the finite element method by applying the Galerkin finite-element technique is also used to obtain the numerical results. Finally, the analytical results obtained solutions are validated by the numerical results from the finite element method. The influences of surface stress and nonlocal elasticity on the natural frequencies under various boundary conditions are also the highlight of this research.

1.6 Outstanding Features of Research

- The governing differential equation of nanobeams including surface stress and nonlocal elasticity is presented originally.
- The modal shape functions of nanobeams and MATLAB code (finite element analysis) developed in this study are very useful for engineers designing the nanobeams as well as the researchers who want to predict the behavior of nanobeams in NEMs technology applications.

CHAPTER 2 THEORETICAL MODEL FORMULATION

In this chapter, theory and model formulation of nanobeams are presented. The model of nanobeams with surface stress and nonlocal elasticity are described individually. A variational formulation is developed based on the combination of surface stress and nonlocal elasticity. The governing equation of nanobeams including surface stress and nonlocal elasticity effects is derived to be the new model of nanobeams including all effects. The present model formulation is applied for free vibration of nanobeams.

2.1 Surface Stress Model for Nanobeams

In the theory of surface elasticity purposed by Gurtin and Murdoch (1987), the influence of surfaces can be expressed as surface stress that surface stress is related to the surface density. If both the surface layer and the bulk of the material are isotropic and linearly elastic, the one-dimensional form of surface stress is given by

$$\sigma^s = \tau^0 + E^s \varepsilon \tag{2.1}$$

where τ^0 is the residual surface tension, E^s is the surface elasticity which can be determined by atomistic simulations or the experimental measurements and ε is the longitudinal strain of the surface caused by an applied force to the nanobeams.

The assumption that the thickness of the surface layer *t* is much smaller than the beam thickness is applied. The effects of surface elasticity on the bending beam can be accounted by replacing the flexural rigidity (*EI*) for the bulk material with that of the effective flexural rigidity (*EI*)* and can be presented as

$$(EI)^* = \begin{cases} E\dfrac{bh^3}{12} + E^s\left(\dfrac{bh^2}{2} + \dfrac{h^3}{6}\right) & \text{(rectangular cross section)} \\ E\dfrac{\pi D^4}{64} + E^s\dfrac{\pi D^3}{8} & \text{(circular cross section)} \end{cases} \tag{2.2}$$

where *b* and *h* are the width and depth of rectangular cross section while *D* is the diameter of circular cross section as presented in Figure 2.1.

According to the generalized Young-Laplace equation, the uniform distributed transverse force q_n resulting from the surface stress along nanobeams in longitudinal direction and depending on the current surface curvature can be obtained as

$$q_n = H\kappa \tag{2.3}$$

where κ is the surface curvature approximated by d^2v/dx^2 for small deformation of nanobeams in which v is the transverse displacement of nanobeams. The parameter *H* is a constant determined by the residual surface tension depending on the shape of cross section. For a rectangular cross section with the width of *b* and a circular cross section with the diameter of *D*, the parameter *H* is given by

$$H = \begin{cases} 2\tau^0 b & \text{(rectangular cross section)} \\ 2\tau^0 D & \text{(circular cross section)} \end{cases} \tag{2.4}$$

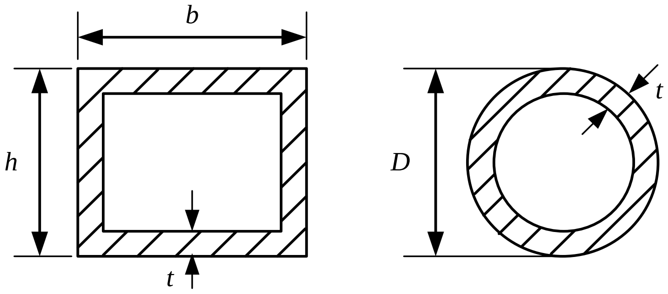


Figure 2.1 A rectangular and circular cross section of nanobeams with a surface layer

2.2 Surface Stress Including Nonlocal Elasticity Model for Nanobeams

According to the theory of nonlocal elasticity, stress at a reference point depends on the strain at the same point and also at all other points in the continuum. The nonlocal bending moment constitutive relations for one-dimensional of nanobeams including the effects of nonlocal elasticity (Eringen, 1983) and surface stress can be expressed as

$$M = -(EI)^* \frac{d^2 v}{dx^2} + \mu \frac{d^2 M}{dx^2} \tag{2.5}$$

where μ is the parameter of nonlocal scale revealing the effect of small-scale on the response of nanostructures while $(EI)^*$ is the effective flexural rigidity including surface stress mentioned above. Besides, Yang and Lim (2012) estimated the values of $\bar{\mu}$ as $\bar{\mu} \leq 0.04$ by matching the analytical nonlocal parameter of Timoshenko nanobeam model and molecular dynamic simulation solutions. The value of $\bar{\mu}$ is the dimensionless nonlocal parameter which $\bar{\mu} = \mu / L^2$ where L is the total length of nanobeams.

2.3 Governing Equation of Nanobeams

By considering the equilibrium of forces and moments on the nanobeams segments as shown in Figure 2.2, the equations of shear force Q and the bending moment M can be written as

$$\frac{\partial Q}{\partial x} = \rho A \frac{\partial^2 v}{\partial t^2} - H \frac{\partial^2 v}{\partial x^2} \tag{2.6}$$

$$\frac{\partial M}{\partial x} = Q \tag{2.7}$$

where ρ is the mass density and A is the cross section area of the nanobeams, respectively. By substituting of equation (2.6) into equation (2.7), the following equation can be obtained

$$\frac{\partial^2 M}{\partial x^2} = \rho A \frac{\partial^2 v}{\partial t^2} - H \frac{\partial^2 v}{\partial x^2} \tag{2.8}$$

Then, substituting equation (2.8) into equation (2.5), one obtains:

$$M = -(EI)^* \frac{\partial^2 v}{\partial x^2} + \mu \left\{ \rho A \frac{\partial^2 v}{\partial t^2} - H \frac{\partial^2 v}{\partial x^2} \right\} \tag{2.9}$$

By setting equation (2.8) equal to the second derivative of equation (2.9), the equation of motion for the nanobeams with consideration of both surface stress and nonlocal elasticity can be expressed as

$$\left[(EI)^* + \mu H\right] \frac{\partial^4 v}{\partial x^4} - H \frac{\partial^2 v}{\partial x^2} - \mu \rho A \frac{\partial^4 v}{\partial x^2 \partial t^2} + \rho A \frac{\partial^2 v}{\partial t^2} = 0 \tag{2.10}$$

In case of the surface stress and the nonlocal elasticity effects are completely neglected which E^s, τ^0, H and μ are all set to zero, equation (2.10) can be reduced to the classical Euler beam equation (Karnovsky, 1985).

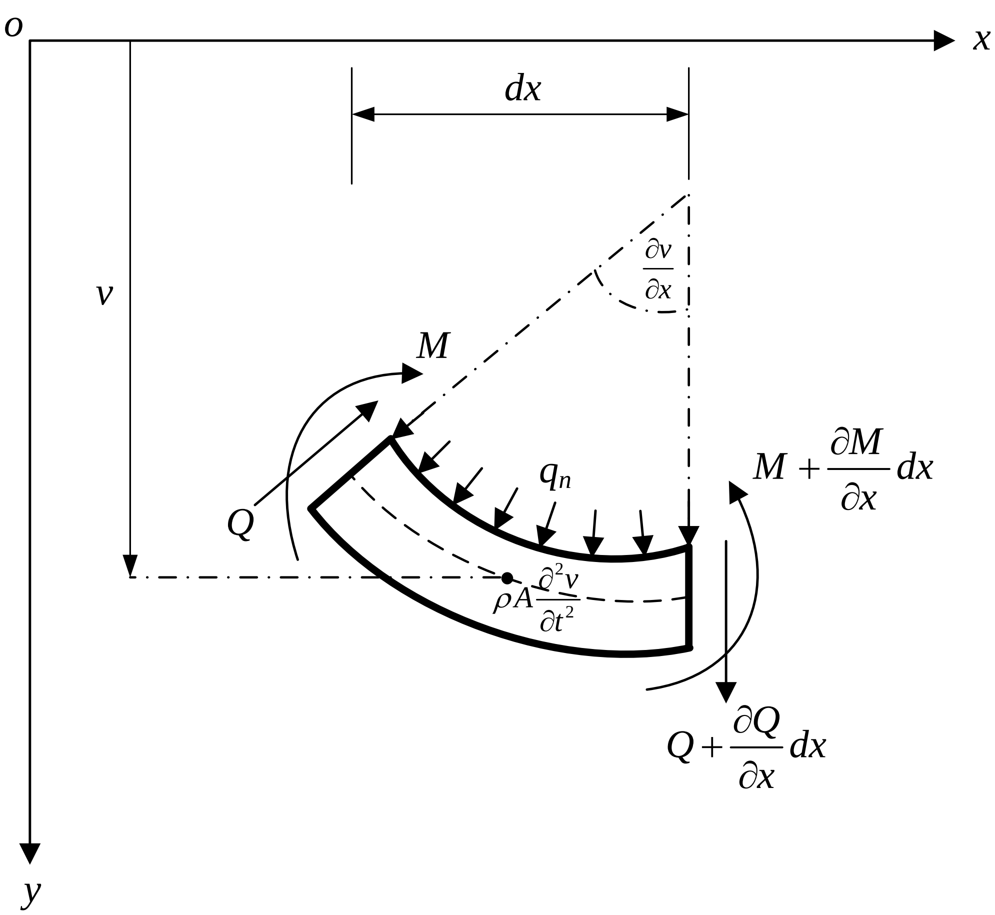


Figure 2.2 Free body of an infinitesimal segment of nanobeams

CHAPTER 3 SOLUTION METHODS

This chapter presents two methods to determine the natural frequencies of nanobeams with various boundary conditions including surface stress and nonlocal elasticity effects. For the first method, the governing equation of nanobeams is derived and solved analytically to obtain the natural frequencies and corresponding mode shapes of nanobeams. For the second method, Galerkin finite element technique is applied to obtain the modified mass and stiffness. The equation of motion is solved for the numerical results.

3.1 Analytical Solution

3.1.1 Free Vibration of Nanobeams

For determining the natural frequencies and corresponding mode shapes, the solution of partial differential equation of motion, equation (2.10), can be obtained by assuming the displacement of the nanobeams as

$$v(x,t) = V(x)e^{i\omega t}, \quad i = \sqrt{-1} \tag{3.1}$$

where ω is the eigenvalue and $V(x)$ is the eigenfuction, respectively. Substituting equation (2.10) into equation (3.1), the governing equation for transverse free vibration of nanobeams considering surface stress and nonlocal elasticity can be obtained as

$$\left[(EI)^* + \mu H)\right]\frac{d^4V}{dx^4} - \left(H - \mu\rho A\omega^2\right)\frac{d^2V}{dx^2} - \rho A\omega^2 V = 0 \tag{3.2}$$

The bending stiffness of nanobeams can be defined by using the parameter α where $\alpha = [(EI)^* + (\mu H)]$. For convenience, the following dimensionless parameters are introduced:

$$\xi = x/L, \; \bar{\mu} = \mu / L^2, \; \eta = \frac{(H - \mu\rho A\omega^2)L^2}{\alpha} \text{ and } \beta = \frac{\rho A\omega^2 L^4}{\alpha}. \tag{3.3a-3.3d}$$

Therefore, the governing equations for transverse free vibration of nanobeams can be simply expressed in dimensionless form as

$$\frac{d^4V}{d\xi^4} - \eta\frac{d^2V}{d\xi^2} - \beta V = 0 \tag{3.4}$$

Finally, the solution of equation (3.4) can be expressed as

$$V(\xi) = C_1 \sinh\lambda\xi + C_2\cosh\lambda\xi + C_3\sin\gamma\xi + C_4\cos\gamma\xi \tag{3.5}$$

where $\lambda^2 = \frac{1}{2}(\eta + \sqrt{\eta^2 + 4\beta^2})$, $\gamma^2 = \frac{1}{2}(-\eta + \sqrt{\eta^2 + 4\beta^2})$ and the unknown constants C_1, C_2, C_3 and C_4 for each case of the support condition can be determined below.

3.1.1.1 Pinned-Pinned nanobeams

The boundary conditions for pinned-pinned nanobeams are given as $V(0) = V(1) = 0$ and $M(0) = M(1) = 0$. By substituting the general solution of nanobeams, equation (3.5), into the boundary conditions, the characteristic equation can be obtained as

$$\sin(\gamma) = 0 \tag{3.6}$$

Then, the unknown constants for pinned-pinned nanobeams are given as

$$C_1 = 0, \tag{3.7a}$$

$$C_2 = 0, \tag{3.7b}$$

$$C_3 = 1, \text{ and} \tag{3.7c}$$

$$C_4 = 0. \tag{3.7d}$$

Next, the eigenvalue of natural frequencies is also obtained in form of

$$\omega^2 = \frac{\alpha(n\pi / L)^4 + H(n\pi / L)^2}{\rho A(1 + \bar{\mu}(n\pi)^2)}, \quad n = 1, 2, 3, \ldots \tag{3.8}$$

Finally, the corresponding modal shape function is

$$V(\xi) = \sin(\gamma\xi) \tag{3.9}$$

3.1.1.2 Clamped-Clamped nanobeams

The boundary conditions of clamped-clamped nanobeams are given as $V(0) = V'(0) = 0$ and $V(1) = V'(1) = 0$ which are the zero displacement and rotation at beam's support. Then, the characteristic equation for determining the natural frequencies can be obtained as

$$\beta + \eta \sinh\lambda \sin\gamma - \beta \cosh\lambda \cos\gamma = 0 \tag{3.10}$$

Then, the unknown constants for clamped-clamped nanobeams are given as

$$C_1 = 1, \tag{3.11a}$$

$$C_2 = \frac{\lambda \sin\gamma - \gamma \sinh\lambda}{\gamma(\cosh\lambda - \cos\gamma)}, \tag{3.11b}$$

$$C_3 = -\frac{\lambda}{\gamma}, \text{ and} \tag{3.11c}$$

$$C_4 = -\frac{\lambda \sin\gamma - \gamma \sinh\lambda}{\gamma(\cosh\lambda - \cos\gamma)}. \tag{3.11d}$$

Finally, the modal shape function of clamped-clamped nanobeams is

$$V(\xi) = \sinh(\lambda\xi) + \frac{\lambda\sin\gamma - \gamma\sinh\lambda}{\gamma(\cosh\lambda - \cos\gamma)}\cosh(\lambda\xi) - \frac{\lambda}{\gamma}\sin(\gamma\xi) - \frac{\lambda\sin\gamma - \gamma\sinh\lambda}{\gamma(\cosh\lambda - \cos\gamma)}\cos(\gamma\xi) \tag{3.12}$$

3.1.1.3 Clamped-Free nanobeams

The boundary conditions for clamped-free nanobeams are given as $\bar{v}(0) = \bar{v}'(0) = 0$ and $M(1) = Q(1) = 0$ in which they are corresponding to the displacement and rotation at clamp end to be zero while the moment and shear at free end to be vanished, respectively. Next, the characteristic equation for determining the natural frequencies is obtained as follow:

$$\beta^2 + \beta\eta\sinh\lambda\sin\gamma + (2\eta^2 + \beta^2)\cosh\lambda\cos\gamma = 0 \tag{3.13}$$

