Chapter 4
Equilibrium Problems and Fixed Point Problem for Nonspreading Mappings in Hilbert Space

In this chapter, we introduce a strong convergence theorem for a nonspreading-type mappings and equilibrium problem in Hilbert spaces by using an idea of mean convergence. The main result of this paper extend the results obtained by Osilike and Isiogugu (Nonlinear Analysis 74 (2011) 1814-1822) and Kurokawa and Takahashi (Nonlinear Analysis 73 (2010) 1562-1568). Moreover, example and numerical results are also given.

4.1 Strong Convergence Theorem for Nonspreading-type Mappings and Equilibrium Problem in Hilbert Spaces

We first prove a strong convergence theorem.

Theorem 4.1.1. *Let C be a nonempty closed convex subset of of a real Hilbert space. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $T : C \to C$ be a k-strictly pseudononspreading mapping with a nonempty fixed point set and $F(T)\bigcap EP(f) \neq \emptyset$. Let $\beta \in [k, 1)$ and let $T_\beta := \beta I + (1-\beta)T$. Let $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1)$ and $\{r_n\}_{n=1}^{\infty} \subset (0,\infty)$ satisfying the conditions:*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \liminf_{n\to\infty} r_n > 0.$$

Let $u \in C$ and let $\{x_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in C generated from an arbitrary $x_1 \in C$ by

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ z_n = \frac{1}{n}\sum_{i=0}^{n-1} T_\beta^i u_n, \ n \geq 1 \\ x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \ n \geq 1. \end{cases} \tag{4.1.1}$$

Then $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ converge strongly to $P_{F(T)\bigcap EP(f)}u$, where $P_{F(T)\bigcap EP(f)} : H \to F(T)\bigcap EP(f)$ is the metric projection of H onto $F(T)\bigcap EP(f)$.

Proof. Let $T_\beta x := \beta x + (1-\beta)Tx$. It is clear that $F(T_\beta) = F(T)$ and for all $x, y \in C$, we have

$$\begin{aligned}
\|T_\beta x - T_\beta y\|^2 &= \|\beta(x-y) + (1-\beta)(Tx-Ty)\|^2 \\
&= \beta\|x-y\|^2 + (1-\beta)\|Tx-Ty\|^2 - \beta(1-\beta)\|x-Tx-(y-Ty)\|^2 \\
&\leq \beta\|x-y\|^2 - \beta(1-\beta)\|x-Tx-(y-Ty)\|^2 \\
&\quad +(1-\beta)\left[\|x-y\|^2 + k\|x-Tx-(y-Ty)\|^2 + 2\langle x-Tx, y-Ty\rangle\right] \\
&= \|x-y\|^2 - \beta(1-\beta)\|x-Tx-(y-Ty)\|^2 \\
&\quad +k(1-\beta)\|x-Tx-(y-Ty)\|^2 + 2(1-\beta)\langle x-Tx, y-Ty\rangle \\
&= \|x-y\|^2 - (1-\beta)(\beta-k)\|x-Tx-(y-Ty)\|^2 \\
&\quad +2(1-\beta)\langle x-Tx, y-Ty\rangle \\
&\leq \|x-y\|^2 + 2(1-\beta)\langle x-Tx, y-Ty\rangle \\
&= \|x-y\|^2 + \frac{2}{(1-\beta)}\langle x-T_\beta x, y-T_\beta y\rangle. \qquad (4.1.2)
\end{aligned}$$

It follows from (4.1.2) that T_β is quasi-nonexpansive. Let $p \in F(T)\bigcap EP(f)$. Then from $u_n = T_{r_n}x_n$, we obtain

$$\begin{aligned}
\|z_n - p\| &= \|\frac{1}{n}\sum_{i=0}^{n-1} T_\beta^i u_n - p\| \\
&\leq \frac{1}{n}\sum_{i=0}^{n-1}\|T_\beta^i u_n - p\| \\
&\leq \frac{1}{n}\sum_{i=0}^{n-1}\|u_n - p\|| = \|u_n - p\| = \|T_{r_n}x_n - p\| \leq \|x_n - p\|.
\end{aligned} \qquad (4.1.3)$$

Thus

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n u + (1-\alpha_n)z_n - p\| \\
&\leq \alpha_n\|u-p\| + (1-\alpha_n)\|z_n - p\| \\
&\leq \alpha_n\|u-p\| + (1-\alpha_n)\|x_n - p\|. \qquad (4.1.4)
\end{aligned}$$

By (4.1.4) and induction, we can conclude that for all $n \in \mathbb{N}$

$$\|x_n - p\| \leq \max\{\|u-p\|, \|x_1 - p\|\}.$$

Thus $\{x_n\}$ is bounded and also $\{u_n\}$ and $\{z_n\}$ are bounded.
Since $\|T_\beta^n u_n - p\| \leq \|u_n - p\|$, we have that $\{T_\beta^n u_n\}$ is also bounded.
Observe that since $\{z_n\}$ is bounded and $\lim_{n\to\infty}\alpha_n = 0$, we obtain

$$\begin{aligned}
\|x_{n+1} - z_n\| &= \|\alpha_n u + (1-\alpha_n)z_n - z_n\| \\
&= \alpha_n\|u - z_n\| \to 0 \text{ as } n \to \infty. \qquad (4.1.5)
\end{aligned}$$

Put $\Omega := F(T)\bigcap EP(f)$. We may assume without loss of generality that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - P_\Omega u, x_n - P_\Omega u\rangle = \lim_{j\to\infty}\langle u - P_\Omega u, x_{n_j} - P_\Omega u\rangle,$$

and $x_{n_j} \rightharpoonup w$ as $j \to \infty$. Since $\|x_{n+1} - z_n\| \to 0$ as $n \to \infty$, it follows that $z_{n_j} \rightharpoonup w$ as $j \to \infty$. Next, we will show that $w \in F(T)$. Using (4.1.2) we obtain for all $i = 0, 1, 2, ..., n-1$ and for arbitrary $y \in C$

$$\begin{aligned}
\|T_\beta^{i+1}u_n - T_\beta y\|^2 &= \|T_\beta(T_\beta^i u_n) - T_\beta y\|^2 \\
&\leq \|T_\beta^i u_n - y\|^2 + \frac{2}{1-\beta}\langle T_\beta^i u_n - T_\beta^{i+1}u_n, y - T_\beta y\rangle \\
&= \|T_\beta^i u_n - T_\beta y + T_\beta y - y\|^2 + \frac{2}{1-\beta}\langle T_\beta^i u_n - T_\beta^{i+1}u_n, y - T_\beta y\rangle \\
&= \|T_\beta^i u_n - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle T_\beta^i u_n - T_\beta y, T_\beta y - y\rangle \\
&\quad + \frac{2}{1-\beta}\langle T_\beta^i u_n - T_\beta^{i+1}u_n, y - T_\beta y\rangle \qquad (4.1.6)
\end{aligned}$$

