

Chapter 3

Approximation Method for Finite and Infinite Family of Nonexpansive Mappings

3.1 A New General Iterative Method for a Finite Family of Nonexpansive Mappings in Hilbert Spaces

In this section, we prove a strong convergence theorem of a new general iterative method by using the K - mapping for finding a common fixed point of a finite family of nonexpansive mappings in the framework of Hilbert spaces. A strong convergence theorem of the purposed iterative method is established under some certain control conditions. Our results improve and extend the results announced by many others.

Theorem 3.1.1. *Let H be a Hilbert space, C a closed convex nonempty subset of H . Let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ and let $f : C \rightarrow C$ be a contraction. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and let K_n be defined by (1.1.16). Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $x_0 \in C$ and given $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $(0, 1)$ and suppose that the following conditions are satisfied:*

- (C1) $\alpha_n \rightarrow 0$;
- (C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (C3) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;
- (C4) $\sum_{n=1}^{\infty} |\gamma_{n,i} - \gamma_{n-1,i}| < \infty$, for all $i = 1, 2, \dots, N$ and $\{\gamma_{n,i}\}_{i=1}^N \subset [a, b]$, where $0 < a \leq b < 1$;
- (C5) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
- (C6) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

If $\{x_n\}_{n=1}^{\infty}$ is the composite process defined by

$$\begin{aligned} x_0 &= x \in C \text{ arbitrarily chosen,} \\ y_n &= \beta_n x_n + (1 - \beta_n) K_n x_n, \\ x_{n+1} &= P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n). \end{aligned} \tag{3.1.1}$$

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $q \in F$, which also solves the following variational inequality:

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in F.$$

Proof. First, we observe that $\{x_n\}_{n=0}^{\infty}$ is bounded. Indeed, take a point $u \in F$ and notice that

$$\|y_n - u\| \leq \beta_n \|x_n - u\| + (1 - \beta_n) \|K_n x_n - u\| \leq \|x_n - u\|. \tag{3.1.2}$$

Since $\alpha_n \rightarrow 0$, we may assume that $\alpha_n \leq \|A\|^{-1}$ for all n . By Lemma 2.7.12, we have $\|I - \alpha_n A\| \leq 1 - \alpha_n \bar{\gamma}$ for all n .

It follows that

$$\begin{aligned}
\|x_{n+1} - u\| &= \|P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n) - P_C(u)\| \\
&\leq \|\alpha_n(\gamma f(x_n) - Au) + (I - \alpha_n A)(y_n - u)\| \\
&\leq \alpha_n \|\gamma f(x_n) - Au\| + (1 - \alpha_n \bar{\gamma}) \|y_n - u\| \\
&\leq \alpha_n \|\gamma f(x_n) - \gamma f(u)\| + \alpha_n \|\gamma f(u) - Au\| + (1 - \alpha_n \bar{\gamma}) \|y_n - u\| \\
&\leq \alpha \gamma \alpha_n \|x_n - u\| + \alpha_n \|\gamma f(u) - Au\| + (1 - \alpha_n \bar{\gamma}) \|x_n - u\| \\
&= (1 - (\bar{\gamma} - \gamma \alpha) \alpha_n) \|x_n - u\| + \alpha_n \|\gamma f(u) - Au\| \\
&= (1 - (\bar{\gamma} - \gamma \alpha) \alpha_n) \|x_n - u\| + (\bar{\gamma} - \gamma \alpha) \alpha_n \frac{\|\gamma f(u) - Au\|}{\bar{\gamma} - \gamma \alpha} \\
&\leq \max \left\{ \|x_n - u\|, \frac{\|\gamma f(u) - Au\|}{\bar{\gamma} - \gamma \alpha} \right\}. \tag{3.1.3}
\end{aligned}$$

By simple inductions, we have

$$\|x_n - u\| \leq \max \left\{ \|x_0 - u\|, \frac{\|\gamma f(u) - Au\|}{\bar{\gamma} - \gamma \alpha} \right\}, n \geq 0.$$

