Chapter 5
Conclusion

In this chapter, we conclude all main result obtained in this research. It is organized by dividing in to 2 sections and each section gives the main result obtained in this study.

5.1 Approximation Method for Finite and Infinite Family of Nonexpansive Mappings

(1) Let H be a Hilbert space, C a closed convex nonempty subset of H. Let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ and let $f \in \prod_c$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and let K_n be defined by (1.1.16). Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $x_0 \in C$ and given $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are sequences in $(0,1)$ and suppose that the following conditions are satisfied:
$(C1)$ $\alpha_n \to 0$;
$(C2)$ $\sum_{n=0}^\infty \alpha_n = \infty$;
$(C3)$ $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
$(C4)$ $\sum_{n=1}^\infty |\gamma_{n,i} - \gamma_{n-1,i}| < \infty,\ for\ all\ i = 1, 2, ..., N$ and $\{\gamma_{n,i}\}_{i=1}^N \subset [a,b]$, where $0 < a \leq b < 1$;
$(C5)$ $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
$(C6)$ $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$.
If $\{x_n\}_{n=1}^\infty$ is the composite process defined by (3.1.1), then $\{x_n\}_{n=1}^\infty$ converges strongly to $q \in F$, which also solves the following variational inequality:

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad p \in F.$$

(2) Let E be a uniformly smooth Banach space and C a nonempty closed convex subset of E. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^\infty F(T_n)$ is nonempty and let $f : C \to C$ be a contraction with coefficient $\alpha \in (0,1)$. Let $\{x_n\}$ be the iteration sequence defined by (3.2.1). Suppose that $\{T_n\}$ satisfies the $AKTT - condition$. Let T be a mapping of C into itself defined by $Tz = \lim\limits_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^\infty F(T_n)$. If the following conditions are satisfied:
$(C1)$ $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^\infty \beta_n = \infty$ and $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$;
$(C2)$ $\sum_{n=0}^\infty \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$,
then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$.

(3) Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm and C be a nonexpty closed convex subset of E. Let $\{T_n\}_{n=1}^{\infty}$ be a countable family of nonexpansive mappings from C into itself such that the common fixed point set $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, and let $f : C \to C$ be a contractive mapping with a contractive constant $\alpha \in (0,1)$. For any give $x_0 \in C$, let $\{x_n\}$ be the iteration sequence defined by (3.2.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Suppose that $\{T_n\}$ satisfies *condition*(B) and the following conditions are satisfied:
$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
$(C2)$ $\lim_{n\to\infty} \beta_n = 0$.
Then $\{x_n\}$ converges strongly to a common fixed point $\{T_n\}$.

5.2 Equilibrium Problems and Fixed Point for Nonspresding Mappings in Hilbert Spaces

5.2.1 Strong Convergence Theorem of Two Iterative Methods of Nonspreading-type Mapping and Equilibrium Problems

(1) Let C be a nonempty closed convex subset of a real Hilbert space. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $T : C \to C$ be a k-strictly pseudononspreading mapping with a nonempty fixed point set and $F(T) \bigcap EP(f) \neq \emptyset$. Let $\beta \in [k,1)$ and let $T_\beta := \beta I + (1-\beta)T$. Let $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1)$ and $\{r_n\}_{n=1}^{\infty} \subset (0,\infty)$ satisfying the conditions:

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \ \textit{and} \ \liminf_{n\to\infty} r_n > 0.$$

Let $u \in C$ and let $\{x_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in C generated from an arbitrary $x_1 \in C$ by

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \ n \geq 1, \\ z_n = \frac{1}{n}\sum_{i=0}^{n-1} T_\beta^i u_n, \ n \geq 1. \end{cases} \tag{5.2.1}$$

Then $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ converge strongly to $P_{F(T)\bigcap EP(f)}u$, where $P_{F(T)\bigcap EP(f)} : H \to F(T)\bigcap EP(f)$ is the metric projection of H onto $F(T)\bigcap EP(f)$.

(2) Let C be a nonempty closed convex subset of of a real Hilbert space. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $T : C \to C$ be a k-strictly pseudononspreading mapping with a nonempty fixed point set and $F(T)\bigcap EP(f) \neq \emptyset$. Let $\beta \in [k,1)$ and let $T_\beta := \beta I + (1-\beta)T$. Let $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1)$ and $\{r_n\}_{n=1}^{\infty} \subset (0,\infty)$ satisfying the conditions:

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \ \textit{and} \ \liminf_{n\to\infty} r_n > 0.$$

Let $u \in C$ and let $\{x_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequence in C generated from an arbitrary $x_1 \in C$ by

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \geq 0, \ \ \forall y \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)u_n, \ n \geq 1, \\ z_n = \frac{1}{n}\sum_{m=0}^{n-1} T_\beta^m x_n, \ n \geq 1. \end{cases} \quad (5.2.2)$$

Then $\{x_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ converge strongly to $P_{F(T)\bigcap EP(f)}u$, where $P_{F(T)\bigcap EP(f)} : H \to F(T)\bigcap EP(f)$ is the metric projection of H onto $F(T)\bigcap EP(f)$.

5.2.2 Common Fixed Points for Two Nonspreading-type Mappings

(1) Let C be a nonempty closed convex subset of of a real Hilbert space. Let $T, S : C \to C$ be k-strictly pseudononspreading mappings with $F(T)\bigcap F(S) \neq \emptyset$. Let $\beta \in [k, 1)$ and let $T_\beta := \beta I + (1-\beta)T$ and $S_\beta := \beta I + (1-\beta)S$. Let $u \in C$ and let $\{x_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in C generated from an arbitrary $x_1 \in C$ by

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n)[\beta_n z_n + (1-\beta_n)w_n], \ n \geq 1, \\ w_n = \frac{1}{n}\sum_{k=0}^{n-1} S_\beta^k x_n \ \textit{and} \ z_n = \frac{1}{n}\sum_{m=0}^{n-1} T_\beta^m x_n, \ n \geq 1. \end{cases} \quad (5.2.3)$$

Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subset [0, 1)$ satisfying the conditions:
$(i)\ \lim_{n\to\infty} \alpha_n = 0, \ \ \sum_{n=1}^{\infty} \alpha_n = \infty;$
$(ii)\ \lim_{n\to\infty} \beta_n = 0 \ \textit{and} \ \sum_{n=1}^{\infty} \beta_n = \infty.$

Then $\{x_n\}$, $\{w_n\}$ and $\{z_n\}$ converge strongly to $P_{F(T)\bigcap F(S)}u$, where P is the metric projection of H onto $F(T)\bigcap F(S)$.