Then, the unknown constants for clamped-free nanobeams can be expressed as

$$C_1 = 1, \tag{3.14a}$$

$$C_2 = -\frac{\lambda^2\sinh\lambda + \lambda\gamma\sin\gamma}{\lambda^2\cosh\lambda + \gamma^2\cos\gamma}, \tag{3.14b}$$

$$C_3 = -\frac{\lambda}{\gamma}, \text{ and} \tag{3.14c}$$

$$C_4 = \frac{\lambda^2\sinh\lambda + \lambda\gamma\sin\gamma}{\lambda^2\cosh\lambda + \gamma^2\cos\gamma}. \tag{3.14d}$$

Finally, the modal shape function of clamped-free nanobeams is given as follow:

$$V(\xi) = \sinh(\lambda\xi) - \frac{\lambda^2\sinh\lambda + \lambda\gamma\sin\gamma}{\lambda^2\cosh\lambda + \gamma^2\cos\gamma}\cosh(\lambda\xi) - \frac{\lambda}{\gamma}\sin(\gamma\xi) + \frac{\lambda^2\sinh\lambda + \lambda\gamma\sin\gamma}{\lambda^2\cosh\lambda + \gamma^2\cos\gamma}\cos(\gamma\xi) \tag{3.15}$$

3.1.1.4 Clamped-Pinned nanobeams

The boundary conditions for clamp-pinned nanobeams are given as $V(0) = V'(0) = V(1) = 0$ and $M(1) = 0$. Next, the characteristic equation for determining the natural frequencies is obtained as

$$\lambda\cosh\lambda\sin\gamma - \gamma\sinh\lambda\cos\gamma = 0 \tag{3.16}$$

Then, the unknown constants for clamped-pinned nanobeams are obtained as

$$C_1 = 1, \tag{3.17a}$$

$$C_2 = -\tanh\lambda, \tag{3.17b}$$

$$C_3 = \frac{\lambda}{\gamma}, \text{ and} \tag{3.17c}$$

$$C_4 = \frac{\lambda}{\gamma}\tan\gamma \; . \tag{3.17d}$$

Finally, the modal shape function of clamped-free nanobeams is given as follow:

$$V(\xi) = \sinh(\lambda\xi) - \tanh\lambda\cosh(\lambda\xi) - \frac{\lambda}{\gamma}\sin(\gamma\xi) + \frac{\lambda}{\gamma}\tanh\lambda\cos(\gamma\xi) \tag{3.18}$$

3.1.1.5 Clamped-Sliding nanobeams

The boundary conditions for clamp-sliding nanobeams are given as $V(0) = V'(0) = 0$ and $V'(1) = Q(1) = 0$. Next, the characteristic equation for determining the natural frequencies is obtained as

$$\gamma\cosh\lambda\sin\gamma + \lambda\sinh\lambda\cos\gamma = 0 \tag{3.19}$$

Then, the unknown constants for clamped-sliding nanobeams are obtained as

$$C_1 = 1 \, , \tag{3.20a}$$

$$C_2 = -\frac{\lambda(\cosh\lambda - \cos\gamma)}{\lambda\sinh\lambda + \gamma\sin\gamma} \, , \tag{3.20b}$$

$$C_3 = -\frac{\lambda}{\gamma} \, , \text{ and} \tag{3.20c}$$

$$C_4 = \frac{\lambda(\cosh\lambda - \cos\gamma)}{\lambda\sinh\lambda + \gamma\sin\gamma} \; . \tag{3.20d}$$

Finally, the modal shape function of clamped-free nanobeams can be expressed as

$$V(\xi) = \sinh(\lambda\xi) - \frac{\lambda(\cosh\lambda - \cos\gamma)}{\lambda\sinh\lambda + \gamma\sin\gamma}\cosh(\lambda\xi) - \frac{\lambda}{\gamma}\sin(\gamma\xi) + \frac{\lambda(\cosh\lambda - \cos\gamma)}{\lambda\sinh\lambda + \gamma\sin\gamma}\cos(\gamma\xi) \tag{3.21}$$

3.1.1.6 Sliding-Pinned nanobeams

The boundary conditions for sliding-pinned nanobeams are given as $V'(0) = Q(0) = 0$ and $V(1) = M(1) = 0$. Next, the characteristic equation for determining the natural frequencies is obtain as

$$\cos\gamma = 0 \tag{3.22}$$

Then, the unknown constants for sliding-pinned nanobeams are obtained as

$$C_1 = 0 \, , \tag{3.23a}$$

$$C_2 = 0 \, , \tag{3.23b}$$

$$C_3 = 0 \, , \text{ and} \tag{3.23c}$$

$$C_4 = 1. \tag{3.23d}$$

Finally, the modal shape function of sliding-pinned nanobeams is

$$V(\xi) = \cos(\gamma\xi) \tag{3.24}$$

To check the validity of natural frequencies and vibration mode shapes of nanobeams, the Galerkin finite element method is used to verify these results. A brief detail of the method is given in the section 3.2.

3.2 Finite Element Solution

In By applying the Galerkin's weighted residual method to equation (3.2), the weighted residual equation can be written as

$$\int_0^L \left\{ \left[(EI)^* - \mu(P-H)\right]\frac{d^4v}{dx^4} + (P-H)\frac{d^2v}{dx^2} + \left(\mu\frac{d^2}{dx^2} - 1\right)q(x)\right\}\bar{v}(x)dx = 0 \tag{3.25}$$

where L is the total length of nanobeams and $\bar{v}(x)$ is the weight functions. Integrating by parts of equation (3.25) twice, one yields the following equation:

$$\int_0^L \left\{ \left[(EI)^* + \mu H\right]\frac{d^2V}{dx^2}\frac{d^2\bar{v}}{dx^2} + (H - \mu\rho A\omega^2)\frac{dV}{dx}\frac{d\bar{v}}{dx} - \rho A\omega^2 V\bar{v}\right\}dx + \left(M\frac{d\bar{v}}{dx} - Q\bar{v}\right)\Bigg|_0^L = 0$$

(3.26)

where $M = -\left[(EI)^* + \mu H\right]\frac{d^2V}{dx^2} - \mu\rho A\omega^2 V$ and $Q = -\left[(EI)^* + \mu H\right]\frac{d^3V}{dx^3} - \mu\rho A\omega^2\frac{dV}{dx}$ $+ H\frac{dV}{dx}$ are the bending moment and shear force terms, respectively.

For two-node finite element with two nodal degrees of freedom per node, the transverse displacement is interpolated in terms of shape functions and degrees of freedom as

$$\bar{v}(x) = [\mathbf{N}]\{\mathbf{d}\} \tag{3.27}$$

where $\{\mathbf{d}\} = \begin{bmatrix} V_1 & \frac{dV_1}{dx} & V_2 & \frac{dV_2}{dx} \end{bmatrix}^T$ is the nodal degree of freedom and $\mathbf{N}_i(x)$ is the cubic polynomial shape functions which can be given as

$$\mathrm{N}_1 = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}, \tag{3.28a}$$

$$\mathrm{N}_2 = x - \frac{2x^2}{l} + \frac{x^3}{l^2}, \tag{3.28b}$$

$$\mathrm{N}_3 = \frac{3x^2}{l^2} - \frac{2x^3}{l^3}\text{, and} \tag{3.28c}$$

$$\mathrm{N}_4 = \frac{x^3}{l^2} - \frac{x^2}{l}\,. \tag{3.28d}$$

where l is the length of nanobeams-element. Substitution of equation (3.27) into equation (3.26), one yields

$$\int_0^l \left\{ \begin{aligned} &\left[(EI)^* + \mu H\right]\left[\mathbf{N}''\right]^T\left[\mathbf{N}''\right] + H\left[\mathbf{N}'\right]^T\left[\mathbf{N}'\right] - \omega^2(\rho A\left[\mathbf{N}\right]^T\left[\mathbf{N}\right] \\ &+ \mu\rho A\left[\mathbf{N}'\right]^T\left[\mathbf{N}'\right]) \end{aligned} \right\}\{\mathbf{d}\}dx = 0 \tag{3.29}$$

The element stiffness matrix is then obtained as

$$[\mathbf{k_e}] = \int_0^l \left\{\left[(EI)^* + \mu H\right]\left[\mathbf{N}''\right]^T\left[\mathbf{N}''\right] + H\left[\mathbf{N}'\right]^T\left[\mathbf{N}'\right]\right\}dx\text{, or} \tag{3.30a}$$

$$[\mathbf{k_e}] = \frac{\left[(EI)^* + \mu H\right]}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} + \frac{H}{30l}\begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix} \tag{3.30b}$$

It can be seen from equation (3.30b) that the modified stiffness presents classical stiffness matrix including the both effects of surface stress and nonlocal elasticity. Then, the element mass matrix is given by

$$[\mathbf{m_e}] = \int_0^l \left\{\rho A\left[\mathbf{N}\right]^T\left[\mathbf{N}\right] + \mu\rho A\left[\mathbf{N}'\right]^T\left[\mathbf{N}'\right]\right\}dx\text{, or} \tag{3.31a}$$

$$[\mathbf{m_e}] = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} + \frac{\mu\rho A}{30l}\begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix} \tag{3.31b}$$

Finally, the element stiffness matrices, mass matrices, and nodal degree of freedom are assembled to obtain the global equilibrium equation of motion as

$$\left([\mathbf{K}] - \omega^2[\mathbf{M}]\right)\{\mathbf{D}\} = \{\mathbf{0}\} \tag{3.32}$$

where $[\mathbf{K}] = \sum_{i=1}^{N_{\text{els}}}[\mathbf{k_e}]_i$, $[\mathbf{M}] = \sum_{i=1}^{N_{\text{els}}}[\mathbf{m_e}]_i$, and $\{\mathbf{D}\} = \sum_{i=1}^{N_{\text{els}}}\{\mathbf{d}\}_i$ are the global structural stiffness matrix, global consistent mass matrix, global nodal displacement, respectively. N_{els} is the number of elements in the structure. Then, equation (3.32) has the form of the algebraic eigenvalue problem.

For nontrivial solution, the determinant of the coefficient matrix is equal to zero.

$$\left|[\mathbf{K}]-\omega^2[\mathbf{M}]\right| = 0 \tag{3.33}$$

By expansion of equation (3.33), one yields a polynomial of order n of characteristic equation. The n roots of ω_i^2 are the eigenvalue which it is ordered from lowest to highest mode of vibration for each boundary condition.

$$0 \le \omega_1^2 \le \omega_2^2 \le \ldots \le \omega_i^2 \le \ldots \omega_n^2 \tag{3.34}$$

CHAPTER 4 RESULTS AND DISCUSSIONS

The model formulation presented in chapter 2 can be applied to investigate the behavior of nanobeams with surface stress and nonlocal elasticity effects. The natural frequencies of nanobeams obtained from analytical solution have been validated by finite element method as shown in section 4.1. In section 4.2, the effects of surface stress and nonlocal elasticity on the natural frequency under various boundary conditions are considered in this analysis. Moreover, the variations of mode shapes are also investigated the behavior of nanobeams as shown in section 4.3.

4.1 Results Verification

In this section, the results of natural frequencies and corresponding mode shapes of nanobeams with various boundary conditions are presented using the nanobeams with the span length (*L*) of 1,000 nm and the diameter (*D*) of 50 nm as the case study. The material properties as used in the work of He and Lilley (2008) are performed. These material properties are E = 76 GPa, E^s = 1.22 N/m, τ^0 = 0.89 N/m, and ρ = 10.5×10^3 kg/m^3, respectively. In case of nonlocal parameter, the authors use the parameter presented in the work of Yang and Lim (2012) which $\overline{\mu}$ = 0.04.

The analytical results for natural frequencies of nanobeams are verified with that of finite element analysis results by using 20 elements discretized along the beam's span length. As presented in Tables 4.1-4.2, the analytical and numerical results for all boundary conditions of nanobeams are identical for the nanobeams with classical Euler beam theory, nanobeams with nonlocal elasticity, nanobeams with surface stress, and nanobeams with combined effects of nonlocal elasticity and surface stress. Moreover, the obtained results also demonstrate high accuracy incomparison with those of previous analysis results (He and Lilley, (2008) and Lu et al., (2006)).

Table 4.1. Analytical and numerical results of eigenvalue, $\omega(\times 10^{9})$ of pinned-pinned (P-P), clamped-clamped (C-C), and clamped-free (C-F) nanobeams

Boundary condition		Classical Euler beam		Surface stress effect		Nonlocal elasticity effect		Surface and Nonlocal effects	
		LT[a] (Karnovsky, 2004)	FEM[b]	LT[a] (He and Lilley, 2008)	FEM[b]	LT[a] (Lu et al., 2006)	FEM[b]	LT[a]	FEM[b]
P-P	Mode 1	0.3319	0.3319	0.3912	0.3912	0.2810	0.2810	0.3489	0.3489
	Mode 2	1.3276	1.3276	1.3919	1.3919	0.8267	0.8267	0.9249	0.9249
	Mode 3	2.9872	2.9873	3.0545	3.0545	1.3999	1.3999	1.5324	1.5325
	Mode 4	5.3106	5.3111	5.3811	5.3817	1.9633	1.9635	2.1322	2.1324
C-C	Mode 1	0.7524	0.7524	0.7877	0.7877	0.6151	0.6151	0.6878	0.6878
	Mode 2	2.0740	2.0741	2.1240	2.1240	1.2249	1.2249	1.3431	1.3431
	Mode 3	4.0660	4.0662	4.1233	4.1235	1.8336	1.8338	1.9922	1.9923
	Mode 4	6.7212	6.7223	6.7846	6.7858	2.4083	2.4088	2.6065	2.6070
C-F	Mode 1	0.1182	0.1182	0.1801	0.1801	0.1085	0.1085	0.1728	0.1728
	Mode 2	0.7410	0.7410	0.8301	0.8301	0.4902	0.4902	0.5806	0.5806
	Mode 3	2.0748	2.0749	2.1563	2.1563	1.0553	1.0553	1.1671	1.1671
	Mode 4	4.0659	4.0660	4.1462	4.1464	1.6234	1.6235	1.7698	1.7698

LT[a] = Linear Beam Theory (Analytical solution)
FEM[b] = Finite Element Method

Table 4.2. Analytical and numerical results of eigenvalue, $\omega(\times 10^9)$ of clamped-pinned (C-P), clamped-sliding (C-S), and sliding-pinned (S-P) nanobeames