Therefore $\{x_n\}$ is bounded, so are $\{y_n\}$, and $\{f(x_n)\}$. Since K_n is nonexpansive and $y_n = \beta_n x_n + (1 - \beta_n) K_n x_n$, we also have

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leq \|(\beta_{n+1} x_{n+1} + (1 - \beta_{n+1}) K_{n+1} x_{n+1}) - (\beta_n x_n + (1 - \beta_n) K_n x_n)\| \\
&= \|\beta_{n+1} x_{n+1} - \beta_{n+1} x_n + \beta_{n+1} x_n - \beta_n x_n \\
&\quad + (1 - \beta_{n+1})(K_{n+1} x_{n+1} - K_{n+1} x_n) \\
&\quad + (1 - \beta_{n+1})(K_{n+1} x_n - K_n x_n) \\
&\quad + (1 - \beta_{n+1}) K_n x_n - (1 - \beta_n) K_n x_n\| \\
&\leq \beta_{n+1} \|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| \|x_n\| \\
&\quad + (1 - \beta_{n+1}) \|K_{n+1} x_{n+1} - K_{n+1} x_n\| \\
&\quad + (1 - \beta_{n+1}) \|K_{n+1} x_n - K_n x_n\| + |\beta_n - \beta_{n+1}| \|K_n x_n\| \\
&\leq \beta_{n+1} \|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| \|x_n\| + (1 - \beta_{n+1}) \|x_{n+1} - x_n\| \\
&\quad + (1 - \beta_{n+1}) \|K_{n+1} x_n - K_n x_n\| + |\beta_n - \beta_{n+1}| \|K_n x_n\| \\
&= \|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| \|x_n\| \\
&\quad + (1 - \beta_{n+1}) \|K_{n+1} x_n - K_n x_n\| + |\beta_n - \beta_{n+1}| \|K_n x_n\|
\end{aligned} \tag{3.1.4}$$

By Lemma 2.6.4, we obtain

$$\|K_n x_{n-1} - K_{n-1} x_{n-1}\| \leq M \sum_{j=1}^N |\gamma_{n,j} - \gamma_{n-1,j}|, \tag{3.1.5}$$

where $M = \sup\{\sum_{j=2}^N (\|T_j U_{n,j-1} x_n\| + \|U_{n,j-1} x_n\|) + \|T_1 x_n\| + \|x_n\|\}$.

By (3.1.4) and (3.1.5), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n) \\
&\quad - (P_C(\alpha_{n-1} \gamma f(x_{n-1}) + (I - \alpha_{n-1} A)y_{n-1}))\| \\
&\leq \|(I - \alpha_n A)(y_n - y_{n-1}) - (\alpha_n - \alpha_{n-1})Ay_{n-1} \\
&\quad + \gamma \alpha_n (f(x_n) - f(x_{n-1})) + \gamma (\alpha_n - \alpha_{n-1})f(x_{n-1})\| \\
&\leq (1 - \alpha_n \bar{\gamma})\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Ay_{n-1}\| \\
&\quad + \gamma \alpha_n \|x_n - x_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\leq (1 - \alpha_n \bar{\gamma})(\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| \\
&\quad + |1 - \beta_n|\|K_n x_{n-1} - K_{n-1} x_{n-1}\| + |\beta_{n-1} - \beta_n|\|K_{n-1} x_{n-1}\|) \\
&\quad + |\alpha_n - \alpha_{n-1}|\|Ay_{n-1}\| + \gamma \alpha_n \|x_n - x_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\leq (1 - \alpha_n \bar{\gamma})\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| \\
&\quad + |1 - \beta_n|\|K_n x_{n-1} - K_{n-1} x_{n-1}\| + |\beta_{n-1} - \beta_n|\|K_{n-1} x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\|Ay_{n-1}\| + \gamma \alpha_n \|x_n - x_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\leq (1 - (\bar{\gamma} - \gamma \alpha) \alpha_n)\|x_n - x_{n-1}\| + L|\beta_{n-1} - \beta_n| + M'|\alpha_n - \alpha_{n-1}| \\
&\quad + |1 - \beta_n| M \sum_{j=1}^N |\gamma_{n,j} - \gamma_{n-1,j}|,
\end{aligned}$$