Summing (4.1.6) from $i = 0$ to $n-1$ and dividing by n we obtain

$$\begin{aligned}
\frac{1}{n}\|T_\beta^n u_n - T_\beta y\|^2 &\leq \frac{1}{n}\|u_n - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle z_n - T_\beta y, T_\beta y - y\rangle \\
&\quad + \frac{2}{n(1-\beta)}\langle u_n - T_\beta^n u_n, y - T_\beta y\rangle \qquad (4.1.7)
\end{aligned}$$

Replacing n by n_j in (4.1.7) we obtain

$$\begin{aligned}
\frac{1}{n_j}\|T_\beta^{n_j} u_{n_j} - T_\beta y\|^2 &\leq \frac{1}{n_j}\|u_{n_j} - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle z_{n_j} - T_\beta y, T_\beta y - y\rangle \\
&\quad + \frac{2}{n_j(1-\beta)}\langle u_{n_j} - T_\beta^{n_j} u_{n_j}, y - T_\beta y\rangle \qquad (4.1.8)
\end{aligned}$$

Since $\{u_n\}$ and $\{T_\beta^n u_n\}$ are bounded, letting $j \to \infty$ in (4.1.8) yields

$$0 \leq \|T_\beta y - y\|^2 + 2\langle w - T_\beta y, T_\beta y - y\rangle. \qquad (4.1.9)$$

Since $y \in C$ was arbitrary, if we set $y = w$ in (4.1.9) we obtain

$$0 \leq \|T_\beta w - w\|^2 - 2\|T_\beta w - w\|^2,$$

from which it follows that $w \in F(T_\beta) = F(T)$.
Since $P_\Omega : H \to \Omega$ is the metric projection, we have

$$\lim_{j\to\infty}\langle u - P_\Omega u, x_{n_j} - P_\Omega u\rangle = \langle u - P_\Omega u, w - P_\Omega u\rangle \leq 0.$$

Hence we have $\limsup_{n\to\infty}\langle u - P_\Omega u, x_n - P_\Omega u\rangle \leq 0$. Using Lemma 2.2.44 (ii) and (4.1.3) we have

$$\begin{aligned}
\|x_{n+1} - P_\Omega u\|^2 &= \|\alpha_n u + (1-\alpha_n)z_n - P_\Omega u\|^2 \\
&= \|\alpha_n u - \alpha_n P_\Omega u + (1-\alpha_n)z_n - (1-\alpha_n)P_\Omega u\|^2 \\
&= \|\alpha_n(u - P_\Omega u) + (1-\alpha_n)(z_n - P_\Omega u)\|^2 \\
&\leq (1-\alpha_n)^2\|z_n - P_\Omega u\|^2 + 2\alpha_n\langle u - P_\Omega u, x_{n+1} - P_\Omega u\rangle \\
&\leq (1-\alpha_n)^2\|x_n - P_\Omega u\|^2 + 2\alpha_n\langle u - P_\Omega u, x_{n+1} - P_\Omega u\rangle \\
&\qquad (4.1.10)
\end{aligned}$$

Since $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty}\langle u - P_\Omega u, x_{n+1} - P_\Omega u\rangle \leq 0$, it follows from Lemma 2.7.15 that $\lim_{n\to\infty} \|x_n - P_\Omega u\| = 0$.

$$0 \leq \|z_n - P_\Omega u\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - P_\Omega u\| \to 0 \text{ as } n \to \infty.$$

Hence $\lim_{n\to\infty} \|z_n - P_\Omega u\| = 0$.
Since $\|x_n - P_\Omega u\| \to 0$, we have $\|x_{n+1} - x_n\| \to 0$.
In sequence, we show that $\|x_n - u_n\| \to 0$, as $n \to \infty$. For $p \in F(T)\bigcap EP(f)$, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\|^2 \\ &\leq \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle \\ &= \langle u_n - p, x_n - p\rangle \\ &= \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2) \end{aligned} \tag{4.1.11}$$

and hence $\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2$. Therefore, from the convexity of $\|\cdot\|^2$ and (4.1.3), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n u + (1-\alpha_n)z_n - p\|^2 \\ &\leq \alpha_n\|u - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\ &\leq \alpha_n\|u - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\ &\leq \alpha_n\|u - p\|^2 + (1-\alpha_n)(\|x_n - p\|^2 - \|x_n - u_n\|^2) \\ &\leq \alpha_n\|u - p\|^2 + \|x_n - p\|^2 - (1-\alpha_n)\|x_n - u_n\|^2 \end{aligned}$$

and hence

$$\begin{aligned} (1-\alpha_n)\|x_n - u_n\|^2 &\leq \alpha_n\|u - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\leq \alpha_n\|u - p\|^2 + \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|) \end{aligned}$$

So, we have $\|x_n - u_n\| \to 0$. This implies $u_{n_j} \rightharpoonup w$ as $j \to \infty$.
Finally, we prove that $w \in EP(f)$. By $u_n = T_{r_n}x_n$, we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in C.$$

From (A2), we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq f(y, u_n), \ \ \forall y \in C.$$

and hence

$$\langle y - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}}\rangle \geq f(y, u_{n_j}), \ \ \forall y \in C.$$

Since $\frac{u_{n_j} - x_{n_j}}{r_{n_j}} \to 0$ and $u_{n_j} \rightharpoonup w$, from (A4) we have $0 \geq f(y, w)$ for all $y \in C$.
For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $f(y_t, w) \leq 0$. So, from (A1) and (A4) we have

$$\begin{aligned} 0 &= f(y_t, y_t) \\ &\leq tf(y_t, y) + (1-t)f(y_t, w) \\ &\leq tf(y_t, y) \end{aligned}$$

and hence $0 \leq f(y_t, y)$. From (A3), we have $0 \leq f(w, y)$ for all $y \in C$ and hence $w \in EP(F)$. Therefore $w \in F(T) \bigcap EP(f)$.

□

If $f(x, y) = 0, \forall (x, y) \in C \times C$, we have that $u_n = x_n$ for all $n \in \mathbb{N}$. Hence the following Corollary is directly obtained by Theorem 4.1.1 of M.O. Osilike and F.O. Isiogugu [37]

Corollary 4.1.2. ([37], Theorem 3.2) Let C be a nonempty closed convex subset of of a real Hilbert space. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $T : C \to C$ be a k-strictly pseudononspreading mapping with a nonempty fixed point set and $F(T) \bigcap EP(f) \neq \emptyset$. Let $\beta \in [k, 1)$ and let $T_\beta := \beta I + (1-\beta)T$. Let $\{\alpha_n\}_{n=1}^\infty \subset [0, 1)$ and $\{r_n\}_{n=1}^\infty \subset (0, \infty)$ satisfying the conditions:

$$\lim_{n\to\infty} \alpha_n = 0 \ \textit{and} \ \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Let $u \in C$ and let $\{x_n\}_{n=1}^\infty$, and $\{z_n\}_{n=1}^\infty$ be sequences in C generated from an arbitrary $x_1 \in C$ by

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n, \ n \geq 1, \\ z_n = \frac{1}{n} \sum_{i=0}^{n-1} T_\beta^i x_n, \ n \geq 1. \end{cases} \tag{4.1.12}$$

Then $\{x_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ converge strongly to $P_{F(T)\bigcap EP(f)} u$, where $P_{F(T)\bigcap EP(f)} : H \to F(T) \bigcap EP(f)$ is the metric projection of H onto $F(T) \bigcap EP(f)$.