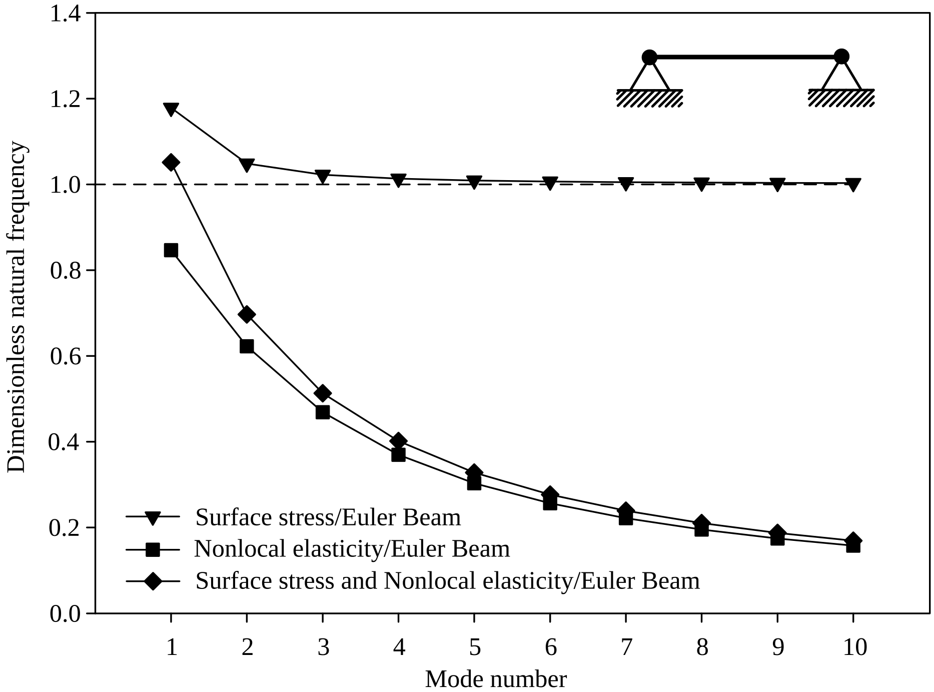
Boundary condition		Classical Euler beam		Surface stress effect		Nonlocal elasticity effect		Nonlocal and Surface effects	
		LT[a] (Karnovsky, 2004)	FEM[b]	LT[a]	FEM[b]	LT[a]	FEM[b]	LT[a]	FEM[b]
C-P	Mode 1	0.5185	0.5185	0.5649	0.5649	0.4286	0.4286	0.4965	0.4965
	Mode 2	1.6803	1.6803	1.7366	1.7366	1.0202	1.0202	1.1271	1.1271
	Mode 3	3.5058	3.5059	3.5677	3.5678	1.6105	1.6101	1.7554	1.7554
	Mode 4	5.9951	5.9960	6.0618	6.0626	2.1832	2.1836	2.3663	2.3667
C-S	Mode 1	0.1881	0.1881	0.2205	0.2205	0.1775	0.1775	0.2195	0.2195
	Mode 2	1.0165	1.0165	1.0689	1.0689	0.7206	0.7206	0.8084	0.8084
	Mode 3	2.5101	2.5101	2.5693	2.5694	1.3178	1.3178	1.4436	1.4436
	Mode 4	4.6675	4.6679	4.7318	4.7321	1.8989	1.8991	2.0628	2.0629
S-P	Mode 1	0.0083	0.0083	0.1325	0.1325	0.0079	0.0079	0.1301	0.1301
	Mode 2	0.7468	0.7468	0.8093	0.8093	0.5434	0.5435	0.6261	0.6261
	Mode 3	2.0744	2.0745	2.1402	2.1403	1.1140	1.1140	1.2290	1.2290
	Mode 4	4.0659	4.0662	4.1347	4.1349	1.6830	1.6831	1.8335	1.8336

LT[a] = Linear Beam Theory (Analytical solution)
FEM[b] = Finite Element Method

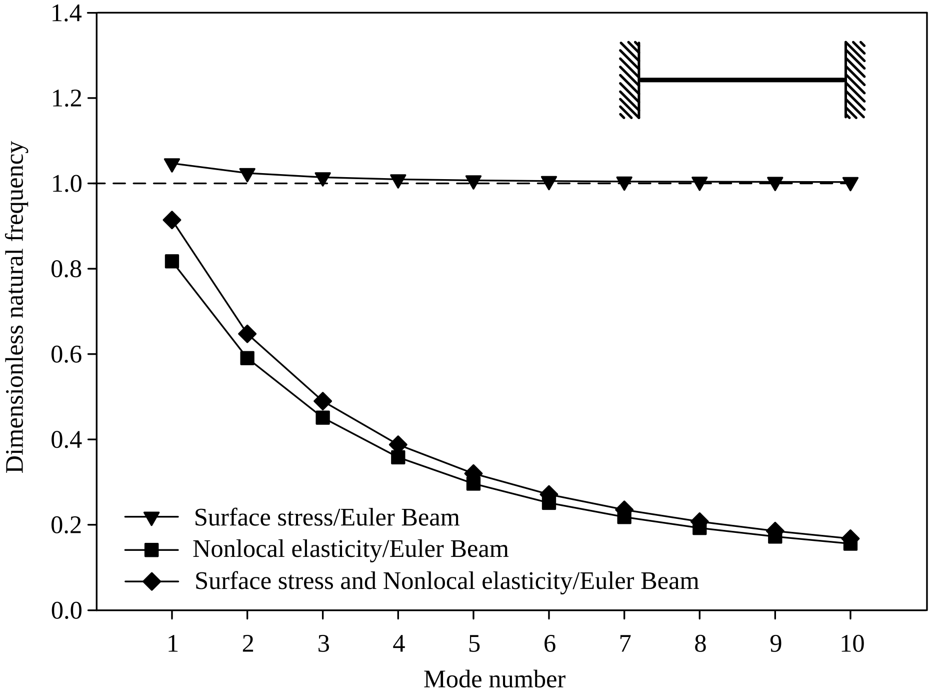
4.2 Natural Frequency of Nanobeams

In this section, the variation of dimensionless natural frequencies of nanobeams under various boundary conditions are illustrated as presented in Figure 4.1. The comparison between the nanobeams with surface stress effect and Euler beam, the nanobeams with nonlocal elasticity effect and Euler beam, and the nanobeams with combined effects of surface stress and nonlocal elasticity and Euler beam are also shown.

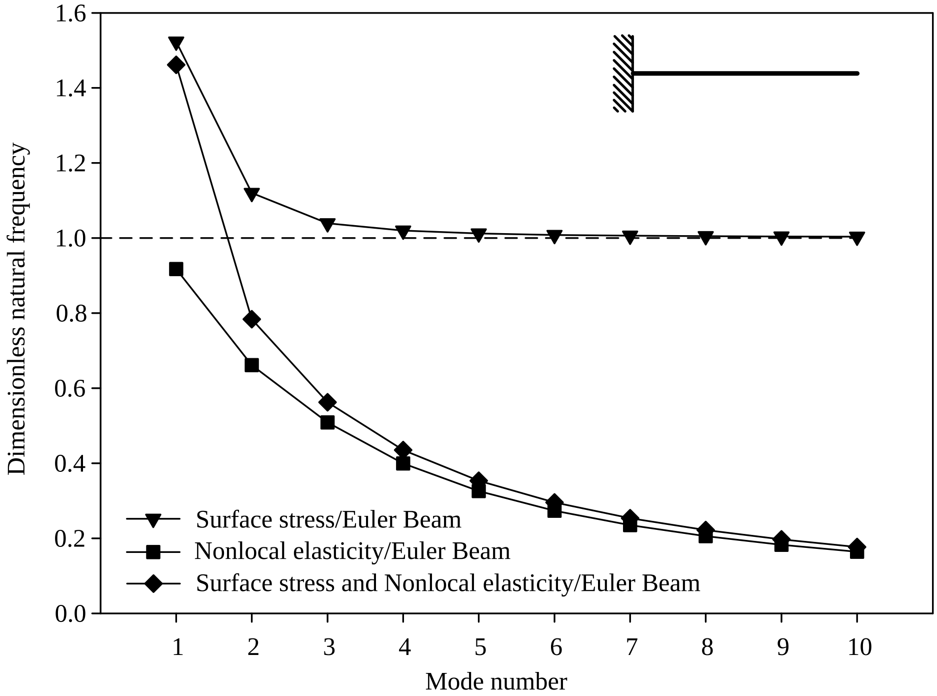
According to Figure 4.1, it can be observed that the surface stress results the increment of natural frequencies but its influence is reduced and then has no effect when the mode number is higher than the fourth mode. The decrement of natural frequencies of nanobeams with accounting only surface stress obtained in this work has a similar trend in comparison with the work of Wang and Feng (2007). For the nanobeams including nonlocal elasticity, there can be seen that the natural frequencies of nanobeams for all boundary conditions are decreased significantly for all modes number. These obtained results give the similar behaviors as presented in the previous work (Lu et al., 2006). It is also observed that the effect of the nonlocal elasticity increases for the higher modes of vibration. When the surface stress is combined with that of nonlocal elasticity, the obtained natural frequencies are in between the one of nanobeams with surface stress and the one with nonlocal elasticity. For the higher modes of vibration, the natural frequencies are converted to the nanobeams with nonlocal elasticity. The reasons are based on the fact that the influence of surface stress is negligible small for the modes number which is higher than the fourth mode.


1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
Dimensionless natural frequency
1
2
3
4
5
6
7
8
9
10
Mode number
Surface stress/Euler Beam
Nonlocal elasticity/Euler Beam
Surface stress and Nonlocal elasticity/Euler Beam

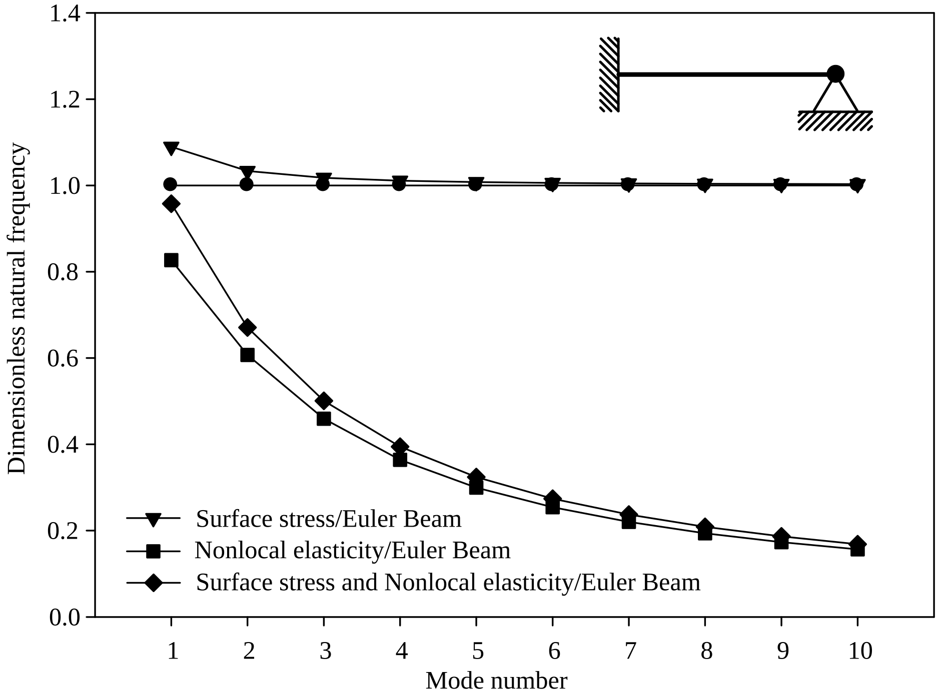
(a)


1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
Dimensionless natural frequency
1
2
3
4
5
6
7
8
9
10
Mode number
Surface stress/Euler Beam
Nonlocal elasticity/Euler Beam
Surface stress and Nonlocal elasticity/Euler Beam

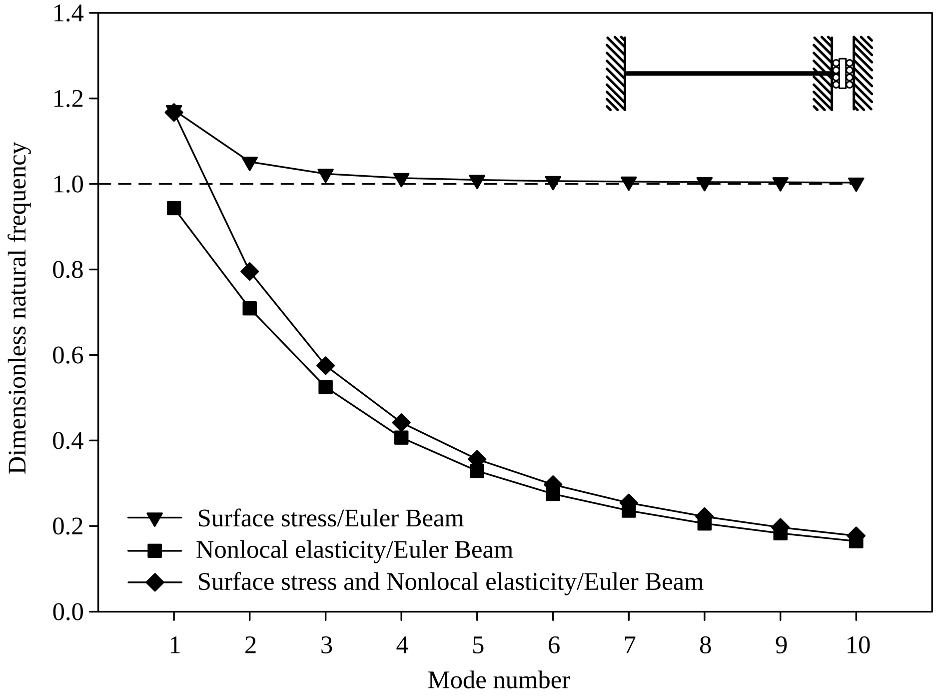
(b)


Dimensionless natural frequency
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
Surface stress/Euler Beam
Nonlocal elasticity/Euler Beam
Surface stress and Nonlocal elasticity/Euler Beam
1
2
3
4
5
6
7
8
9
10
Mode number

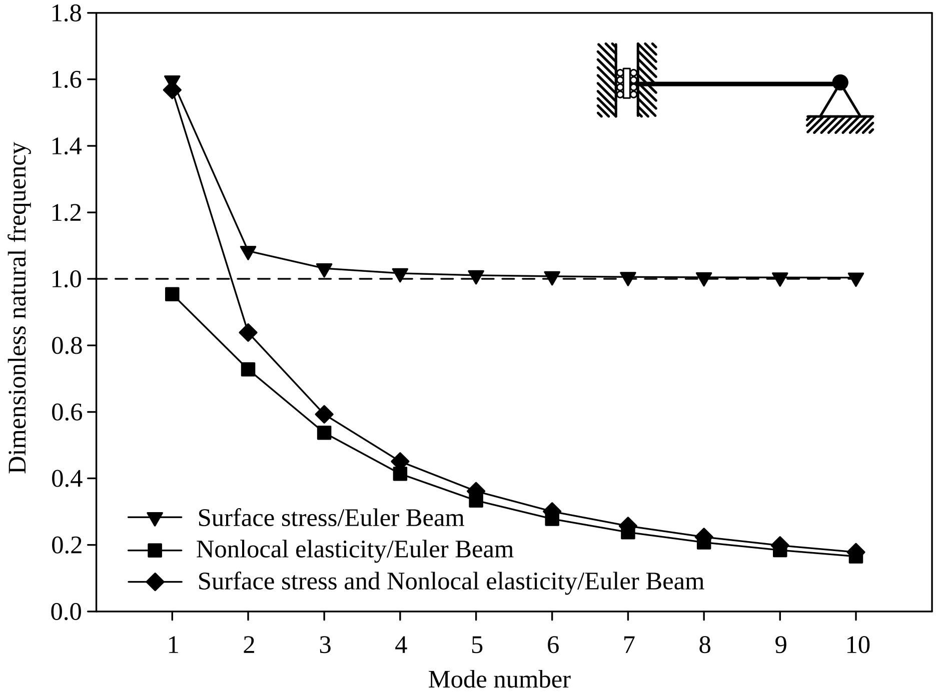
(c)