where $L = \sup\{\|x_{n-1}\| + \|K_{n-1} x_{n-1}\| : n \in \mathbb{N}\}$, $M' = \sup\{\|Ay_{n-1}\| + \gamma \|f(x_{n-1})\| : n \in \mathbb{N}\}$. Since $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$ and $\sum_{j=1}^{\infty} |\gamma_{n,j} - \gamma_{n-1,j}| < \infty$, for all $j = 1, 2, \dots, N$, by Lemma 2.7.1, we obtain $\|x_{n+1} - x_n\| \rightarrow 0$. It follows that

$$\begin{aligned}
\|x_{n+1} - y_n\| &= \|P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n) - P_C(y_n)\| \\
&\leq \|\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n - y_n\| \\
&= \alpha_n \|\gamma f(x_n) + Ay_n\|.
\end{aligned}$$

Since $\alpha_n \rightarrow 0$ and $\{f(x_n)\}, \{Ay_n\}$ are bounded, we have $\|x_{n+1} - y_n\| \rightarrow 0$ as $n \rightarrow \infty$. Since

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|,$$

it implies that $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$.

On the other hand, we have

$$\|K_n x_n - x_n\| \leq \|x_n - y_n\| + \|y_n - K_n x_n\| = \|x_n - y_n\| + \beta_n \|x_n - K_n x_n\|,$$

which implies $(1 - \beta_n)\|K_n x_n - x_n\| \leq \|x_n - y_n\|$.

From condition (C3) and $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$, we obtain

$$\|K_n x_n - x_n\| \rightarrow 0. \tag{3.1.6}$$

By (C4), we have $\lim_{n \rightarrow \infty} \gamma_{n,i} = \gamma_i \in [a, b]$ for all $i = 1, 2, \dots, N$. Let K be the K -mapping generated by T_1, \dots, T_N and $\gamma_1, \dots, \gamma_N$. Next, we show that

$$\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle \leq 0, \tag{3.1.7}$$

where $q = \lim_{t \rightarrow 0} x_t$ with x_t being the fixed point of the contraction $x \mapsto t\gamma f(x) + (I - tA)Kx$. Thus, x_t solves the fixed point equation $x_t = t\gamma f(x_t) + (I - tA)Kx_t$. By Lemma 2.7.13 and Lemma 2.6.3, we have $q \in F$ and $\langle \gamma f(q) - Aq, p - q \rangle \leq 0$ for all $p \in F$. It follows by (3.1.6) and Lemma 2.6.5 that $\|Kx_n - x_n\| \rightarrow 0$. Thus, we have $\|x_t - x_n\| = \|(I - tA)(Kx_t - x_n) + t(\gamma f(x_t) - Ax_n)\|$. It follows from Lemma 2.2.44 (ii) that for $0 < t < \|A\|^{-1}$,

$$\begin{aligned}
\|x_t - x_n\|^2 &= \|(I - tA)(Kx_t - x_n) + t(\gamma f(x_t) - Ax_n)\|^2 \\
&\leq (1 - \bar{\gamma}t)^2 \|Kx_t - x_n\|^2 + 2t \langle \gamma f(x_t) - Ax_n, x_t - x_n \rangle \\
&\leq (1 - \bar{\gamma}t)^2 (\|Kx_t - Kx_n\|^2 + 2\|Kx_t - Kx_n\| \|Kx_n - x_n\| \\
&\quad + \|Kx_n - x_n\|^2) + 2t (\langle \gamma f(x_t) - Ax_t, x_t - x_n \rangle \\
&\quad + \langle Ax_t - Ax_n, x_t - x_n \rangle) \\
&\leq (1 - 2\bar{\gamma}t + (\bar{\gamma}t)^2) \|x_t - x_n\|^2 + f_n(t) \\
&\quad + 2t \langle \gamma f(x_t) - Ax_t, x_t - x_n \rangle + 2t \langle Ax_t - Ax_n, x_t - x_n \rangle,
\end{aligned} \tag{3.1.8}$$

where

$$f_n(t) = (2\|x_t - x_n\| + \|x_n - Kx_n\|) \|x_n - Kx_n\| \rightarrow 0, \text{ as } n \rightarrow \infty. \tag{3.1.9}$$