Remark 4.1.3. *If T is nonspreading, then T is 0-strictly pseudononspreading. By putting $\beta = 0$, we obtain $T_0 = T$. By Theorem 4.1.1, we obtain the result of Kurokawa and Takahashi* ([29], Theorem 4.1).

Example and numerical results

In this section, we give examples and numerical results for our main theorem.

Example 4.1.4. *Let $T : [-4, 2] \to [-4, 2]$ be define by*

$$Tx = \begin{cases} x, & x \in [-4, 0); \\ -2x, & x \in [0, 2]. \end{cases}$$

Let $H = \mathbb{R}$ and $C = [-4, 2]$, and let $f(x, y) = -9x^2 + xy + 8y^2$. Find $\hat{x} \in [-4, 2]$ such that

$$f(\hat{x}, y) + \frac{1}{r}\langle y - \hat{x}, \hat{x} - z\rangle \geq 0, \ \forall y \in [-4, 2]$$

Solution. By Example 1 [37], we know that T is $\frac{1}{3}$−strictly pseudononspreading. Observe that $F(T) = [-4, 0]$.

For $r > 0$ and $x \in [-4, 2]$, by Lemma 2.5.2, there exists $z \in [-4, 2]$ such that for each $y \in [-4, 2]$

$$\begin{aligned}
& f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \\
& \Leftrightarrow -9z^2 + yz + 8y^2 + \frac{1}{r}(z - x)(y - z) \geq 0 \\
& \Leftrightarrow -9rz^2 + ryz + 8ry^2 + zy - z^2 - xy + xz \geq 0 \\
& \Leftrightarrow 8ry^2 + (rz + z - x)y - (9rz^2 + z^2 - xz) \geq 0
\end{aligned}$$

Put $G(y) = 8ry^2 + (rz + z - x)y - (9rz^2 + z^2 - xz)$. Then G is a quadratic function of y with coefficient $a = 8r, b = rz + z - x$ and $c = -(9rz^2 + z^2 + xz)$. We next compute the discriminant Δ of G as follows:

$$\begin{aligned}
\Delta &= b^2 - 4ac \\
&= (rz + z - x)^2 + 32r(9rz^2 + z^2 - xz) \\
&= [(r + 1)z - x]^2 + 32r(9rz^2 + z^2 - xz) \\
&= x^2 - 2(r + 1)xz + (r + 1)^2z^2 + 288r^2z^2 + 32rz^2 - 32rxz \\
&= x^2 - 34rxz - 2zx + 289r^2z^2 + 34rz^2 + z^2 \\
&= x^2 - 2(17rz + z)x + (289r^2z^2 + 34rz^2 + z^2) \\
&= [x - (17rz + z)]^2
\end{aligned}$$

We know that $G(y) \geq 0$ for all $y \in [-4, 2]$. If it has most one solution in $[-4, 2]$, so $\Delta \leq 0$ and hence $x = 17rz + z$. Now we have $z = T_r x = \frac{x}{17r+1}$.
Since $T_\beta := \beta I + (1 - \beta)T$, we obtain

$$T_\beta x = \begin{cases} x, & x \in [-4, 0); \\ (3\beta - 2)x, & x \in [0, 2]. \end{cases}$$

Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated by $x_1 = x \in [-4, 2]$ and

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in C, \\ z_n = \frac{1}{n}\sum_{i=0}^{n-1} T_\beta^i u_n, \ \ n \geq 1 \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)z_n, \ \ n \geq 1. \end{cases} \tag{4.1.13}$$

We next give two numerical results for algorithm (4.1.13).

Let $\alpha_n = \frac{1}{50n}$ and $r_n = \frac{n}{3n+1}$. Choose $\beta = \frac{5}{6}$ and $x_1 = u = 1$. Then algorithm (4.1.13) becomes

$$\begin{cases} x_{n+1} = \frac{1}{50n} + \left(1 - \frac{1}{50n}\right) z_n, \ \ n \geq 1, \\ z_n = \frac{1}{n}\sum_{i=0}^{n-1} T_\beta^i \left(\frac{x_n}{17r_n+1}\right), \ \ n \geq 1. \end{cases} \tag{4.1.14}$$

n	x_n	z_n
1	1.000000	0.190476
2	0.206667	0.026463
3	0.036199	0.003462
4	0.010105	0.000760
5	0.005756	0.000353
$\vdots$	$\vdots$	$\vdots$
110	0.000184	0.000001
111	0.000182	0.000000

Table 1:

Let $\alpha_n = \frac{1}{50n}$ and $r_n = \frac{n}{n+1}$. Choose $\beta = \frac{5}{6}$ and $x_1 = u = -1$. Then algorithm (4.1.13) becomes

$$\begin{cases} x_{n+1} = -\frac{1}{50n} + \left(1 - \frac{1}{50n}\right) z_n, \; n \geq 1, \\ z_n = \frac{1}{n} \sum_{i=0}^{n-1} T_\beta^i \left(\frac{x_n}{17r_n+1}\right), \; n \geq 1. \end{cases} \tag{4.1.15}$$

n	x_n	z_n
1	-1.000000	-0.105263
2	-0.123158	-0.009986
3	-0.019886	-0.001446
4	-0.008103	-0.000555
5	-0.005552	-0.000366
$\vdots$	$\vdots$	$\vdots$
125	-0.000171	-0.000010
126	-0.000170	-0.000009

Table 2:

Conclusion. Table 1 and Table 2 show that the sequence $\{x_n\}$ and $\{z_n\}$ converge to 0 which both solves the equilibrium problem of f and the fixed point problem of T. On the other hand, using Lemma 2.5.2 (3) , we can check that $F(T_r) = EP(f) = \{0\}$.

4.2 Equilibrium Problems and Fixed Point Problems for Nonspreading-type Mappings in Hilbert Space

We first prove a strong convergence theorem.

Theorem 4.2.1. *Let C be a nonempty closed convex subset of of a real Hilbert space. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $T : C \to C$ be a k-strictly pseudononspreading mapping with a nonempty fixed point* *set and* $F(T)\bigcap EP(f) \neq \emptyset$. *Let* $\beta \in [k,1)$ *and let* $T_\beta := \beta I + (1-\beta)T$. *Let* $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1)$ *and* $\{r_n\}_{n=1}^{\infty} \subset (0,\infty)$ *satisfying the conditions:*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \liminf_{n\to\infty} r_n > 0.$$

Let $u \in C$ *and let* $\{x_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$ *and* $\{z_n\}_{n=1}^{\infty}$ *be sequence in* C *generated from an arbitrary* $x_1 \in C$ *by*

$$\begin{cases} z_n = \frac{1}{n}\sum_{m=0}^{n-1} T_\beta^m x_n, \ n \geq 1 \\ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \geq 0, \ \ \forall y \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)u_n, \ n \geq 1. \end{cases} \tag{4.2.1}$$

Then $\{x_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$ *and* $\{z_n\}_{n=1}^{\infty}$ *converge strongly to* $P_{F(T)\bigcap EP(f)}u$, *where* $P_{F(T)\bigcap EP(f)} : H \to F(T)\bigcap EP(f)$ *is the metric projection of* H *onto* $F(T)\bigcap EP(f)$.