Dimensionless natural frequency
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
Surface stress/Euler Beam
Nonlocal elasticity/Euler Beam
Surface stress and Nonlocal elasticity/Euler Beam
1
2
3
4
5
6
7
8
9
10
Mode number

(d)



(e)



(f)

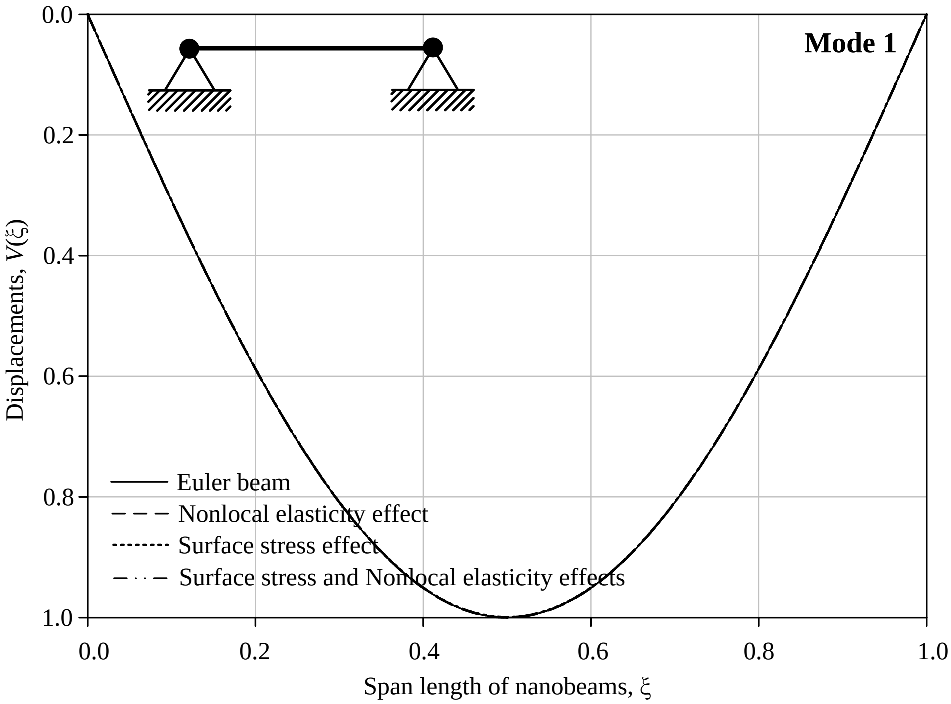
Figure 4.1 Variation of dimensionless natural frequency with surface stress and nonlocal elasticity effects for different boundary condition: (a) pinned-pinned, (b) clamped-clamped, (c) clamped-free, (d) clamped-pinned, (e) clamped-sliding, and (f) sliding-pinned.

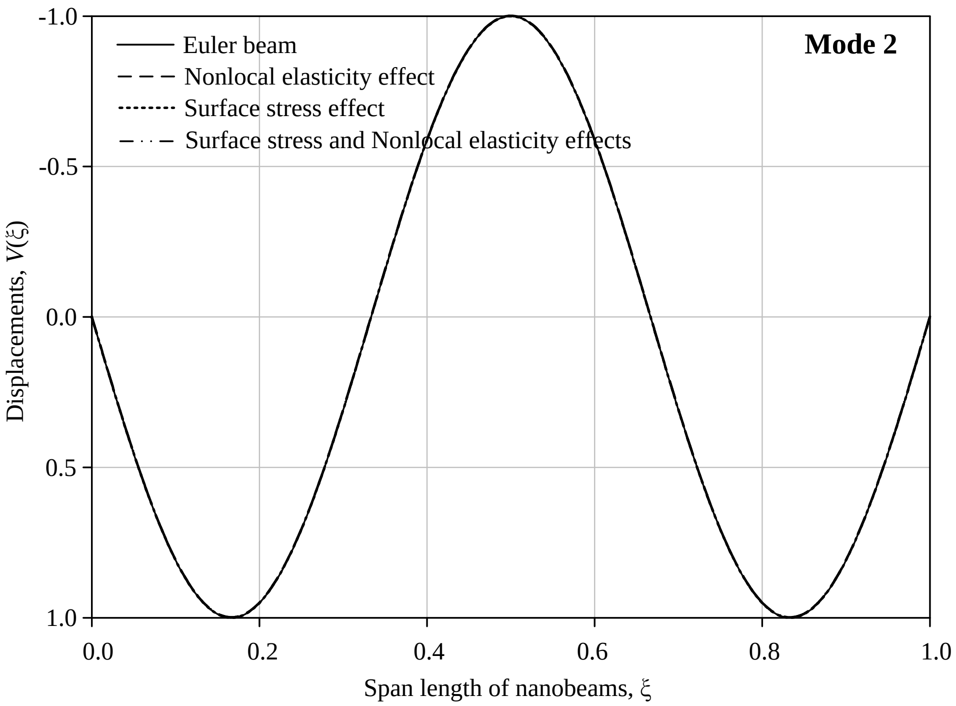
4.3 Mode shapes of Nanobeams

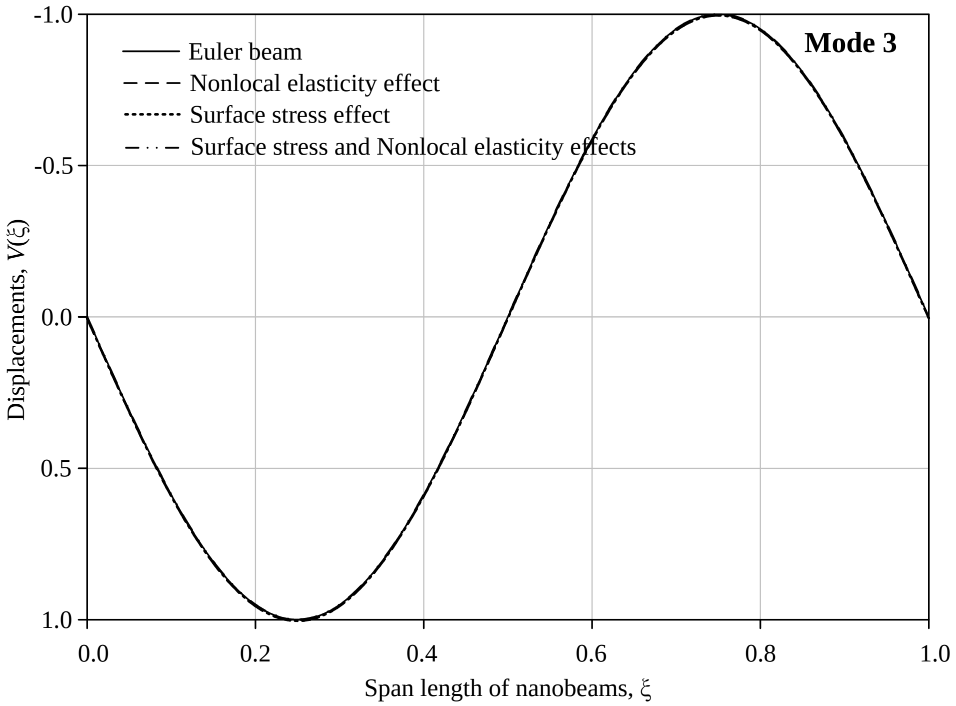
In this section, the variations of mode shapes of nanobeams for all boundary conditions are also presented. The variations of mode shapes of pinned-pinned nanobeams are shown in Figure 4.2. It is interesting to note that the nonlocal elasticity and surface stress have no effect on the vibration mode shapes since the modal shape function, equation (3.9), of pinned-pinned nanobeams is independent from both effects. In other words, the nonlocal elasticity and surface stress effects affect only on natural frequencies in case of pinned-pinned nanobeams.

In Figures 4.3-4.6, the variations of mode shapes of clamped-clamped, clamped-free, clamped-pinned and clamped-sliding nanobeams are presented. It can be seen that surface stress and nonlocal elasticity terms affects directly on the vibration mode shapes of nanobeams. The surface stress increases the vibration amplitudes. The influence of this effect is also described by considering the modified stiffness of equation (3.30), which the vibration amplitude basically increases when the stiffness matrix increases. Consequently, the nonlocal elasticity decreases the vibration amplitudes of nanobeams. This is due to the effect of nonlocal elasticity term in the mass matrix of equation (3.31). When both effects are combined, the similar results of vibration mode shapes with that of the nanobeams including only the nonlocal elasticity are obtained. This gives a practical implication that the surface stress has no significantly effect on the vibration amplitude.

The mode shapes of sliding-pinned nanobeams is presented in Figure 4.7 and also found that the results of nanobeams including both effects of surface stress and nonlocal elasticity is not different from the Euler beam. In the other words, the sliding-pinned nanobeams give similar vibration behavior as pinned-pinned nanobeams. The boundary condition can be satisfied by the following vibration of $\cos\gamma = 0$ or $\gamma = (2n-1)\pi/2$ and the mode number is independent on both nonlocal elasticity and surface stress effects.


Mode 1
Euler beam
Nonlocal elasticity effect
Surface stress effect
Surface stress and Nonlocal elasticity effects
Displacements, V(ξ)
Span length of nanobeams, ξ
0.0
0.2
0.4
0.6
0.8
1.0


Mode 2
Euler beam
Nonlocal elasticity effect
Surface stress effect
Surface stress and Nonlocal elasticity effects
Displacements, V(ξ)
Span length of nanobeams, ξ
-1.0
-0.5
0.0
0.5
1.0
0.2
0.4
0.6
0.8



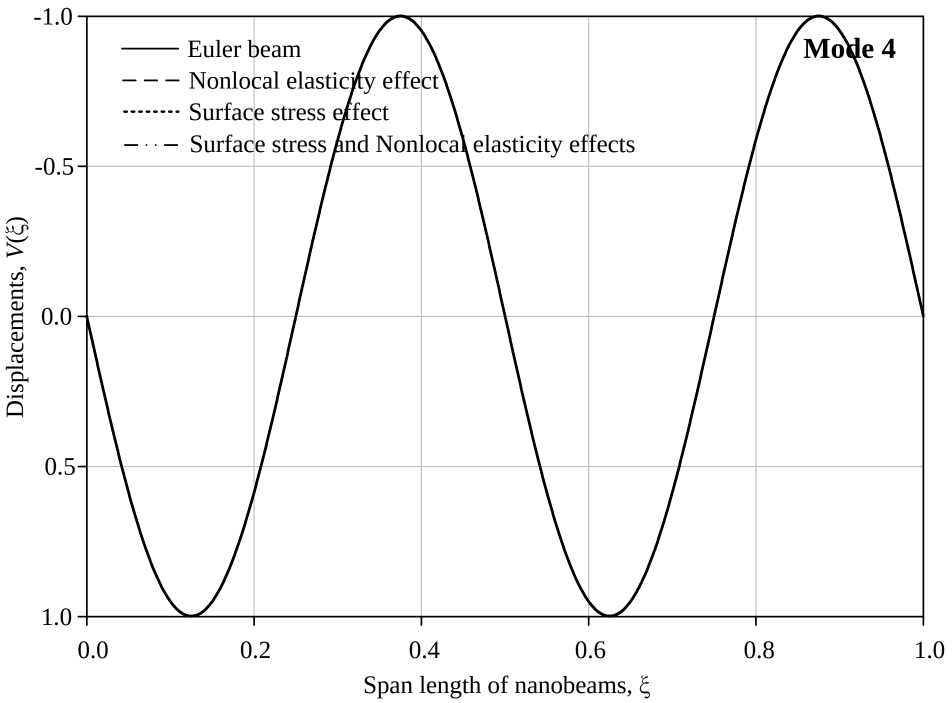
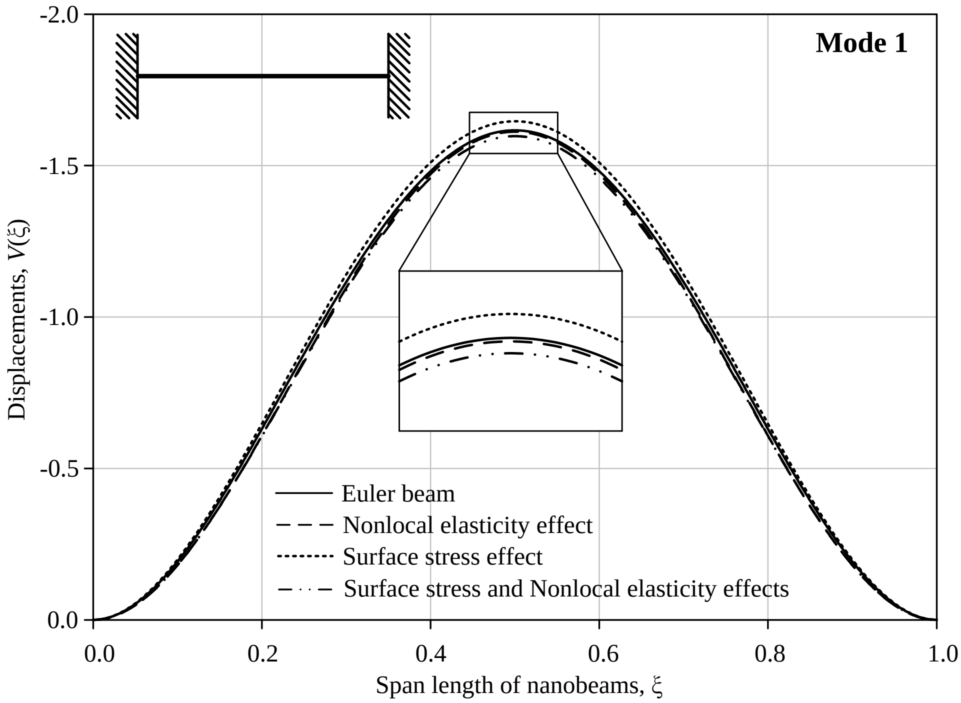
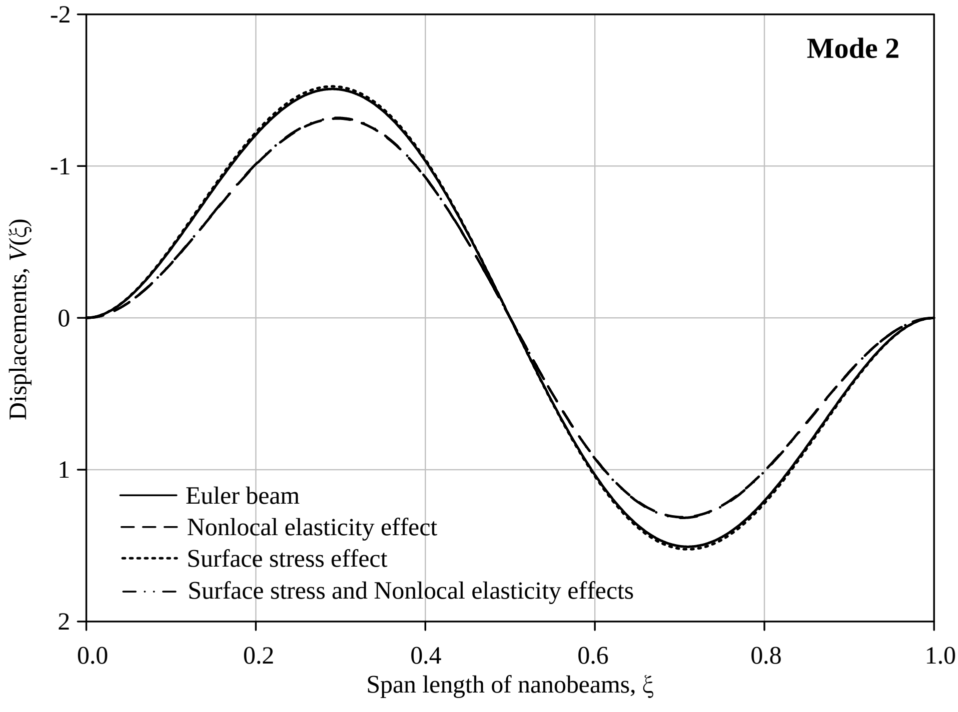


Figure 4.2 Variations of mode shapes of pinned-pinned nanobeams.