It follows that

$$\begin{aligned}
\langle Ax_t - \gamma f(x_t), x_t - x_n \rangle &\leq \left(\frac{-2\bar{\gamma}t + (\bar{\gamma}t)^2}{2t} \right) \|x_t - x_n\|^2 \\
&\quad + \frac{1}{2t} f_n(t) + \langle Ax_t - Ax_n, x_t - x_n \rangle \\
&\leq \left(\frac{-2 + \bar{\gamma}t}{2} \right) \bar{\gamma} \|x_t - x_n\|^2 \\
&\quad + \frac{1}{2t} f_n(t) + \langle Ax_t - Ax_n, x_t - x_n \rangle \\
&\leq \left(-1 + \frac{\bar{\gamma}t}{2} \right) \langle Ax_t - Ax_n, x_t - x_n \rangle \\
&\quad + \frac{1}{2t} f_n(t) + \langle Ax_t - Ax_n, x_t - x_n \rangle \\
&\leq \frac{\bar{\gamma}t}{2} \langle Ax_t - Ax_n, x_t - x_n \rangle + \frac{1}{2t} f_n(t) \\
&\leq \frac{t}{2} M_0 + \frac{1}{2t} f_n(t),
\end{aligned} \tag{3.1.10}$$

where $M_0 > 0$ is a constant such that $M_0 \geq \bar{\gamma} \langle Ax_t - Ax_n, x_t - x_n \rangle$ for all $t \in (0, 1)$ and $n \geq 1$. Letting $n \rightarrow \infty$ in (3.1.10) and (3.1.9), we get

$$\limsup_{n \rightarrow \infty} \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leq \frac{t}{2} M_0, \tag{3.1.11}$$

Taking $t \rightarrow 0$ in (3.1.11), we have

$$\limsup_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leq 0. \tag{3.1.12}$$



On the other hand, one has

$$\begin{aligned}
\langle \gamma f(q) - Aq, x_n - q \rangle &= \langle \gamma f(q) - Aq, x_n - q \rangle - \langle \gamma f(q) - Aq, x_n - x_t \rangle \\
&\quad + \langle \gamma f(q) - Aq, x_n - x_t \rangle - \langle \gamma f(q) - Ax_t, x_n - x_t \rangle \\
&\quad + \langle \gamma f(q) - Ax_t, x_n - x_t \rangle - \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \\
&\quad + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle. \\
&= \langle \gamma f(q) - Aq, x_t - q \rangle + \langle Ax_t - Aq, x_n - x_t \rangle \\
&\quad + \langle \gamma f(q) - \gamma f(x_t), x_n - x_t \rangle + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \\
&\leq \| \gamma f(q) - Aq \| \| x_t - q \| + (\| A \| \| x_t - q \| + \gamma \alpha \| x_t - q \|) \| x_n - x_t \| \\
&\quad + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \\
&= \| \gamma f(q) - Aq \| \| x_t - q \| + (\| A \| + \gamma \alpha) \| x_t - q \| \| x_n - x_t \| \\
&\quad + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle &\leq \| \gamma f(q) - Aq \| \| x_t - q \| \\
&\quad + (\| A \| + \gamma \alpha) \| x_t - q \| \limsup_{n \rightarrow \infty} \| x_n - x_t \| \\
&\quad + \limsup_{n \rightarrow \infty} \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle.
\end{aligned}$$

Therefore, from (3.1.12) and $\lim_{t \rightarrow 0} \| x_t - q \| = 0$, we have

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle &= \limsup_{t \rightarrow 0} (\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle) \\
&\leq \limsup_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \\
&\leq 0.
\end{aligned}$$