Proof. Let $T_\beta x := \beta x + (1-\beta)Tx$. It is clear that $F(T_\beta) = F(T)$ and for all $x, y \in C$, we have

$$\begin{aligned} \|T_\beta x - T_\beta y\|^2 &= \|\beta(x-y) + (1-\beta)(Tx-Ty)\|^2 \\ &= \beta\|x-y\|^2 + (1-\beta)\|Tx-Ty\|^2 - \beta(1-\beta)\|x-Tx-(y-Ty)\|^2 \\ &\leq \beta\|x-y\|^2 - \beta(1-\beta)\|x-Tx-(y-Ty)\|^2 \\ &\quad +(1-\beta)\left[\|x-y\|^2 + k\|x-Tx-(y-Ty)\|^2 + 2\langle x-Tx, y-Ty\rangle\right] \\ &= \|x-y\|^2 - \beta(1-\beta)\|x-Tx-(y-Ty)\|^2 \\ &\quad +k(1-\beta)\|x-Tx-(y-Ty)\|^2 + 2(1-\beta)\langle x-Tx, y-Ty\rangle \\ &= \|x-y\|^2 - (1-\beta)(\beta-k)\|x-Tx-(y-Ty)\|^2 \\ &\quad +2(1-\beta)\langle x-Tx, y-Ty\rangle \\ &\leq \|x-y\|^2 + 2(1-\beta)\langle x-Tx, y-Ty\rangle \\ &= \|x-y\|^2 + \frac{2}{(1-\beta)}\langle x-T_\beta x, y-T_\beta y\rangle. \end{aligned} \tag{4.2.2}$$

It follows from (4.1.2) that T_β is quasi-nonexpansive. Let $p \in F(T)\bigcap EP(f)$. Then from $u_n = T_{r_n}z_n$, 11we obtain

$$\begin{aligned} \|z_n - p\| &= \|\frac{1}{n}\sum_{m=0}^{n-1} T_\beta^m x_n - p\| \\ &\leq \frac{1}{n}\sum_{m=0}^{n-1}\|T_\beta^m x_n - p\| \leq \frac{1}{n}\sum_{m=0}^{n-1}\|x_n - p\|| = \|x_n - p\|. \end{aligned} \tag{4.2.3}$$

Thus

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n u + (1-\alpha_n)u_n - p\| \\ &= \|\alpha_n u + (1-\alpha_n)T_{r_n}z_n - p\| \\ &\leq \alpha_n\|u-p\| + (1-\alpha_n)\|T_{r_n}z_n - p\| \\ &\leq \alpha_n\|u-p\| + (1-\alpha_n)\|z_n - p\| \\ &\leq \alpha_n\|u-p\| + (1-\alpha_n)\|x_n - p\|. \end{aligned} \tag{4.2.4}$$

By (4.2.4) and induction, we can conclude that for all $n \in \mathbb{N}$

$$\|x_n - p\| \le \max\{\|u - p\|, \|x_1 - p\|\}.$$

This implies that $\{x_n\}$ and $\{z_n\}$ are bounded. Since $\|T_\beta^n u_n - p\| \le \|u_n - p\|$ and $\|u_n - p\| = \|T_{r_n} z_n - p\| \le \|z_n - p\| \le \|x_n - p\|$, we have that $\{T_\beta^n u_n\}$ and $\{u_n\}$ are also bounded.
Observe that since $\{u_n\}$ is bounded and $\lim_{n\to\infty} \alpha_n = 0$, we obtain

$$\begin{aligned} \|x_{n+1} - u_n\| &= \|\alpha_n u + (1-\alpha_n)u_n - u_n\| \\ &= \alpha_n \|u - u_n\| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{4.2.5}$$

Put $\Omega := F(T) \bigcap EP(f)$. We may assume without loss of generality that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle u - P_\Omega u, x_n - P_\Omega u\rangle = \lim_{j\to\infty} \langle u - P_\Omega u, x_{n_j} - P_\Omega u\rangle,$$

and $x_{n_j} \rightharpoonup w$ as $j \to \infty$. Since $\|x_{n+1} - u_n\| \to 0$ as $n \to \infty$, it follows that $u_{n_j} \rightharpoonup w$ as $j \to \infty$. Next we will show that $w \in F(T)$. Using (4.2.2) we obtain for all $m = 0, 1, 2, ..., n-1$ and for arbitrary $y \in C$

$$\begin{aligned} \|T_\beta^{m+1} x_n - T_\beta y\|^2 &= \|T_\beta(T_\beta^m x_n) - T_\beta y\|^2 \\ &\le \|T_\beta^m x_n - y\|^2 + \frac{2}{1-\beta}\langle T_\beta^m x_n - T_\beta^{m+1} x_n, y - T_\beta y\rangle \\ &= \|T_\beta^m x_n - T_\beta y + T_\beta y - y\|^2 + \frac{2}{1-\beta}\langle T_\beta^m x_n - T_\beta^{m+1} x_n, y - T_\beta y\rangle \\ &= \|T_\beta^m x_n - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle T_\beta^m x_n - T_\beta y, T_\beta y - y\rangle \\ &\quad + \frac{2}{1-\beta}\langle T_\beta^m x_n - T_\beta^{m+1} x_n, y - T_\beta y\rangle. \end{aligned} \tag{4.2.6}$$

Summing (4.2.6) from $m = 0$ to $n-1$ and dividing by n we obtain

$$\begin{aligned} \frac{1}{n}\|T_\beta^n x_n - T_\beta y\|^2 &\le \frac{1}{n}\|x_n - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle z_n - T_\beta y, T_\beta y - y\rangle \\ &\quad + \frac{2}{n(1-\beta)}\langle x_n - T_\beta^n x_n, y - T_\beta y\rangle. \end{aligned} \tag{4.2.7}$$

Replacing n by n_j in (4.2.7) we obtain

$$\begin{aligned} \frac{1}{n_j}\|T_\beta^{n_j} x_{n_j} - T_\beta y\|^2 &\le \frac{1}{n_j}\|x_{n_j} - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle z_{n_j} - T_\beta y, T_\beta y - y\rangle \\ &\quad \frac{2}{n_j(1-\beta)}\langle x_{n_j} - T_\beta^{n_j} x_{n_j}, y - T_\beta y\rangle. \end{aligned} \tag{4.2.8}$$