Mode 1
-2.0
-1.5
-1.0
-0.5
0.0
Displacements, V(ξ)
Euler beam
Nonlocal elasticity effect
Surface stress effect
Surface stress and Nonlocal elasticity effects
0.0
0.2
0.4
0.6
0.8
1.0
Span length of nanobeams, ξ


Mode 2
-2
-1
0
1
2
Displacements, V(ξ)
Euler beam
Nonlocal elasticity effect
Surface stress effect
Surface stress and Nonlocal elasticity effects
0.0
0.2
0.4
0.6
0.8
1.0
Span length of nanobeams, ξ

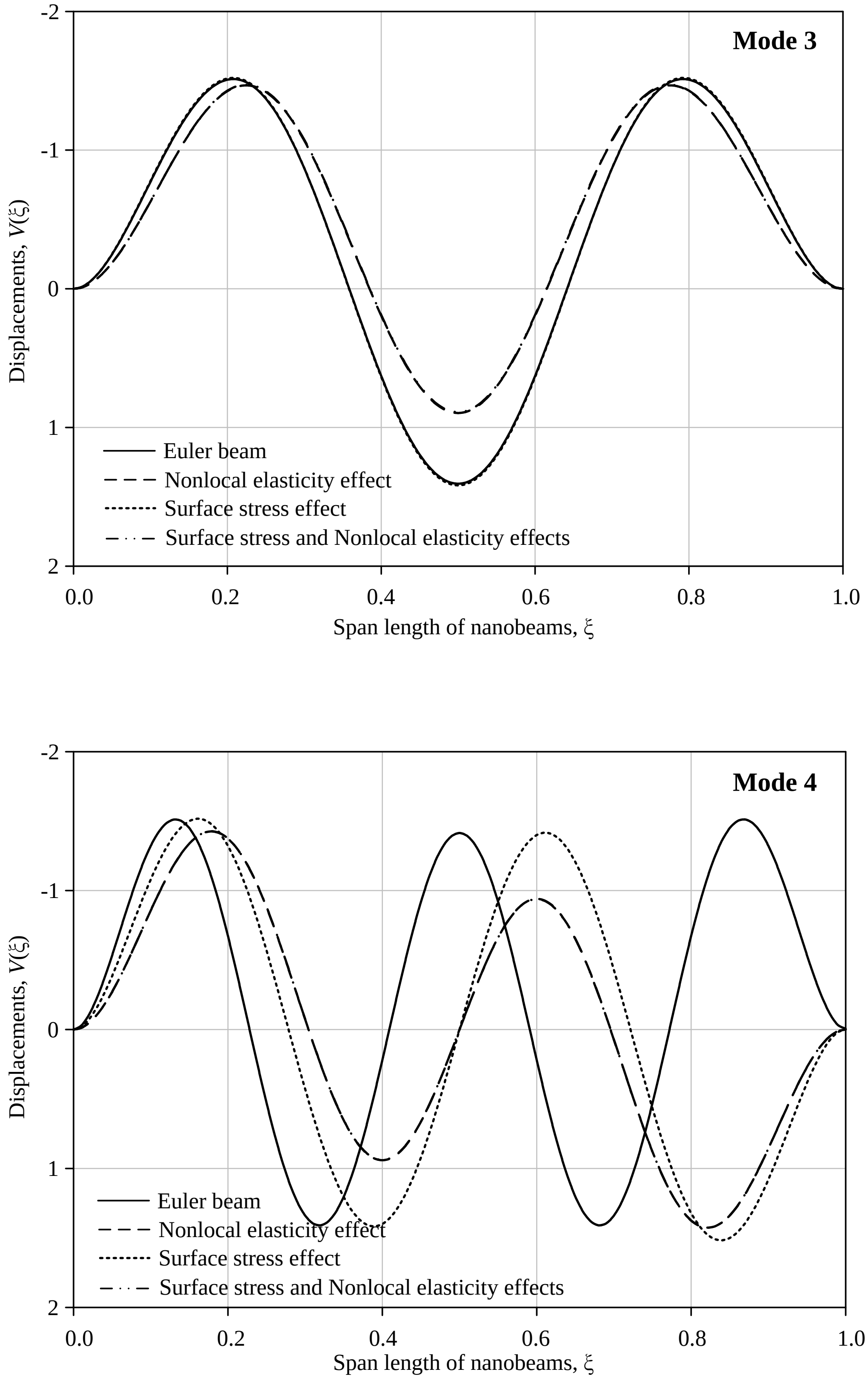
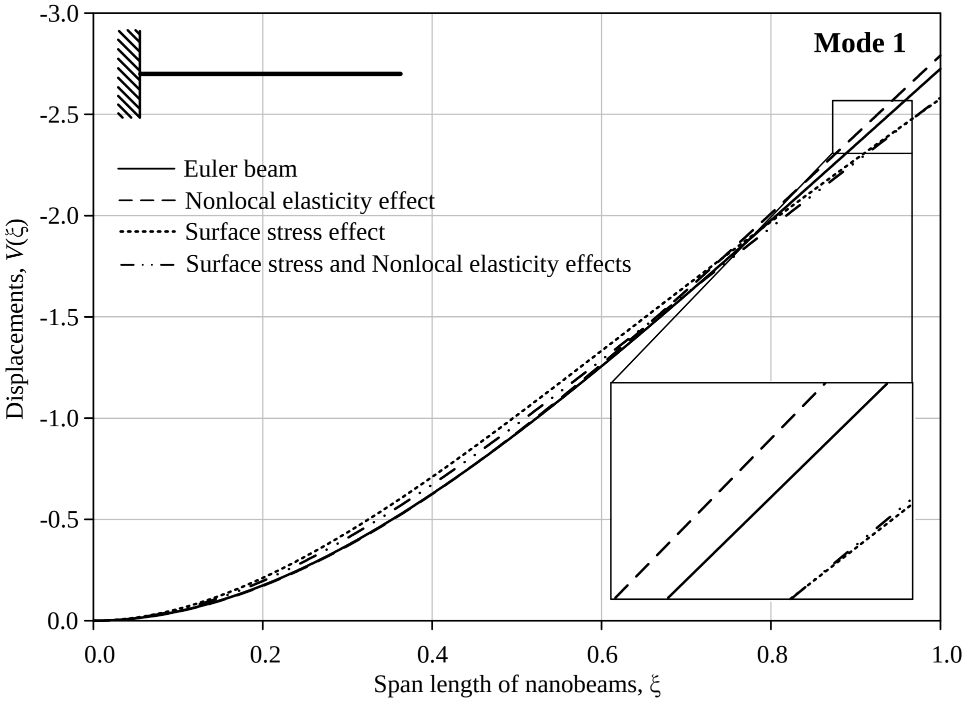
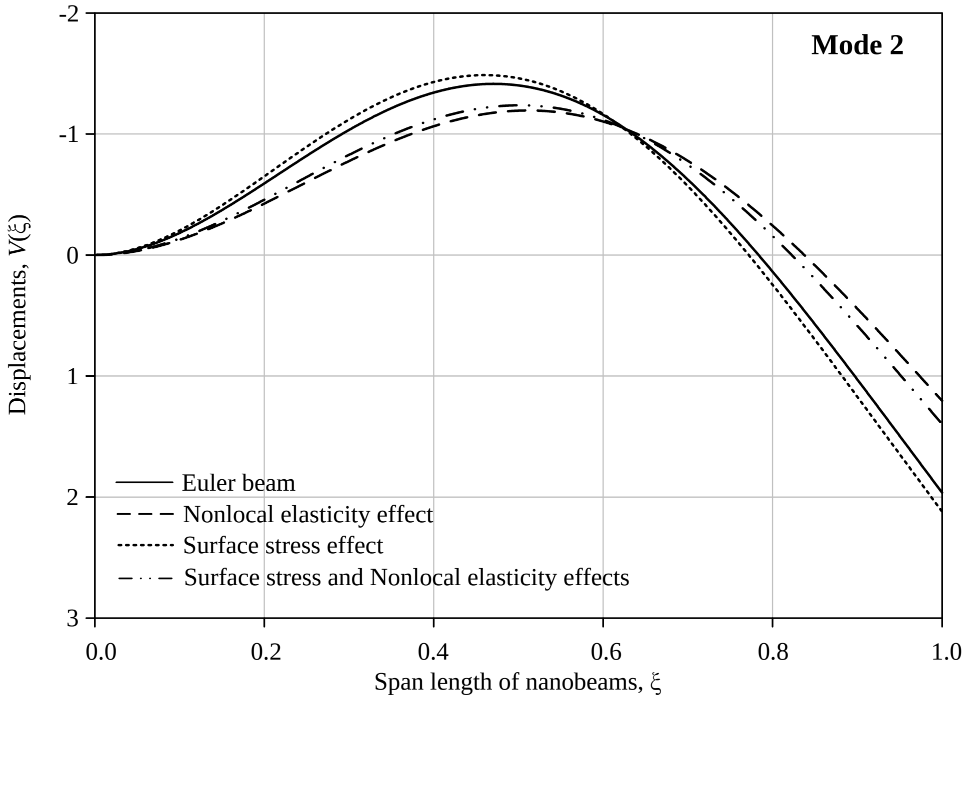


Figure 4.3 Variations of mode shapes of clamped-clamped nanobeams.


Mode 1
Euler beam
Nonlocal elasticity effect
Surface stress effect
Surface stress and Nonlocal elasticity effects
Displacements, V(ξ)
Span length of nanobeams, ξ
-3.0
-2.5
-2.0
-1.5
-1.0
-0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0


Mode 2
Euler beam
Nonlocal elasticity effect
Surface stress effect
Surface stress and Nonlocal elasticity effects
Displacements, V(ξ)
Span length of nanobeams, ξ
-2
-1
0
1
2
3
0.0
0.2
0.4
0.6
0.8
1.0