Hence (3.1.7) holds. Finally, we prove $x_n \rightarrow q$. By using (3.1.2) and together with Schwarz inequality, we have

$$\begin{aligned}
\| x_{n+1} - q \|^2 &= \| P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n) - P_C(q) \|^2 \\
&\leq \| \alpha_n (\gamma f(x_n) - Aq) + (I - \alpha_n A)(y_n - q) \|^2 \\
&= \| (I - \alpha_n A)(y_n - q) \|^2 + \alpha_n^2 \| \gamma f(x_n) - Aq \|^2 \\
&\quad + 2\alpha_n \langle (I - \alpha_n A)(y_n - q), \gamma f(x_n) - Aq \rangle \\
&\leq (1 - \alpha_n \bar{\gamma})^2 \| y_n - q \|^2 + \alpha_n^2 \| \gamma f(x_n) - Aq \|^2 \\
&\quad + 2\alpha_n \langle y_n - q, \gamma f(x_n) - Aq \rangle - 2\alpha_n^2 \langle A(y_n - q), \gamma f(x_n) - Aq \rangle \\
&\leq (1 - \alpha_n \bar{\gamma})^2 \| x_n - q \|^2 + \alpha_n^2 \| \gamma f(x_n) - Aq \|^2 \\
&\quad + 2\alpha_n \langle y_n - q, \gamma f(x_n) - \gamma f(q) \rangle + 2\alpha_n \langle y_n - q, \gamma f(q) - Aq \rangle \\
&\quad - 2\alpha_n^2 \langle A(y_n - q), \gamma f(x_n) - Aq \rangle \\
&\leq (1 - \alpha_n \bar{\gamma})^2 \| x_n - q \|^2 + \alpha_n^2 \| \gamma f(x_n) - Aq \|^2 \\
&\quad + 2\alpha_n \| y_n - q \| \| \gamma f(x_n) - \gamma f(q) \| + 2\alpha_n \langle y_n - q, \gamma f(q) - Aq \rangle \\
&\quad - 2\alpha_n^2 \langle A(y_n - q), \gamma f(x_n) - Aq \rangle
\end{aligned}$$

$$\begin{aligned}
&\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\gamma \alpha_n \|y_n - q\| \|x_n - q\| + 2\alpha_n \langle y_n - q, \gamma f(q) - Aq \rangle \\
&\quad - 2\alpha_n^2 \langle A(y_n - q), \gamma f(x_n) - Aq \rangle \\
&\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\gamma \alpha_n \|x_n - q\|^2 + 2\alpha_n \langle y_n - q, \gamma f(q) - Aq \rangle \\
&\quad - 2\alpha_n^2 \langle A(y_n - q), \gamma f(x_n) - Aq \rangle \\
&\leq ((1 - \alpha_n \bar{\gamma})^2 + 2\gamma \alpha_n) \|x_n - q\|^2 + 2\alpha_n \langle y_n - q, \gamma f(q) - Aq \rangle \\
&\quad + \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 + 2\alpha_n^2 \|A(y_n - q)\| \|\gamma f(x_n) - Aq\| \\
&= (1 - 2(\bar{\gamma} - \gamma \alpha) \alpha_n) \|x_n - q\|^2 + \alpha_n (2 \langle y_n - q, \gamma f(q) - Aq \rangle \\
&\quad + \alpha_n (\|\gamma f(x_n) - Aq\|^2 + 2 \|A(y_n - q)\| \|\gamma f(x_n) - Aq\| \\
&\quad + \bar{\gamma}^2 \|x_n - q\|^2))
\end{aligned}$$

