บทที่ 4

ผลการวิจัย

(MAIN RESULT)

In this study, we use Catalan's conjecture. It is proved there that the only solution in integers $a>1, b>1, x>1$ and $y>1$ of the equation $a^x + b^y = 1$ is $a = y = 3$ and $b = x = 2$. Now we have the followings.

Theorem 1. The Diophantine equation $4^x + 13^y = z^2$ has no solution in non-negative integers.

Proof. From the Diophantine equation $4^x + 13^y = z^2$, we consider in 3 cases.

Case 1: $x = 0$.

We have $13^y = z^2 - 1 = (z-1)(z+1)$. Then there are non-negative integers a and b such that $13^a = z-1, 13^b = z+1$, $a < b$ and $a + b = y$. So we have $13^a\left(13^{b-a} - 1\right) = 13^b - 13^a = (z+1) - (z-1) = 2$. Therefore, $13^a = 1$ or $a = 0$. It follows that $z = 2$ and $13^b = z + 1 = 3$. This is impossible.

Case 2: $y = 0$.

We have $2^{2x} = 4^x = z^2 - 1 = (z-1)(z+1)$. Then there are non-negative integers a and b such that $2^a = z-1, 2^b = z+1$, $a < b$ and $a + b = 2x$. Therefore $2^a\left(2^{b-a} - 1\right) = 2^b - 2^a = (z+1) - (z-1) = 2$. It follows that $2a = 1$ or $2a = 2$. That is $a = 0$ or $a = 1$.

If $a=0$ then $z=2$ and $2b=3$. This is impossible. Thus $a=1$. This implies that $z=3$ and $b=2$. Thus $2x=a+b=3$. That is, x is not integer which is a contradiction.

Case 3: $x>0$ and $y>0$.

We have $13^y = z^2 - 4^x = (z-2^x)(z+2^x)$. Then there are non-negative integers a and b such that $13^a = z-2^x$, $13^b = z+2^x$, $a<b$ and $a+b=y$.

Therefore $13^a(13^{b-a}-1) = 13^b - 13^a = (z+2^x)-(z-2^x) = 2^{x+1}$. It follows that $13^a = 1$ and $13^{b-a} - 2^{x+1} = 1$. By Catalan's conjecture, we can conclude that this Diophantine equation has no solution. The theorem is proved.

Theorem 2. The Diophantine equation $4^x + 17^y = z^2$ has no solution in non-negative integers.

Proof. From the Diophantine equation $4^x + 17^y = z^2$, we consider in 3 cases.

Case 1: $x=0$.

We have $17^y = z^2 - 1 = (z-1)(z+1)$. Then there are non-negative integers a and b such that $17^a = z-1, 17^b = z+1$, $a<b$ and $a+b=y$. So we have $17^a(17^{b-a}-1) = 17^b - 17^a = (z+1)-(z-1) = 2$. Therefore, $17^a = 1$ or $a=0$. It follows that $z=2$ and $17^b = z+1 = 3$. This is impossible.

Case 2: $y = 0$.

We have $2^{2x} = 4^x = z^2 - 1 = (z-1)(z+1)$. Then there are non-negative integers a and b such that $2^a = z - 1, 2^b = z + 1,$ $a < b$ and $a + b = 2x$.

Therefore $2^a(2^{b-a} - 1) = 2^b - 2^a = (z+1) - (z-1) = 2$. It follows that $2a = 1$ or $2a = 2$. That is $a = 0$ or $a = 1$.

If $a = 0$ then $z = 2$ and $2b = 3$. This is impossible. Thus $a = 1$. This implies that $z = 3$ and $b = 2$. Thus $2x = a + b = 3$. That is, x is not integer which is a contradiction.

Case 3: $x > 0$ and $y > 0$.

We have $17^y = z^2 - 4^x = (z - 2^x)(z + 2^x)$. Then there are non-negative integers a and b such that $17^a = z - 2^x$, $17^b = z + 2^x$, $a < b$ and $a + b = y$.

Therefore $17^a(17^{b-a} - 1) = 17^b - 17^a = (z + 2^x) - (z - 2^x) = 2^{x+1}$. It follows that $17^a = 1$ and $17^{b-a} - 2^{x+1} = 1$. By Catalan's conjecture, we can conclude that this Diophantine equation has no solution. The theorem is proved.

Theorem 3. Consider the Diophantine equation

$$p^x + p^y = q^z \qquad \text{-----} \qquad (1)$$

where p and q are distinct prime numbers and x, y, z are non-negative integers. We get

(i) $(x,y,z)\in\{(0,3,2),(3,0,2)\}$ is a solution of the Diophantine equation (1) for $p=2$ and $q=3$.

(ii) $(x,y,z)\in\{(0,0,1)\}$ is a solution of the Diophantine equation (1) for $q=2$.

(iii) $(x,y,z)\in\{(0,k,1)|\, k\in N^{*}\}$ is a solution of the Diophantine equation (1) for $q=p^{k}+1$.

(iv) $(x,y,z)\in\{(0,1,k)|\, k\in N^{*}\}$ is a solution of the Diophantine equation (1) for $p=q^{k}-1$.

Proof. Suppose that p and q are distinct prime numbers. Consider the Diophantine equation $p^{x}+p^{y}=q^{z}$ in 2 cases..