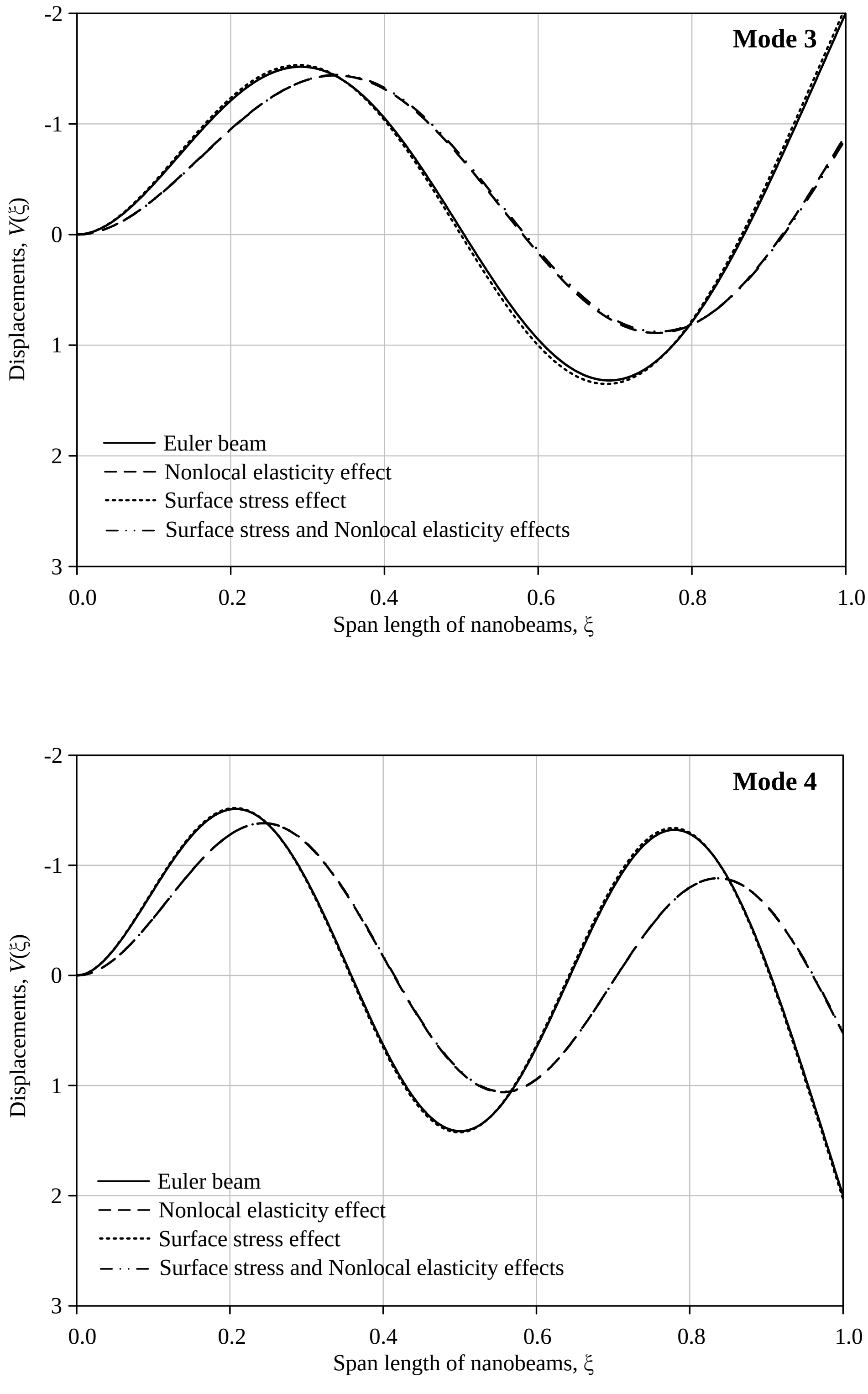
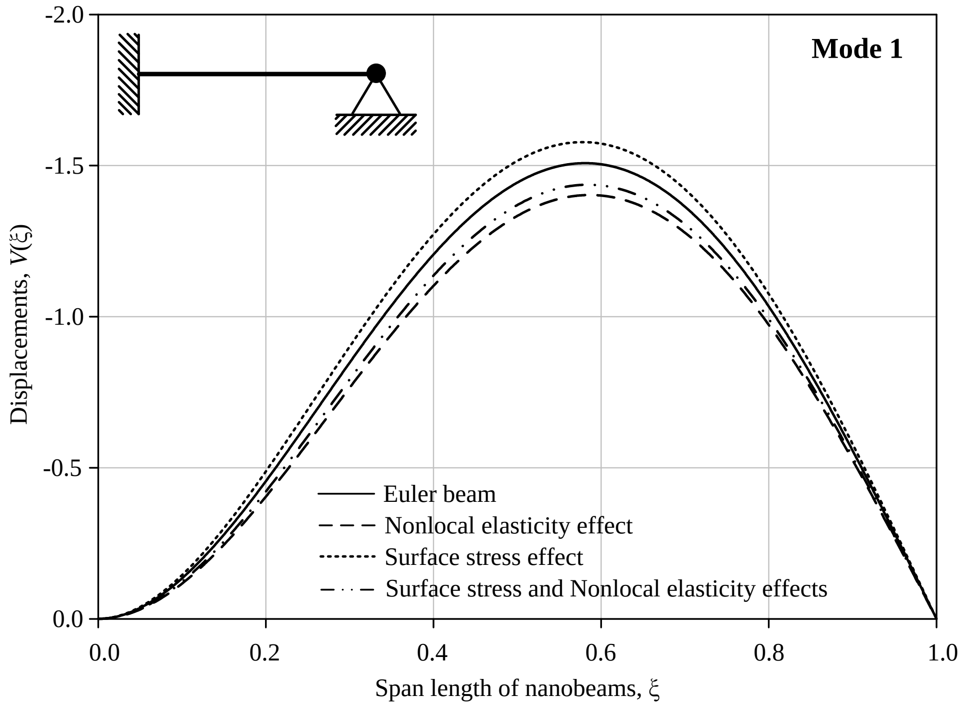
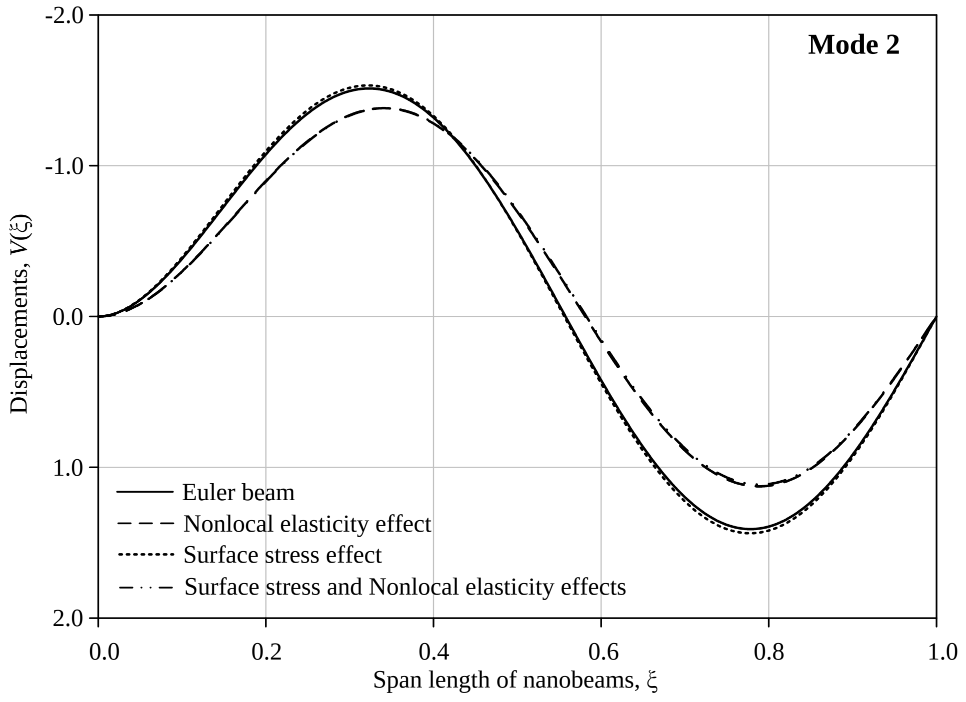
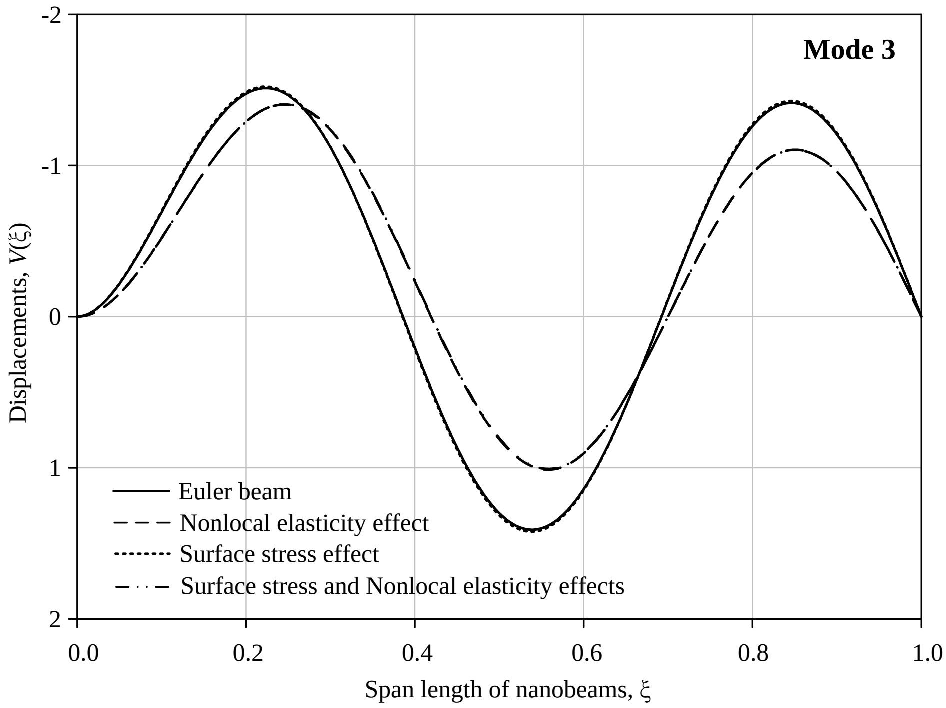


Figure 4.4 Variations of mode shapes of clamped-free nanobeams.


Mode 1
Displacements, V(ξ)
-2.0
-1.5
-1.0
-0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
Span length of nanobeams, ξ
Euler beam
Nonlocal elasticity effect
Surface stress effect
Surface stress and Nonlocal elasticity effects


Mode 2
Displacements, V(ξ)
-2.0
-1.0
0.0
1.0
2.0
0.0
0.2
0.4
0.6
0.8
1.0
Span length of nanobeams, ξ
Euler beam
Nonlocal elasticity effect
Surface stress effect
Surface stress and Nonlocal elasticity effects



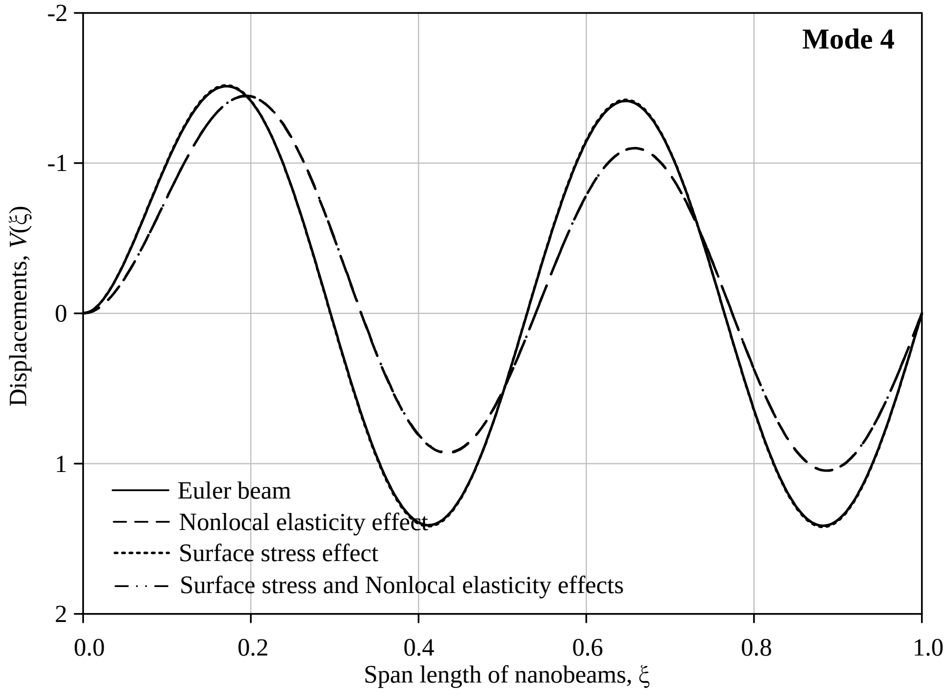
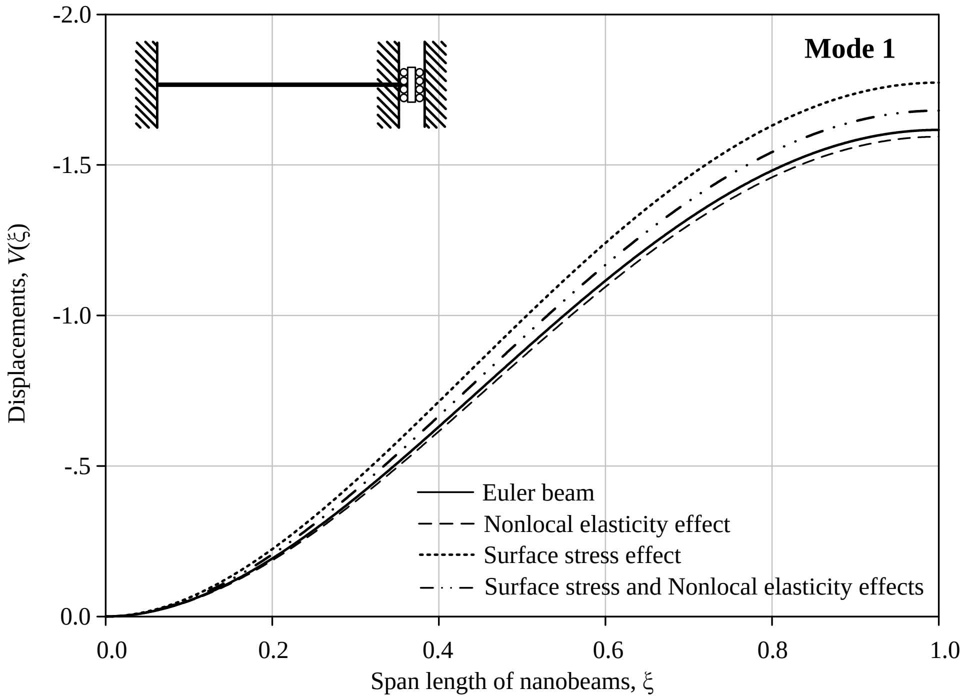
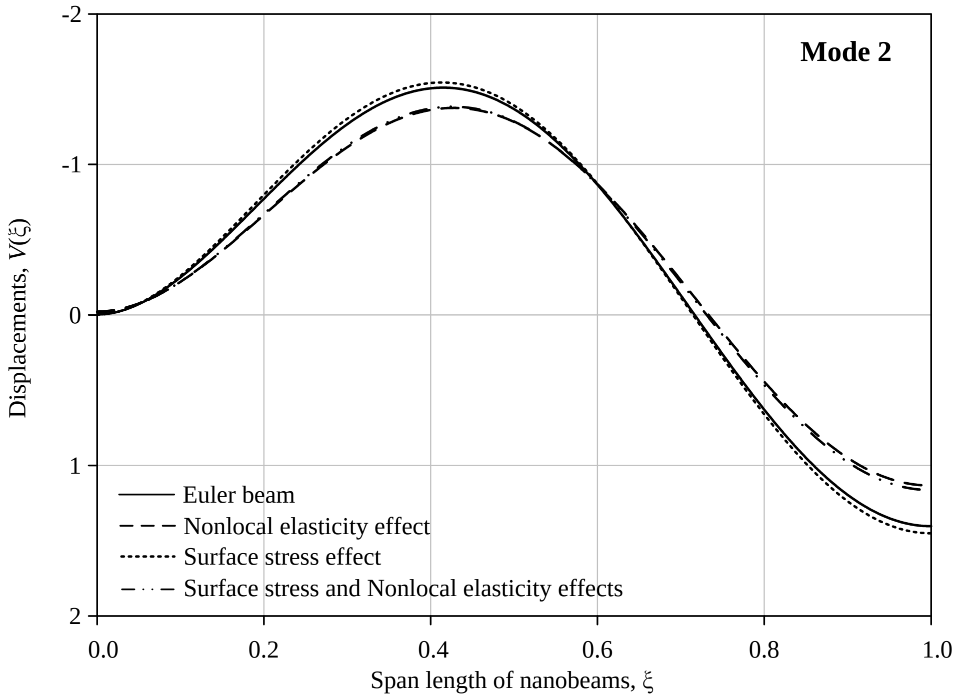
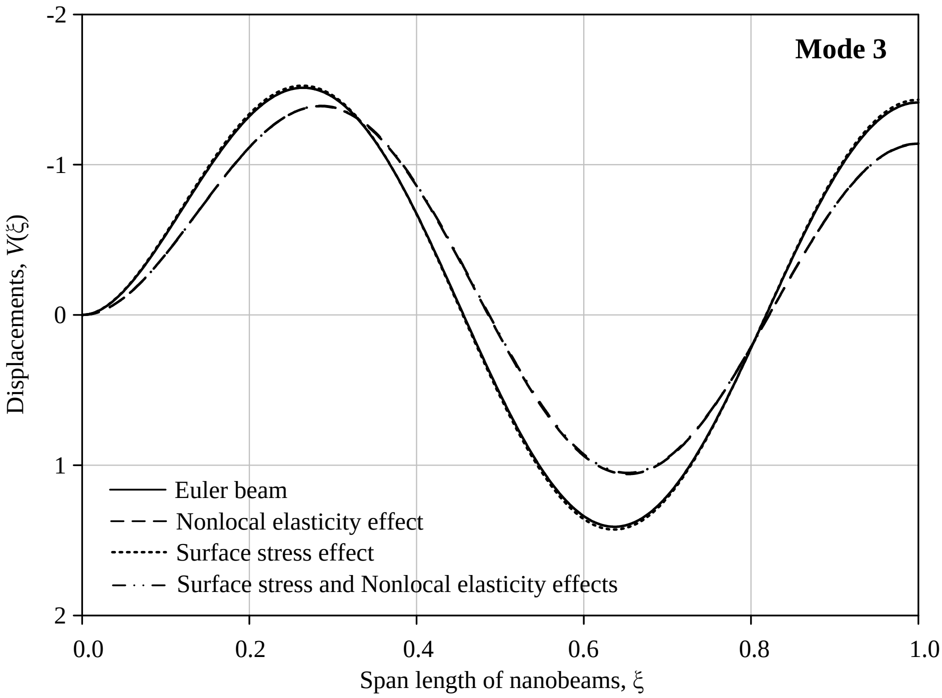


Figure 4.5 Variations of mode shapes of clamped-pinned nanobeams.