Since $\{x_n\}$, $\{f(x_n)\}$ and $\{y_n\}$ are bounded, we can take a constant $\eta > 0$ such that

$$\eta \geq \|\gamma f(x_n) - Aq\|^2 + 2 \|A(y_n - q)\| \|\gamma f(x_n) - Aq\| + \bar{\gamma}^2 \|x_n - q\|^2$$

for all $n \geq 0$. It then follows that

$$\|x_{n+1} - q\|^2 \leq (1 - 2(\bar{\gamma} - \gamma \alpha) \alpha_n) \|x_n - q\|^2 + \alpha_n \beta_n \quad (3.1.13)$$

where $\beta_n = 2 \langle y_n - q, \gamma f(q) - Aq \rangle + \eta \alpha_n$. By $\limsup_{n \rightarrow \infty} \langle (\gamma f - A)q, y_n - q \rangle \leq 0$, we get $\limsup_{n \rightarrow \infty} \beta_n \leq 0$. By applying Lemma 2.7.15 to (3.1.13), we can conclude that $x_n \rightarrow q$. This completes the proof. \square

If $A = I$ and $\gamma = 1$ in Theorem 3.1.1, we obtain the following result.

Corollary 3.1.2. *Let H be a Hilbert space, C a closed convex nonempty subset of H , and let $f : C \rightarrow C$ be a contraction. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and let K_n be defined by (1.1.16). Assume that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $x_0 \in C$ and given $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are sequences in $(0, 1)$ and suppose that the following conditions are satisfied:*

- (C1) $\alpha_n \rightarrow 0$;
- (C2) $\sum_{n=0}^\infty \alpha_n = \infty$;
- (C3) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;
- (C4) $\sum_{n=1}^\infty |\gamma_{n,i} - \gamma_{n-1,i}| < \infty$, for all $i = 1, 2, \dots, N$ and $\{\gamma_{n,i}\}_{i=1}^N \subset [a, b]$, where $0 < a \leq b < 1$;
- (C5) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
- (C6) $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$.

If $\{x_n\}_{n=1}^\infty$ is the composite process defined by

$$\begin{aligned}
y_n &= \beta_n x_n + (1 - \beta_n) K_n x_n \\
x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) y_n,
\end{aligned}$$

then $\{x_n\}_{n=1}^\infty$ converges strongly to $q \in F$, which also solves the following variational inequality:

$$\langle (f - I)q, p - q \rangle \leq 0, \quad \forall p \in F.$$

If $N = 1$, $A = I$, $\gamma = 1$ and $f(x) = u$ for all $x \in C$ in Theorem 3.1.1, we get the results of Kim and Xu [25]

Corollary 3.1.3. *Let H be a Hilbert space, C a closed convex nonempty subset of H , and let $f : C \rightarrow C$ be a contraction. Let T be a nonexpansive mapping of C into itself. $F(T) \neq \emptyset$. Let $x_0 \in C$ and given $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are sequences in $(0, 1)$ and suppose that the following conditions are satisfied:*

- (C1) $\alpha_n \rightarrow 0$;
- (C2) $\sum_{n=0}^\infty \alpha_n = \infty$;
- (C3) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;
- (C4) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
- (C5) $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$.

If $\{x_n\}_{n=1}^\infty$ is the composite process defined by

$$\begin{aligned} y_n &= \beta_n x_n + (1 - \beta_n) T x_n \\ x_{n+1} &= \alpha_n u + (I - \alpha_n) y_n, \end{aligned}$$

then $\{x_n\}_{n=1}^\infty$ converges strongly to $q \in F$, which also solves the following variational inequality:

$$\langle u - q, p - q \rangle \leq 0, \quad \forall p \in F.$$

3.2 Strong Convergence Theorems for a Countable Family of Nonexpansive Mappings in Banach Spaces

Throughout this section, we assume that C is a nonempty closed convex subset of a uniformly smooth Banach space E , f is a contraction mapping of C onto itself with coefficient $\alpha \in (0, 1)$ and $\{T_n\}$ is a sequence of nonexpansive mappings of C into itself with $F := \bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. To obtain our main result, we need the following lemmas.

Lemma 3.2.1. *Suppose that $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\beta_n\}$ is a sequence in $[0, 1)$ and let $\{x_n\}$ be the iteration sequence defined by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) T_n x_n, \quad n \geq 1, \quad (3.2.1)$$

where $\{T_n\}$ is a sequence of nonexpansive mappings of C into itself with $F := \bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Then $\{x_n\}$ is bounded.