Since $\{x_n\}$ and $\{T_\beta^n x_n\}$ are bounded, letting $j \to \infty$ in (4.2.8) yields

$$0 \le \|T_\beta y - y\|^2 + 2\langle w - T_\beta y, T_\beta y - y\rangle. \tag{4.2.9}$$

Since $y \in C$ was arbitrary, if we set $y = w$ in (4.2.9) we obtain

$$0 \le \|T_\beta w - w\|^2 - 2\|T_\beta w - w\|^2,$$

from which it follows that $w \in F(T_\beta) = F(T)$.
Since $P_\Omega : H \to \Omega$ is the metric projection, we have

$$\lim_{j\to\infty} \langle u - P_\Omega u, x_{n_j} - P_\Omega u\rangle = \langle u - P_\Omega u, w - P_\Omega u\rangle \le 0.$$

Hence we have $\limsup_{n\to\infty}\langle u - P_\Omega u, x_n - P_\Omega u\rangle \le 0$. Using Lemma 2.2.44 (ii) and (4.2.3) we have

$$\begin{aligned}
\|x_{n+1} - P_\Omega u\|^2 &= \|\alpha_n u + (1-\alpha_n)u_n - P_\Omega u\|^2 \\
&= \|\alpha_n u + (1-\alpha_n)T_{r_n}z_n - P_\Omega u\|^2 \\
&= \|\alpha_n u - \alpha_n P_\Omega u + (1-\alpha_n)T_{r_n}z_n - (1-\alpha_n)P_\Omega u\|^2 \\
&= \|\alpha_n(u - P_\Omega u) + (1-\alpha_n)(T_{r_n}z_n - P_\Omega u)\|^2 \\
&\le (1-\alpha_n)^2\|T_{r_n}z_n - P_\Omega u\|^2 + 2\alpha_n\langle u - P_\Omega u, x_{n+1} - P_\Omega u\rangle \\
&\le (1-\alpha_n)^2\|z_n - P_\Omega u\|^2 + 2\alpha_n\langle u - P_\Omega u, x_{n+1} - P_\Omega u\rangle \\
&\le (1-\alpha_n)^2\|x_n - P_\Omega u\|^2 + 2\alpha_n\langle u - P_\Omega u, x_{n+1} - P_\Omega u\rangle.
\end{aligned} \tag{4.2.10}$$

Since $\alpha_n \to 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\limsup_{n\to\infty}\langle u - P_\Omega u, x_{n+1} - P_\Omega u\rangle \le 0$, it follows from Lemma 2.7.15 that $\lim_{n\to\infty}\|x_n - P_\Omega u\| = 0$.

$$0 \le \|u_n - P_\Omega u\| \le \|u_n - x_{n+1}\| + \|x_{n+1} - P_\Omega u\| \to 0 \text{ as } n \to \infty.$$

Hence $\lim_{n\to\infty}\|u_n - P_\Omega u\| = 0$.
Since $\|x_n - P_\Omega u\| \to 0$, we have $\|x_{n+1} - x_n\| \to 0$. In sequence, we show that $\|z_n - u_n\| \to 0$, as $n \to \infty$. For $p \in F(T)\bigcap EP(f)$, we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|T_{r_n}z_n - T_{r_n}p\|^2 \\
&\le \langle T_{r_n}z_n - T_{r_n}p, z_n - p\rangle \\
&= \langle u_n - p, z_n - p\rangle \\
&= \frac{1}{2}(\|u_n - p\|^2 + \|z_n - p\|^2 - \|z_n - u_n\|^2)
\end{aligned} \tag{4.2.11}$$

and hence $\|u_n - p\|^2 \le \|z_n - p\|^2 - \|z_n - u_n\|^2$.
Therefore, from the convexity of $\|\cdot\|^2$ and (4.2.3), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n u + (1-\alpha_n)u_n - p\|^2 \\
&= \|\alpha_n u + (1-\alpha_n)T_{r_n}z_n - p\|^2 \\
&\le \alpha_n\|u - p\|^2 + (1-\alpha_n)\|T_{r_n}z_n - p\|^2 \\
&\le \alpha_n\|u - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\
&\le \alpha_n\|u - p\|^2 + (1-\alpha_n)(\|z_n - p\|^2 - \|z_n - u_n\|^2) \\
&\le \alpha_n\|u - p\|^2 + (1-\alpha_n)(\|x_n - p\|^2 - \|z_n - u_n\|^2) \\
&\le \alpha_n\|u - p\|^2 + \|x_n - p\|^2 - (1-\alpha_n)\|z_n - u_n\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
(1-\alpha_n)\|z_n - u_n\|^2 &\le \alpha_n\|u - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\le \alpha_n\|u - p\|^2 + \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|).
\end{aligned}$$

So, we have $\|z_n - u_n\| \to 0$, and $\|z_n - P_\Omega u\| \leq \|z_n - u_n\| + \|u_n - P_\Omega u\| \to 0$ *as* $n \to \infty$. Finally, we prove that $w \in EP(f)$. By $u_n = T_{r_n} z_n$ and $u_{n_j} \rightharpoonup w$ as $j \to \infty$, we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \geq 0, \ \ \forall y \in C.$$

From (A2), we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \geq f(y, u_n), \ \ \forall y \in C,$$

and hence

$$\langle y - u_{n_j}, \frac{u_{n_j} - z_{n_j}}{r_{n_j}}\rangle \geq f(y, u_{n_j}), \ \ \forall y \in C.$$

Since $\frac{u_{n_j} - z_{n_j}}{r_{n_j}} \to 0$ and $u_{n_j} \rightharpoonup w$, from (A4) we have $0 \geq f(y, w)$ for all $y \in C$. For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $f(y_t, w) \leq 0$. So, from (A1) and (A4) we have

$$\begin{aligned} 0 &= f(y_t, y_t) \\ &\leq tf(y_t, y) + (1-t)f(y_t, w) \\ &\leq tf(y_t, y) \end{aligned}$$

and hence $0 \leq f(y_t, y)$. From (A3), we have $0 \leq f(w, y)$ for all $y \in C$ and hence $w \in EP(F)$. Therefore $w \in F(T) \bigcap EP(f)$. □

If $f(x, y) = 0, \forall (x, y) \in C \times C$, we have that $u_n = z_n$ for all $n \in \mathbb{N}$. Hence the following Corollary is directly obtained by Theorem 4.2.1

Corollary 4.2.2. ([37], Theorem 3.2) Let C be a nonempty closed convex subset of of a real Hilbert space. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $T : C \to C$ be a k-strictly pseudononspreading mapping with a nonempty fixed point set and $F(T) \bigcap EP(f) \neq \emptyset$. Let $\beta \in [k, 1)$ and let $T_\beta := \beta I + (1-\beta)T$. Let $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1)$ satisfying the conditions:

$$\lim_{n\to\infty} \alpha_n = 0, \ \ \textit{and} \ \sum_{n=1}^{\infty} \alpha_n = \infty$$

and let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in C generated from an arbitrary $x_1 \in C$ by

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \ n \geq 1, \\ z_n = \frac{1}{n}\sum_{m=0}^{n-1} T_\beta^m x_n, \ n \geq 1, \end{cases} \tag{4.2.12}$$

Then $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ converge strongly to $P_{F(T)\bigcap EP(f)} u$, where $P_{F(T)\bigcap EP(f)} : H \to F(T) \bigcap EP(f)$ is the metric projection of H onto $F(T) \bigcap EP(f)$.