Mode 1
-2.0
-1.5
-1.0
-.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
Displacements, V(ξ)
Span length of nanobeams, ξ
Euler beam
Nonlocal elasticity effect
Surface stress effect
Surface stress and Nonlocal elasticity effects


Mode 2
-2
-1
0
1
2
0.0
0.2
0.4
0.6
0.8
1.0
Displacements, V(ξ)
Span length of nanobeams, ξ
Euler beam
Nonlocal elasticity effect
Surface stress effect
Surface stress and Nonlocal elasticity effects



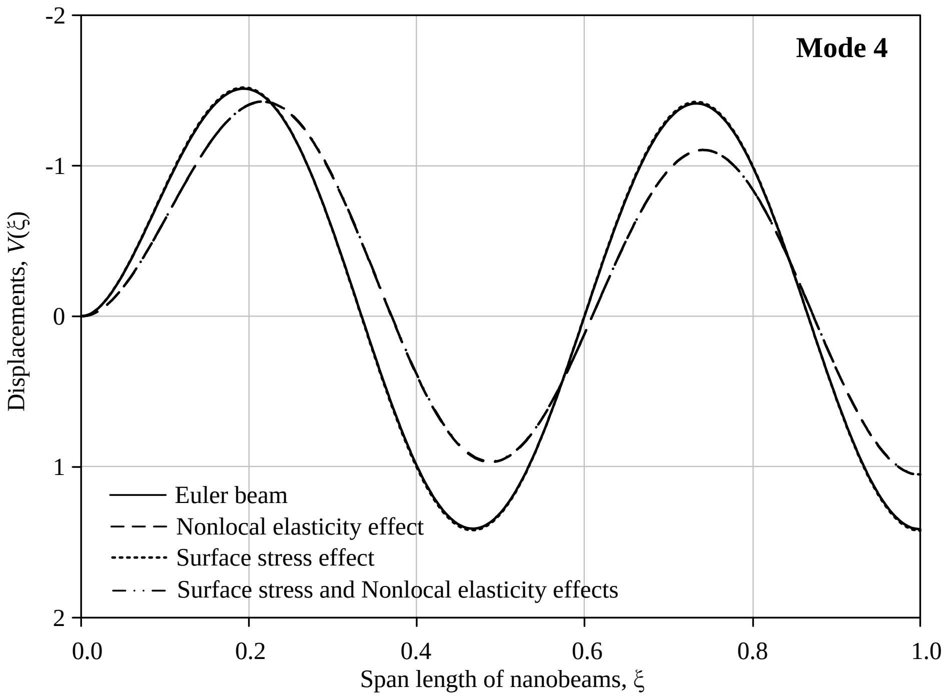
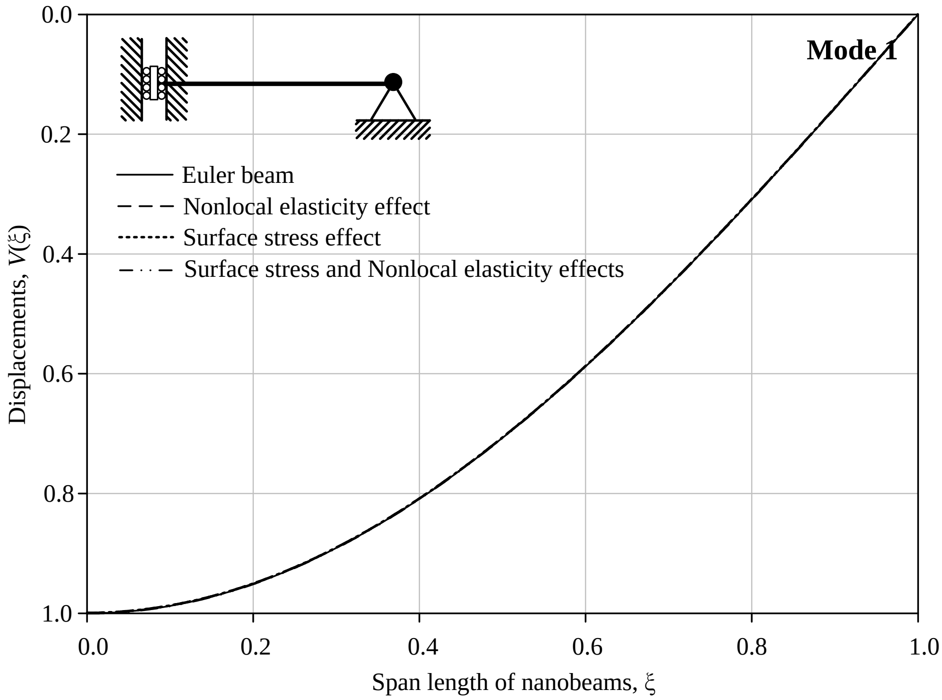
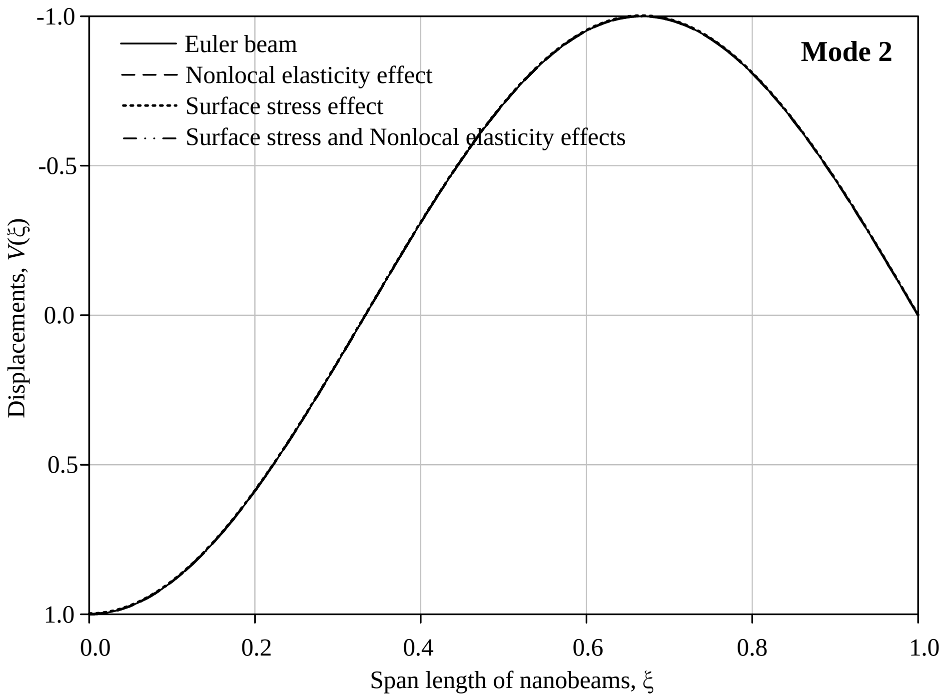
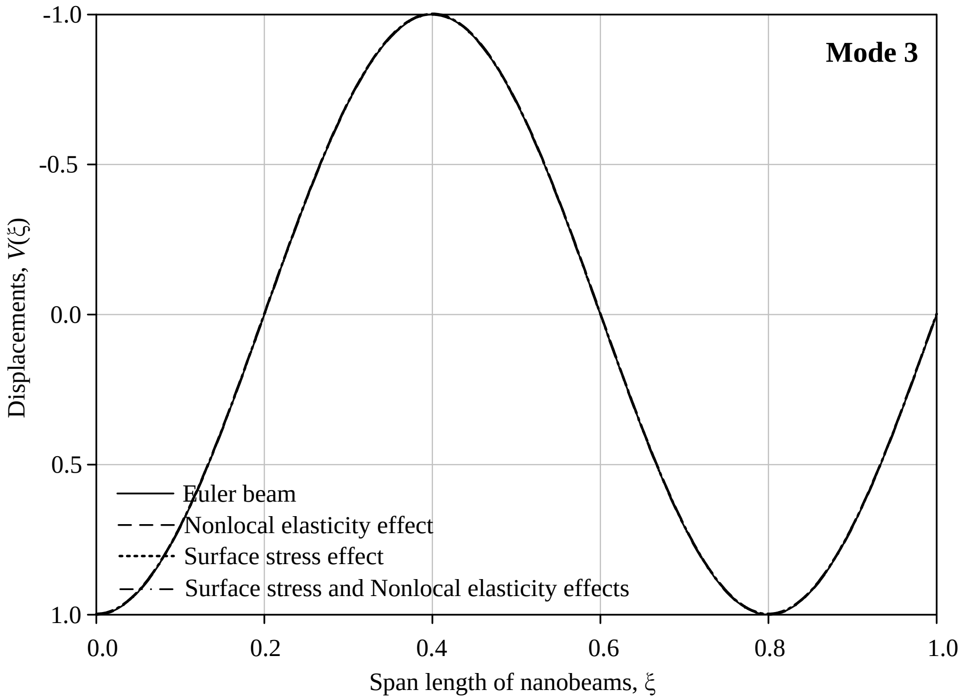


Figure 4.6 Variations of mode shapes of clamped-sliding nanobeams.


Mode 1
Euler beam
Nonlocal elasticity effect
Surface stress effect
Surface stress and Nonlocal elasticity effects
Displacements, V(ξ)
0.0
0.2
0.4
0.6
0.8
1.0
Span length of nanobeams, ξ


Mode 2
Euler beam
Nonlocal elasticity effect
Surface stress effect
Surface stress and Nonlocal elasticity effects
Displacements, V(ξ)
-1.0
-0.5
0.0
0.5
1.0
0.2
0.4
0.6
0.8
Span length of nanobeams, ξ



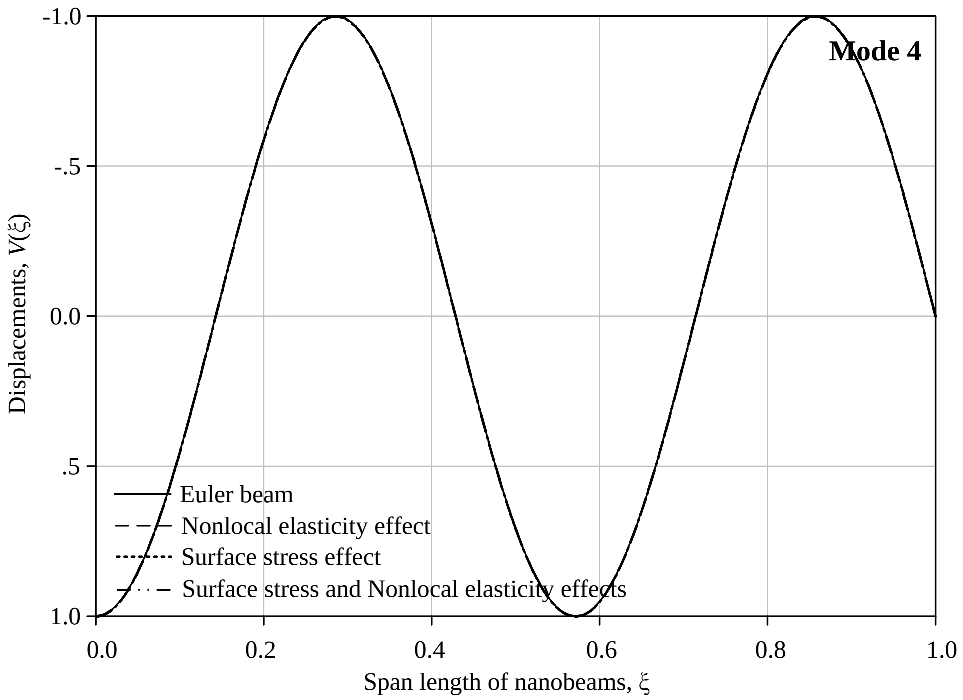


Figure 4.7 Variations of mode shapes of sliding-pinned nanobeams.

4.4 Parametric study

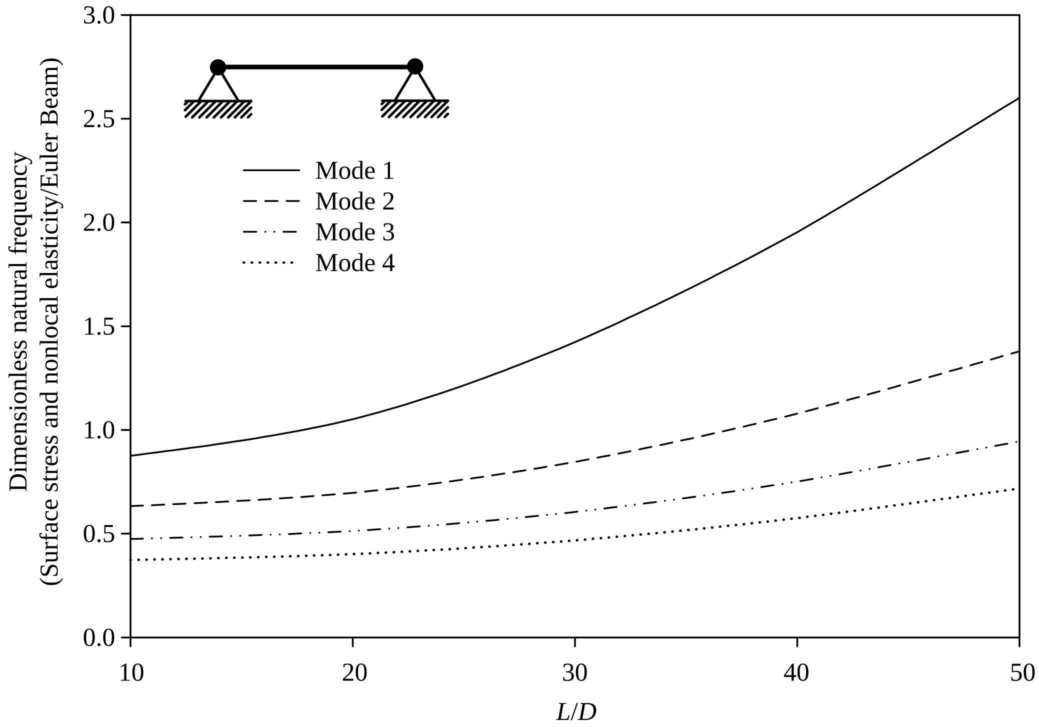
4.4.1 Effect of Aspect Ratio on the dimensionless natural frequencies of nanobeams

The effect of aspect ratio (L/D) on the dimensionless natural frequencies of nanobeams is analyzed by using the input parameters are shown in Table 4.3. The variation of the dimensionless natural frequencies of nanobeams with span length L=1,000 nm by varying the aspect ratio are shown in Figures 4.8 for pinned-pinned, clamped-clamped, clamped-free, clamped-pinned, clamped-sliding, and sliding-pinned boundary conditions, respectively.