Proof. Let $p \in F$. By the definition of x_n , we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)T_n x_n - p\| \\
&= \|\alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)T_n x_n - \alpha_n p + \alpha_n p - \beta_n p + \beta_n p - p\| \\
&= \|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) + (1 - \alpha_n - \beta_n)T_n x_n - (1 - \alpha_n - \beta_n)p\| \\
&\leq \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + (1 - \alpha_n - \beta_n) \|T_n x_n - p\| \\
&\leq \alpha_n (\|f(x_n) - f(p)\| + \|f(p) - p\|) + \beta_n \|x_n - p\| \\
&\quad + (1 - \alpha_n - \beta_n) (\|T_n x_n - T_n p\| + \|T_n p - p\|) \\
&\leq \alpha_n (\|f(x_n) - f(p)\| + \|f(p) - p\|) + (1 - \alpha_n) \|x_n - p\| \\
&\quad + (1 - \alpha_n - \beta_n) \|T_n p - p\| \\
&\leq (1 - \alpha_n) \|x_n - p\| + \alpha_n \alpha \|x_n - p\| + \alpha_n \|f(p) - p\| \\
&= ((1 - \alpha_n) - \alpha_n \alpha) \|x_n - p\| + \alpha_n \|f(p) - p\| \\
&= (1 - (1 - \alpha)\alpha_n) \|x_n - p\| + \alpha_n \|f(p) - p\| \\
&\leq \max \left\{ \|x_n - p\|, \frac{1}{1 - \alpha} \|f(p) - p\| \right\}. \tag{3.2.2}
\end{aligned}$$

By simple inductions, we have

$$\|x_n - p\| \leq \max \left\{ \|x_1 - p\|, \frac{1}{1 - \alpha} \|f(p) - p\| \right\}, \quad n \geq 1.$$

and hence $\{x_n\}$ is bounded. \square

Lemma 3.2.2. *Let $\{\beta_n\}$ and $\{\alpha_n\}$ be sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. If $\{T_n\}$ satisfies the AKTT-condition, then $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ where $\{x_n\}$ is the sequence generated by (3.2.1).*

Proof. By Lemma 3.2.1, $\{x_n\}$ is bounded. It follows that $\{f(x_n) - T_n x_n\}$ and $\{x_n - f(x_n)\}$ are bounded. Let $M = \sup\{\|f(x_n) - T_n x_n\|, \|x_n - f(x_n)\| : n \in \mathbb{N}\}$. From the definition of $\{x_n\}$, we have

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &= \|\alpha_{n+1} f(x_{n+1}) + \beta_{n+1} x_{n+1} + (1 - \alpha_{n+1} - \beta_{n+1})T_{n+1} x_{n+1} \\
&\quad - \alpha_n f(x_n) - \beta_n x_n - (1 - \alpha_n - \beta_n)T_n x_n\| \\
&= \|\alpha_{n+1} f(x_{n+1}) + \beta_{n+1} x_{n+1} + (1 - \alpha_{n+1} - \beta_{n+1})T_{n+1} x_{n+1} \\
&\quad - (1 - \alpha_{n+1} - \beta_{n+1})T_{n+1} x_n + (1 - \alpha_{n+1} - \beta_{n+1})T_{n+1} x_n \\
&\quad - \alpha_{n+1} f(x_n) + \alpha_{n+1} f(x_n) - \beta_{n+1} x_n + \beta_{n+1} x_n \\
&\quad - \alpha_n f(x_n) - \beta_n x_n - (1 - \alpha_n - \beta_n)T_n x_n\| \\
&= \|(1 - \alpha_{n+1} - \beta_{n+1})(T_{n+1} x_{n+1} - T_{n+1} x_n) + \alpha_{n+1}(f(x_{n+1}) - f(x_n)) \\
&\quad + \beta_{n+1}(x_{n+1} - x_n) + (\alpha_{n+1} - \alpha_n)f(x_n) + (\beta_{n+1} - \beta_n)x_n \\
&\quad + (1 - \alpha_{n+1} - \beta_{n+1})T_{n+1} x_n - (1 - \alpha_{n+1} - \beta_{n+1})T_n x_n \\
&\quad + (1 - \alpha_{n+1} - \beta_{n+1})T_n x_n - (1 - \alpha_n - \beta_n)T_n x_n\|
\end{aligned}$$