Remark 4.2.3. *If T is nonspreading, then T is 0-strictly pseudononspreading. By putting $\beta = 0$, then $T_0 = T$. By Theorem 4.2.1, we obtain the result of Kurokawa and Takahashi* ([29], Theorem 4.1).

Example and numerical results

In this section, we give examples and numerical results for our main theorem.

Example 4.2.4. *Let* $T : [-9,3] \to [-9,3]$ *be define by*

$$Tx = \begin{cases} x, & x \in [-9,0); \\ -3x, & x \in [0,3]. \end{cases}$$

Let $H = \mathbb{R}$ *and* $C = [-9,3]$, *and let* $f(x,y) = y^2 + xy - 2x^2$. *Find* $\hat{x} \in [-9,3]$ *such that*

$$f(\hat{x},y) + \frac{1}{r}\langle y - \hat{x}, \hat{x} - z\rangle \geq 0, \ \ \forall y \in [-9,3]$$

Solution. To see that T is $k-$strictly pseudononspreading, if $x, y \in [-9,0)$, then

$$|Tx - Ty|^2 = |x-y|^2 + k|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle \ \ \forall k \in [0,1),$$

since $|Tx - Ty|^2 = |x-y|^2$, and $k|x - Tx - (y-Ty)|^2 = 2\langle x - Tx, y - Ty\rangle = 0$. For all $x, y \in [0,3]$, we have $|Tx-Ty|^2 = 9|x-y|^2$, $|x-Tx-(y-Ty)|^2 = 16|x-y|^2$ and $2\langle x - Tx, y - Ty\rangle = 32xy \geq 0$. Thus

$$\begin{aligned} |Tx - Ty|^2 = 9|x-y|^2 &= |x-y|^2 + \frac{1}{2}|x - Tx - (y-Ty)|^2 \\ &\leq |x-y|^2 + \frac{1}{2}|x - Tx - (y-Ty)|^2 + 2\langle x - Tx, y - Ty\rangle. \end{aligned}$$

Finally for all $x \in [-9,0), y \in [0,3]$ we have $|Tx-Ty|^2 = |x+3y|^2 = x^2+6xy+9y^2$, $2\langle x - Tx, y - Ty\rangle = 0$, and $\frac{1}{2}|x - Tx - (y - Ty)|^2 = 8y^2$. Hence

$$\begin{aligned} &|x-y|^2 + \frac{1}{2}|x - Tx - (y-Ty)|^2 + 2\langle x - Tx, y - Ty\rangle \\ &= x^2 - 2xy + 9y^2 \\ &= x^2 + 6xy + 9y^2 - 8xy \\ &\geq x^2 + 6xy + 9y^2 \ \ (\text{since} - 8xy \geq 0) \\ &= (x+3y)^2 = |x+3y|^2 = |Tx - Ty|^2. \end{aligned}$$

Hence, for all $x, y \in [-9,3]$, we obtain

$$|Tx - Ty|^2 \leq |x-y|^2 + \frac{1}{2}|x - Tx - (y-Ty)|^2 + 2\langle x - Tx, y - Ty\rangle.$$

Thus T is $\frac{1}{2}-$strictly pseudononspreading, observe that $F(T) = [-9,0]$. We observe that if $x = 1$, $y = 0$,

$$|Tx - Ty|^2 = 9|x-y|^2 = 9 > 1 = |x-y|^2 + 2\langle x - Tx, y - Ty\rangle$$

So T is not nonspreading. For $r > 0$ and $z \in [-9,3]$, by Lemma 2.5.2, there exists $x \in [-9,3]$ such that for each $y \in [-9,3]$

$$\begin{aligned} &f(x,y) + \frac{1}{r}\langle y - x, x - z\rangle \geq 0 \\ &\Leftrightarrow y^2 + xy - 2x^2 + \frac{1}{r}(y-x)(x-z) \geq 0 \\ &\Leftrightarrow ry^2 + rxy - 2rx^2 + xy - x^2 - yz + xz \geq 0 \\ &\Leftrightarrow ry^2 + (rx + x - z)y - (2rx^2 + x^2 - xz) \geq 0. \end{aligned}$$

Put $G(y) = ry^2 + (rx + x - z)y - (2rx^2 + x^2 - xz)$. Then G is a quadratic function of y with coefficient $a = r, b = rx + x - z$ and $c = -(2rx^2 + x^2 - xz)$. We next compute the discriminant Δ of G as follows:

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (rx + x - z)^2 + 4r(2rx^2 + x^2 - xz) \\ &= z^2 - 2(rx + x)z + (rx + x)^2 + 8rx^2 + 4rx^2 - 4rxz \\ &= z^2 - 2rxz - 2xz + r^2x^2 + 2rx^2 + x^2 + 8r^2x^2 + 4rx^2 - 4rxz \\ &= z^2 - 6rxz - 2xz + 9r^2x^2 + 6rx^2 + x^2 \\ &= z^2 - 2(3rx + x) + (9r^2 + 6r + 1)x^2 \\ &= [z - (3r + 1)x]^2\end{aligned}$$

We know that $G(y) \geq 0$ for all $y \in [-9, 3]$. If it has most one solution in $[-9, 3]$, so $\Delta \leq 0$ and hence $z = 3rx + x$. Now we have $x = T_r z = \frac{z}{3r+1}$.
Since $T_\beta := \beta I + (1 - \beta)T$, we obtain

$$T_\beta x = \begin{cases} x, & x \in [-9, 0); \\ (4\beta - 3)x, & x \in [0, 3]. \end{cases}$$

Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated by $x_1 = x \in [-9, 3]$ and

$$\begin{cases} z_n = \frac{1}{n}\sum_{m=0}^{n-1} T_\beta^m x_n, \ n \geq 1 \\ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)u_n, \ n \geq 1. \end{cases} \tag{4.2.13}$$

We next give two numerical results for algorithm (4.2.13).