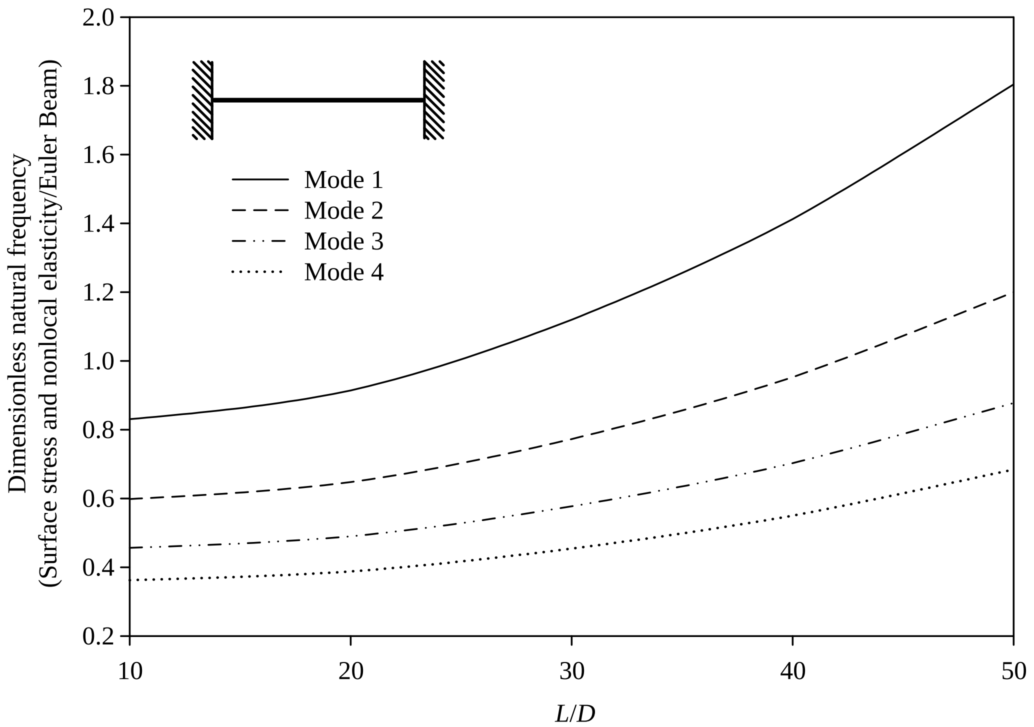
Table 4.3 Input parameters for analysis effect of aspect ratio on the dimensionless natural frequencies of nanobeams

1. Aspect ratio (L/D)	10-50
2. Modulus elasticity (E)	76 GPa
3. Surface elasticity (E^s)	1.22 N/m
4. Residual surface tension (τ^0)	0.89 N/m
5. Nonlocal parameter ($\overline{\mu}$)	0.04

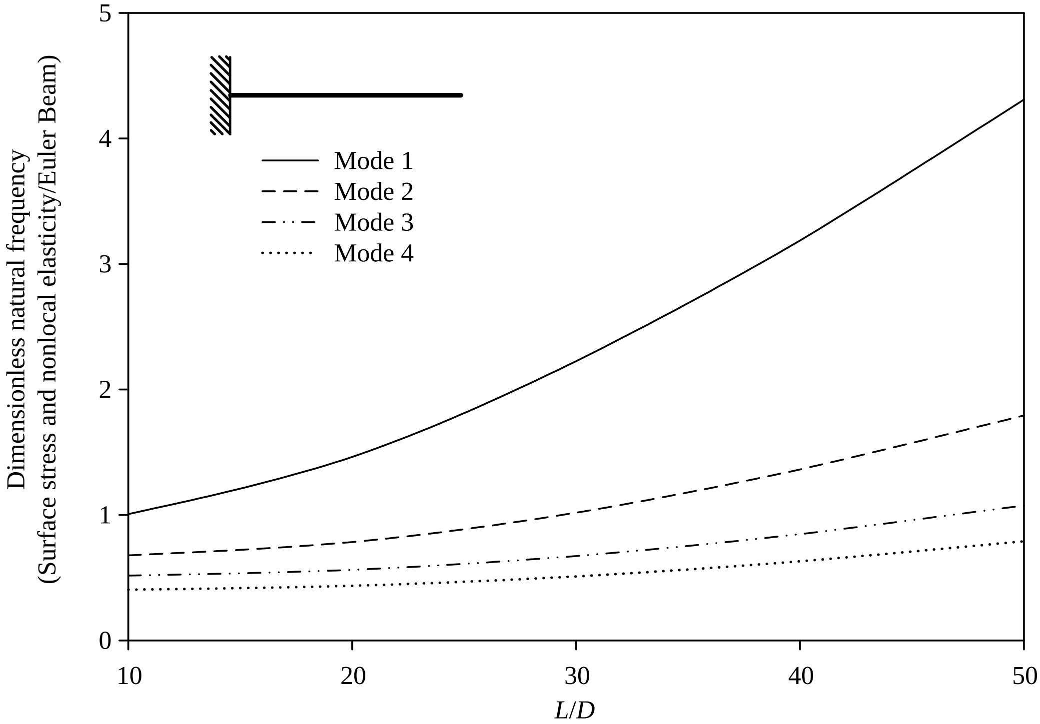
According to Figures 4.8, it can be observed that the dimensionless natural frequencies of nanobeams obtained from this analysis show deviation from that of Euler beam especially for the nanowires with high L/D. The increment of the dimensionless natural frequencies arises from the influence of both surface stress and nonlocal elasticity. Increment of the surface area (small D values and large L values) gives the increment of dimensionless natural frequencies. Thus, the behavior of nanobeams is not only diameter dependent but also the span length dependent. The previous results also indicated that this model provides understanding of observed size-dependent on the free vibration of nanobeams including the effects of both surface stress and nonlocal elasticity.


3.0
2.5
2.0
1.5
1.0
0.5
0.0
Dimensionless natural frequency
(Surface stress and nonlocal elasticity/Euler Beam)
Mode 1
Mode 2
Mode 3
Mode 4
10
20
30
40
50
L/D

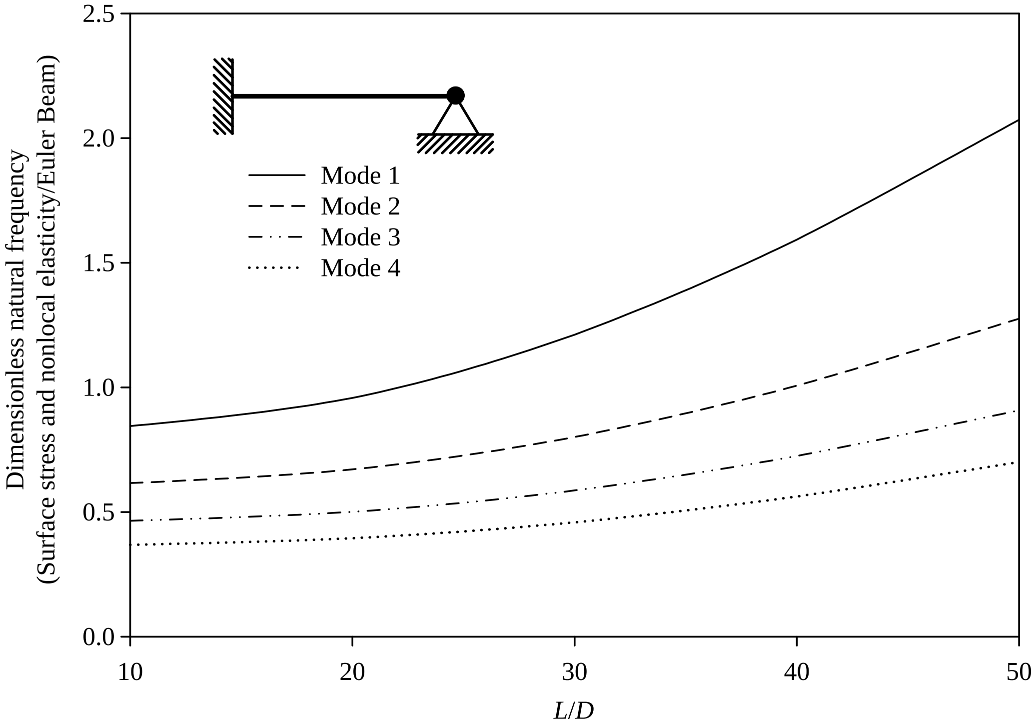
(a)


2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
Dimensionless natural frequency
(Surface stress and nonlocal elasticity/Euler Beam)
Mode 1
Mode 2
Mode 3
Mode 4
10
20
30
40
50
L/D

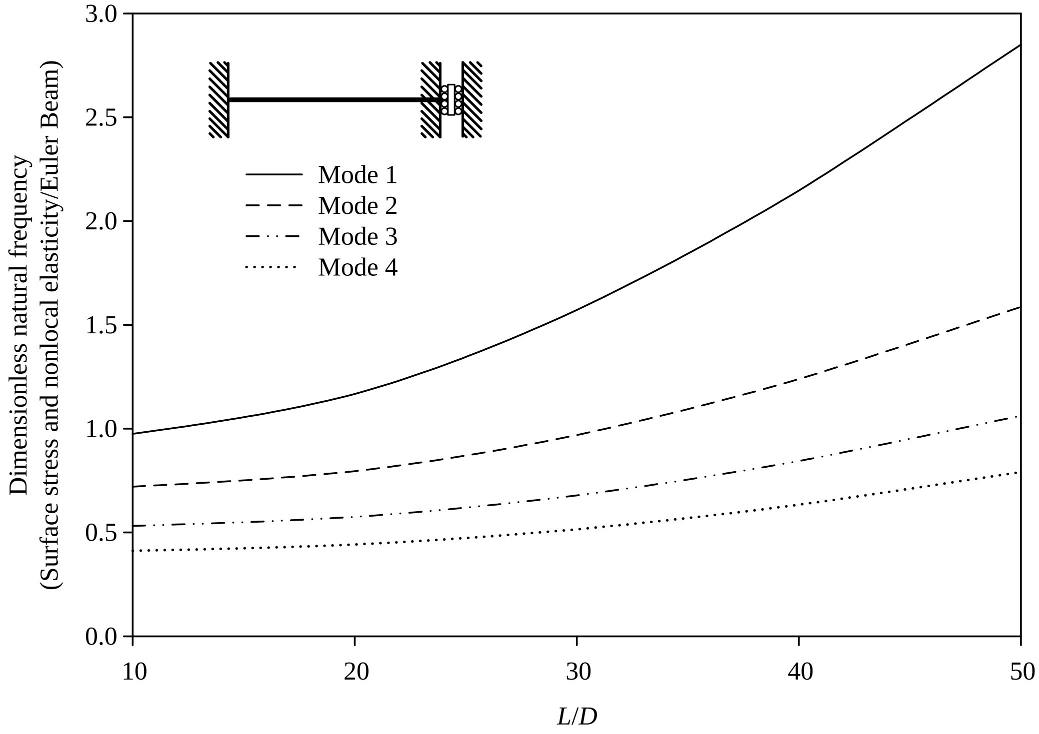
(b)


Dimensionless natural frequency
(Surface stress and nonlocal elasticity/Euler Beam)
0
1
2
3
4
5
10
20
30
40
50
L/D
Mode 1
Mode 2
Mode 3
Mode 4

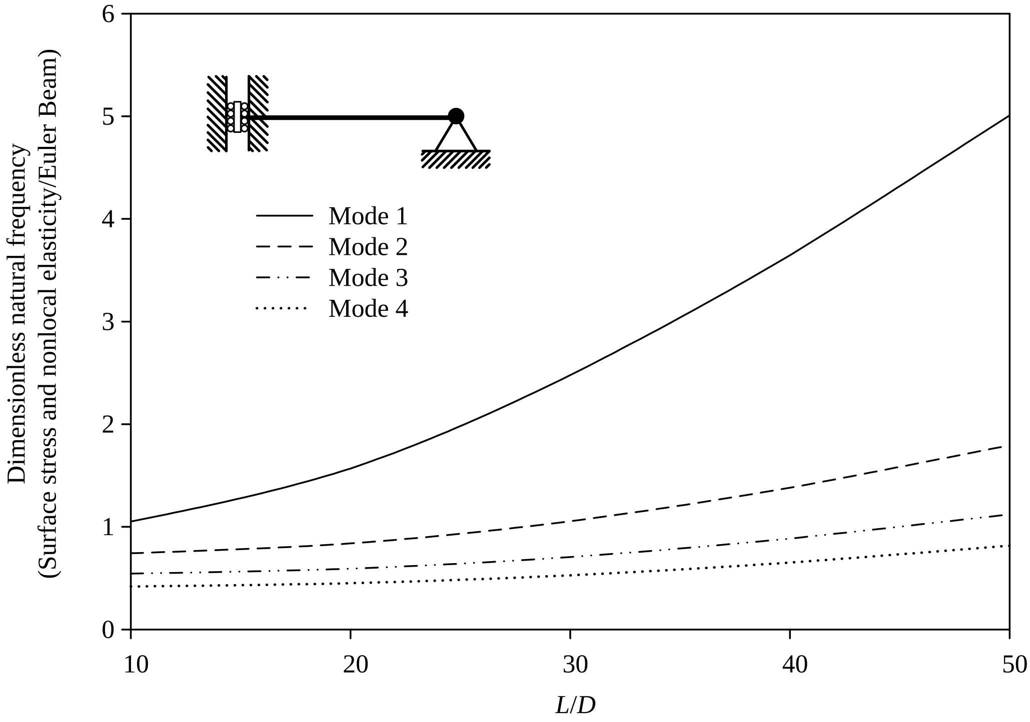
(c)


Dimensionless natural frequency
(Surface stress and nonlocal elasticity/Euler Beam)
0.0
0.5
1.0
1.5
2.0
2.5
10
20
30
40
50
L/D
Mode 1
Mode 2
Mode 3
Mode 4

(d)



(e)



(f)

Figure 4.8 Variation of dimensionless natural frequency by varying *L*/*D* for different boundary condition: (a) pinned-pinned, (b) clamped-clamped, (c) clamped-free, (d) clamped-pinned, (e) clamped-sliding, and (f) sliding-pinned

CHAPTER 5 CONCLUSIONS

In final chapter, summary of all results presented in preceding chapters are presented in conjunction with some suggestions for the future works of nanobeams studies.

5.1 Research Summary

Analytical solutions of nanobeams including both surface stress and nonlocal elasticity for free vibration analysis are presented and verified numerically by using the finite element method. The analytical solutions show identical results with that of finite element method. The obtained results demonstrated that natural frequencies and corresponding modes shapes of nanobeams actually depend on the effects of surface stress and nonlocal elasticity. All of the results in this study are summarized as follows:

- The surface stress increases the natural frequencies, especially, for lower modes number. The effect of surface stress is almost disappeared for the mode number with higher than the fourth mode. Moreover, nonlocal elasticity decreases the natural frequencies of nanobeams and its effect is increased for higher modes of vibration.

- For the nanobeams including the combined effects of surface stress and nonlocal elasticity, the results indicate that the natural frequencies are in between the one of nanobeams with surface stress and the one with nonlocal elasticity. Since the surface effect is reduced for the higher modes number, the natural frequencies of nanobeams including combined effects are converted to nanobeams which only nonlocal elasticity is considered.

- The clamped-clamped, clamped-free, clamped-pinned and clamped-sliding nanobeams exhibit the variation of mode shapes when the effects of surface stress and nonlocal elasticity are included. However, the surface stress and nonlocal elasticity have no effect in case of the pinned-pinned and sliding-pinned nanobeams.

5.2 Recommendations for Further Research

Several areas of research are still open for further theoretical investigation of nanobeams. Additional investigations needed to be made are as follows.

- Large amplitude free and forced vibration of nanobeams with surface stress and nonlocal elasticity effects by using numerical solution.

- Large displacement analysis of nanobeams including the effects of surface stress and nonlocal elasticity.

- The effects of surface stress and nonlocal elasticity on the post-buckling behavior of nanobeams.