Case 1: $x\le y$.

The Diophantine equation (1) becomes $1+p^{y-x}=q^{z}/p^{x}$. Thus q^{z}/p^{x} must be an integer. Then $x=0$. It follows that $q^{z}-p^{y}=1$. By Catalan's conjecture, we get $(x,y,z)=(0,3,2)$ is a solution of the Diophantine equation (1) where $p=2$ and $q=3$.

If $z=1$, then $q=1+p^{y}$. So, $(x,y,z)=(0,k,1)$ is a solution of the Diophantine equation (1) where k is a non-negative integer such that $q=1+p^{k}$.

If $y=0$, then $q^{z}=2$. So, $(x,y,z)=(0,0,1)$ is a solution of the Diophantine equation (1) where $q=2$.

If $y=1$, then $p=q^{y}-1$. So, $(x,y,z)=(0,1,k)$ is a solution of the Diophantine equation (1) where k is a non-negative integer such that $p=q^{k}-1$.

Case 2: x > y.

The Diophantine equation (1) becomes $p^{x-y}+1=q^z/p^y$. So q^z/p^y must be an integer number. Then $y=0$. It follows that $q^z-p^x=1$. By Catalan's conjecture, we get $(x,y,z)=(3,0,2)$ is a solution of the Diophantine equation (1) where $p=2$ and $q=3$.

If $z=1$, then $q=1+p^x$. So, $(x,y,z)=(k,0,1)$ is a solution of the Diophantine equation (1) where k is a non-negative integer such that $q=1+p^k$.

If $x=0$, then $q^z=2$. So, $(x,y,z)=(0,0,1)$ is a solution of the Diophantine equation (1) where $q=2$.

If $x=1$, then $p=q^y-1$. So, $(x,y,z)=(1,0,k)$ is a solution of the Diophantine equation (1) where k is a non-negative integer such that $p=q^k-1$.

Theorem 4. Consider the Diophantine equation

$$p^x+q^y=q^z \qquad \text{-----} \qquad (2)$$

where p and q are distinct prime numbers and x, y, z are non-negative integers. We get

(i) $(x,y,z)\in\{(3,0,2)\}$ is a solution of the Diophantine equation (1) for $p=2$ and $q=3$.

(ii) $(x,y,z)\in\{(0,0,1)\}$ is a solution of the Diophantine equation (1) for $q=2$.

(iii) $(x,y,z)\in\{(k,0,1)\mid k\in N^*\}$ is a solution of the Diophantine equation (1) for $q=p^k+1$.

(iv) $(x,y,z)\in\{(1,0,k)\mid k\in N^*\}$ is a solution of the Diophantine equation (1) for $p=q^k-1$.

Proof. Suppose that p and q are distinct prime numbers. Consider the Diophantine equation $p^x + q^y = q^z$ in 2 cases.

Case 1: $y \geq z$.

We get $q^y \geq q^z$ and $p^x > 0$. So, $p^x + q^y > q^z$. That is the Diophantine equation (2) has no solution.

Case 2: $y < z$.

Since $q^z \equiv 0 (\bmod q)$ and $p^x + q^y \equiv 0 (\bmod q)$ except $y = 0$, $p^x + q^y = q^z$ is impossible except $y = 0$.

If $y = 0$, the Diophantine equation (2) becomes $q^z - p^x = 1$. By Catalan's conjecture, we get $(x, y, z) = (3,0,2)$ is a solution of the Diophantine equation (1) where $p = 2$ and $q = 3$.

If $z = 1$, then $q = 1 + p^x$. So, $(x, y, z) = (k,0,1)$ is a solution of the Diophantine equation (1) where k is a non-negative integer such that $q = 1 + p^k$.

If $x = 0$, then $q^z = 2$. So, $(x, y, z) = (0,0,1)$ is a solution of the Diophantine equation (1) where $q = 2$.

If $x = 1$, then $p = q^y - 1$. So, $(x, y, z) = (1,0,k)$ is a solution of the Diophantine equation (1) where k is a non-negative integer such that $p = q^k - 1$.

Theorem 5. If p, q and r are distinct prime numbers which are not 2, then the Diophantine equation

$$p^x + q^y = r^z \qquad \text{-----} \qquad (3)$$

has no solution.

Proof. Consider the Diophantine equation $p^x + q^y = r^z$.

Since p, q and r are odd, so p^x, q^y and r^z are odd, too. Then, $p^x + q^y$ is even. So, $p^x + q^y$ cannot equal r^z. Hence the Diophantine equation (3) has no solution.

Theorem 6. Consider the Diophantine equation $p^x + p^y = z^2$ where p is a prime number, we get

(i) For $p = 2$, a solution of this Diophantine equation is

$$(x, y, z) \in \{(0,3,3),(3,0,3)\} \cup \{(2k-1, 2k-1, 2^k) \mid k \in N\}.$$

(ii) For $p = 3$, a solution of this Diophantine equation is

$$(x, y, z) \in \{(0,1,2),(1,0,2)\}.$$

(iii) For $p = h^2 + 1$ where $h \in N$, a solution of this Diophantine equation is

$$(x, y, z) \in \{(2k, 2k+1, hk) \mid k \in N\} \cup \{(2k, 2k+1, hk) \mid k \in N\}.$$