Let $\alpha_n = \frac{1}{200n}$ and $r_n = \frac{n}{n+1}$. Choose $\beta = \frac{5}{6}$ and $x_1 = u = 1$. Then algorithm (4.2.13) becomes

$$\begin{cases} x_{n+1} = \frac{1}{200n} + (1 - \frac{1}{200n})\left(\frac{z_n}{3r_n+1}\right) \ n \geq 1, \\ z_n = \frac{1}{n}\sum_{m=0}^{n-1} T_\beta^m x_n, \ n \geq 1. \end{cases} \tag{4.2.14}$$

n	x_n	z_n
1	1.000000	1.000000
2	0.403000	0.268667
3	0.091832	0.044215
4	0.015249	0.005648
5	0.002909	0.000869
⋮	⋮	⋮
123	0.000041	0.000001
124	0.000041	0.000000

Table 1:

Let $\alpha_n = \frac{1}{200n}$ and $r_n = \frac{n}{n+1}$. Choose $\beta = \frac{5}{6}$ and $x_1 = u = -1$. Then algorithm (4.2.13) becomes

$$\begin{cases} x_{n+1} = -\frac{1}{200n} + (1 - \frac{1}{200n})\left(\frac{Z_n}{3r_n+1}\right) \ n \geq 1, \\ z_n = \frac{1}{n}\sum_{m=0}^{n-1} T_\beta^m x_n, \ n \geq 1. \end{cases} \tag{4.2.15}$$

n	x_n	z_n
1	-1.00000	-1.00000
2	-0.40300	-0.40300
3	-0.13650	-0.13650
4	-0.04360	-0.04360
5	-0.01406	-0.01406
⋮	⋮	⋮
68	-0.00001	-0.00001
69	0.00000	0.00000

Table 2:

Conclusion. Table 1 and Table 2 show that the sequence $\{x_n\}$ and $\{z_n\}$ converge to 0 which solves both the equilibrium problem of f and the fixed point problem of T. On the other hand, using Lemma 2.5.2 (3) , we can check that $F(T_r) = EP(f) = \{0\}$.

4.3 Common Fixed Points for Two Nonspreading-type Mappings in Hilbert Spaces

In this section, we prove strong convergence of the sequences $\{x_n\}$, $\{w_n\}$ and $\{z_n\}$ defined by using the idea of mean convergence, we introduce a new iterative scheme for finding a common fixed point of two k-strictly pseudononspreading mapping in Hilbert spaces. A strong convergence theorem of the proposed iteration is obtained. Our main result can be appiled for finding a common fixed point of two nonspreading mappings in Hilbert spaces.

Theorem 4.3.1. *Let C be a nonempty closed convex subset of of a real Hilbert space. Let $T, S : C \to C$ be k-strictly pseudononspreading mappings with $F(T)\bigcap F(S) \neq \emptyset$. Let $\beta \in [k, 1)$ and let $T_\beta := \beta I + (1-\beta)T$ and $S_\beta := \beta I + (1-\beta)S$. Let $u \in C$ and let $\{x_n\}_{n=1}^\infty$, $\{w_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ be sequences in C generated from an arbitrary $x_1 \in C$ by*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n)[\beta_n z_n + (1-\beta_n)w_n], \ n \geq 1, \\ w_n = \frac{1}{n}\sum_{k=0}^{n-1} S_\beta^k x_n \ \text{and} \ z_n = \frac{1}{n}\sum_{m=0}^{n-1} T_\beta^m x_n, \ n \geq 1. \end{cases} \tag{4.3.1}$$

Let $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty \subset [0, 1)$ satisfying the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^\infty \beta_n = \infty$.

Then $\{x_n\}$, $\{w_n\}$ and $\{z_n\}$ converge strongly to $P_{F(T)\bigcap F(S)}u$, where P is the metric projection of H onto $F(T)\bigcap F(S)$.

Proof. Let $T_\beta x := \beta x + (1-\beta)Tx$. It is clear that $F(T_\beta) = F(T)$. For $x, y \in C$, we have

$$\begin{aligned}
\|T_\beta x - T_\beta y\|^2 &= \|\beta(x-y) + (1-\beta)(Tx-Ty)\|^2 \\
&= \beta\|x-y\|^2 + (1-\beta)\|Tx-Ty\|^2 - \beta(1-\beta)\|x-Tx-(y-Ty)\|^2 \\
&\le \beta\|x-y\|^2 - \beta(1-\beta)\|x-Tx-(y-Ty)\|^2 \\
&\quad +(1-\beta)\left[\|x-y\|^2 + k\|x-Tx-(y-Ty)\|^2 + 2\langle x-Tx, y-Ty\rangle\right] \\
&= \|x-y\|^2 - (1-\beta)(\beta-k)\|x-Tx-(y-Ty)\|^2 \\
&\quad +2(1-\beta)\langle x-Tx, y-Ty\rangle \\
&\le \|x-y\|^2 + 2(1-\beta)\langle x-Tx, y-Ty\rangle \\
&= \|x-y\|^2 + \frac{2}{(1-\beta)}\langle x-T_\beta x, y-T_\beta y\rangle. \qquad (4.3.2)
\end{aligned}$$

It follows from (4.3.2) that T_β is quasi-nonexpansive. Similarly, S_β is also quasi-nonexpansive. Let $p \in F(T)\bigcap F(S)$. We have

$$\|w_n - p\| = \|\frac{1}{n}\sum_{k=0}^{n-1} S_\beta^k x_n - p\| \le \frac{1}{n}\sum_{k=0}^{n-1}\|S_\beta^k x_n - p\| \le \frac{1}{n}\sum_{k=0}^{n-1}\|x_n - p\|\| = \|x_n - p\|, \qquad (4.3.3)$$

$$\|z_n - p\| = \|\frac{1}{n}\sum_{m=0}^{n-1} T_\beta^m x_n - p\| \le \frac{1}{n}\sum_{m=0}^{n-1}\|T_\beta^m x_n - p\| \le \frac{1}{n}\sum_{m=0}^{n-1}\|x_n - p\|\| = \|x_n - p\|. \qquad (4.3.4)$$

Thus

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n u + (1-\alpha_n)[\beta_n z_n + (1-\beta_n)w_n] - p\| \\
&\le \alpha_n\|u-p\| + (1-\alpha_n)\|\beta_n z_n + (1-\beta_n)w_n - p\| \\
&\le \alpha_n\|u-p\| + (1-\alpha_n)\left[\beta_n\|z_n - p\| + (1-\beta_n)\|w_n - p\|\right] \\
&\le \alpha_n\|u-p\| + (1-\alpha_n)\left[\beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\|\right] \\
&= \alpha_n\|u-p\| + (1-\alpha_n)\|x_n - p\|. \qquad (4.3.5)
\end{aligned}$$

By (4.3.5) and induction, we can show that for all $n \in \mathbb{N}$

$$\|x_n - p\| \le \max\{\|u-p\|, \|x_1 - p\|\}.$$

This implies that $\{x_n\}$, $\{w_n\}$ and $\{z_n\}$ are bounded. Since $\|S_\beta^n x_n - p\| \le \|x_n - p\|$ and $\|T_\beta^n x_n - p\| \le \|x_n - p\|$, we have that $\{S_\beta^n x_n\}$ and $\{T_\beta^n x_n\}$ are also bounded.

Using (4.3.2) we obtain for all $k = 0, 1, 2, ..., n-1$ and for arbitrary $y \in C$,

$$\begin{aligned} \|S_\beta^{m+1}x_n - S_\beta y\|^2 &= \|S_\beta(S_\beta^m x_n) - S_\beta y\|^2 \\ &\leq \|S_\beta^m x_n - y\|^2 + \frac{2}{1-\beta}\langle S_\beta^m x_n - S_\beta^{m+1}x_n, y - S_\beta y\rangle \\ &= \|S_\beta^m x_n - S_\beta y + S_\beta y - y\|^2 + \frac{2}{1-\beta}\langle S_\beta^m x_n - S_\beta^{m+1}x_n, y - S_\beta y\rangle \\ &= \|S_\beta^m x_n - S_\beta y\|^2 + \|S_\beta y - y\|^2 + 2\langle S_\beta^m x_n - S_\beta y, S_\beta y - y\rangle \\ &\quad + \frac{2}{1-\beta}\langle S_\beta^m x_n - S_\beta^{m+1}x_n, y - S_\beta y\rangle. \end{aligned} \tag{4.3.6}$$

Summing (4.3.6) from $k = 0$ to $n-1$ and dividing by n we obtain

$$\begin{aligned} \frac{1}{n}\|S_\beta^n x_n - S_\beta y\|^2 &\leq \frac{1}{n}\|x_n - S_\beta y\|^2 + \|S_\beta y - y\|^2 + 2\langle w_n - S_\beta y, S_\beta y - y\rangle \\ &\quad + \frac{2}{n(1-\beta)}\langle x_n - S_\beta^n x_n, y - S_\beta y\rangle. \end{aligned} \tag{4.3.7}$$

Replacing n by n_j in (4.3.7) we obtain

$$\begin{aligned} \frac{1}{n_j}\|S_\beta^{n_j} x_{n_j} - S_\beta y\|^2 &\leq \frac{1}{n_j}\|x_{n_j} - S_\beta y\|^2 + \|S_\beta y - y\|^2 + 2\langle w_{n_j} - S_\beta y, S_\beta y - y\rangle \\ &\quad \frac{2}{n_j(1-\beta)}\langle x_{n_j} - S_\beta^{n_j} x_{n_j}, y - S_\beta y\rangle. \end{aligned} \tag{4.3.8}$$

Since $\{x_n\}$ and $\{S_\beta^n x_n\}$ are bounded, letting $j \to \infty$ in (4.3.8) yields

$$0 \leq \|S_\beta y - y\|^2 + 2\langle w - S_\beta y, S_\beta y - y\rangle. \tag{4.3.9}$$

Since $y \in C$ is arbitrary, if we set $y = w$ in (4.3.9) we obtain

$$0 \leq \|S_\beta w - w\|^2 - 2\|S_\beta w - w\|^2,$$

from which it follows that $w \in F(S_\beta) = F(S)$. By using the same arguments as above, we also obtain $w \in F(T_\beta) = F(T)$. Observe that since $\{w_n\}$ and $\{z_n\}$ are bounded and $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \beta_n = 0$, we obtain

$$\begin{aligned} \|x_{n+1} - w_n\| &= \|\alpha_n u + (1-\alpha_n)[\beta_n z_n + (1-\beta_n)w_n] - w_n\| \\ &\leq \alpha_n\|u - w_n\| + (1-\alpha_n)\|\beta_n z_n + (1-\beta_n)w_n - w_n\| \\ &= \alpha_n\|u - w_n\| + (1-\alpha_n)\beta_n\|z_n - w_n\| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Next, we show that $\limsup_{n\to\infty}\langle u - P_{F(T)\bigcap F(S)}u, x_n - P_{F(T)\bigcap F(S)}u\rangle \leq 0$. Put $\Omega := F(T)\bigcap F(S)$. We may assume without loss of generality that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - P_\Omega u, x_n - P_\Omega u\rangle = \lim_{j\to\infty}\langle u - P_\Omega u, x_{n_j} - P_\Omega u\rangle,$$

and $x_{n_j} \rightharpoonup w$ as $j \to \infty$ for some $w \in C$. Since $\|x_{n+1} - w_n\| \to 0$ as $n \to \infty$, it follows that $w_{n_j} \rightharpoonup w$ as $j \to \infty$. Since $P_\Omega : H \to \Omega$ is the metric projection, we have

$$\lim_{j\to\infty}\langle u - P_\Omega u, x_{n_j} - P_\Omega u\rangle = \langle u - P_\Omega u, w - P_\Omega u\rangle \leq 0.$$

Hence we have $\limsup_{n\to\infty}\langle u - P_\Omega u, x_n - P_\Omega u\rangle \le 0$. Using Lemma 2.2.44 (ii) and (4.3.4) we have

$$\begin{aligned}
\|x_{n+1} - P_\Omega u\|^2 &= \|\alpha_n u + (1-\alpha_n)[\beta_n z_n + (1-\beta_n)w_n] - P_\Omega u\|^2 \\
&= \|\alpha_n(u - P_\Omega u) + (1-\alpha_n)[\beta_n z_n + (1-\beta_n)w_n - P_\Omega u]\|^2 \\
&\le (1-\alpha_n)^2\|\beta_n z_n + (1-\beta_n)w_n - P_\Omega u\|^2 \\
&\quad +2\alpha_n\langle u - P_\Omega u, x_{n+1} - P_\Omega u\rangle \\
&\le (1-\alpha_n)\|\beta_n(z_n - P_\Omega u) + (1-\beta_n)(w_n - P_\Omega u)\|^2 \\
&\quad +2\alpha_n\langle u - P_\Omega u, x_{n+1} - P_\Omega u\rangle \\
&\le (1-\alpha_n)\big[\beta_n\|z_n - P_\Omega u\|^2 + (1-\beta_n)\|w_n - P_\Omega u\|^2\big] \\
&\quad +2\alpha_n\langle u - P_\Omega u, x_{n+1} - P_\Omega u\rangle \\
&\le (1-\alpha_n)\big[\beta_n\|x_n - P_\Omega u\|^2 + (1-\beta_n)\|x_n - P_\Omega u\|^2\big] \\
&\quad +2\alpha_n\langle u - P_\Omega u, x_{n+1} - P_\Omega u\rangle \\
&\le (1-\alpha_n)\|x_n - P_\Omega u\|^2 + 2\alpha_n\langle u - P_\Omega u, x_{n+1} - P_\Omega u\rangle.
\end{aligned} \tag{4.3.10}$$

Since $\alpha_n \to 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\limsup_{n\to\infty}\langle u - P_\Omega u, x_{n+1} - P_\Omega u\rangle \le 0$, it follows from Lemma 2.7.15 that $\lim_{n\to\infty}\|x_n - P_\Omega u\| = 0$. By (4.3.3), we obtain

$$0 \le \|w_n - P_\Omega u\| \le \|x_n - P_\Omega u\| \to 0 \text{ as } n \to \infty.$$

That is $\lim_{n\to\infty}\|w_n - P_\Omega u\| = 0$ and by (4.3.4), we have

$$0 \le \|z_n - P_\Omega u\| \le \|x_n - P_\Omega u\| \to 0 \text{ as } n \to \infty.$$

Hence $\lim_{n\to\infty}\|z_n - P_\Omega u\| = 0$. The proof is now complete. □