$$\begin{aligned}
&= \|(1 - \alpha_{n+1} - \beta_{n+1})(T_{n+1}x_{n+1} - T_{n+1}x_n) \\
&\quad + \alpha_{n+1}(f(x_{n+1}) - f(x_n)) + \beta_{n+1}(x_{n+1} - x_n) \\
&\quad + (\alpha_{n+1} - \alpha_n)f(x_n) + (\beta_{n+1} - \beta_n)x_n - (\alpha_{n+1} - \alpha_n)T_nx_n \\
&\quad - (\beta_{n+1} - \beta_n)T_nx_n + (1 - \alpha_{n+1} - \beta_{n+1})(T_{n+1}x_n - T_nx_n)\| \\
&\leq (1 - \alpha_{n+1} - \beta_{n+1})\|x_{n+1} - x_n\| + \alpha_{n+1}\|x_{n+1} - x_n\| \\
&\quad + \beta_{n+1}\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|f(x_n) - T_nx_n\| \\
&\quad + |\beta_{n+1} - \beta_n|\|x_n - f(x_n)\| \\
&\quad + (1 - \alpha_{n+1} - \beta_{n+1})\|T_{n+1}x_n - T_nx_n\| \\
&\leq (1 - \alpha_{n+1}(1 - \alpha))\|x_{n+1} - x_n\| \\
&\quad + (|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|)M + \|T_{n+1}x_n - T_nx_n\|
\end{aligned} \tag{3.2.3}$$

for each $n \in \mathbb{N}$. Putting $\delta_n = (|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|)M + \|T_{n+1}x_n - T_nx_n\|$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \delta_n &= \sum_{n=1}^{\infty} ((|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|)M + \|T_{n+1}x_n - T_nx_n\|) \\
&\leq M \sum_{n=1}^{\infty} (|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|) \\
&\quad + \sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_nz\| : z \in \{x_n\}\} < \infty.
\end{aligned} \tag{3.2.4}$$

It follows by Lemma 2.7.1 that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. \square

Theorem 3.2.3. *Let E be a uniformly smooth Banach space and C a nonempty closed convex subset of E . Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty and let $f : C \rightarrow C$ be a contraction with coefficient $\alpha \in (0, 1)$. Let $\{x_n\}$ be the iteration sequence defined by (3.2.1). Suppose that $\{T_n\}$ satisfies the AKTT-condition. Let T be a mapping of C into itself defined by $Tz = \lim_{n \rightarrow \infty} T_nz$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. If the following conditions are satisfied:*

$$(C1) \quad \lim_{n \rightarrow \infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty;$$

$$(C2) \quad \sum_{n=0}^{\infty} \alpha_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \alpha_n = 0,$$

then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$.

Proof. By Lemma 3.2.1, we have $\{x_n\}$ is bounded which leads to the boundedness of the set $\{f(x_n)\}, \{T_nx_n\}$. Next, we show that $\|Tx_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$.

By Lemma 3.2.2 and 2.7.7, we have $\lim_n \|x_{n+1} - x_n\| = 0$ and $\lim_{n \rightarrow \infty} \sup\{\|Tz - T_nz\| : z \in \{x_n\}\} = 0$. Moreover, we note that

$$\begin{aligned}
\|x_n - T_nx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_nx_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)T_nx_n - T_nx_n\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - T_nx_n\| + \beta_n \|x_n - T_nx_n\|,
\end{aligned}$$

which implies

$$\|x_n - T_n x_n\| \leq \frac{1}{1 - \beta_n} \|x_{n+1} - x_n\| + \frac{\alpha_n}{1 - \beta_n} \|f(x_n) - T_n x_n\|. \quad (3.2.5)$$

Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, by Lemma 3.2.2, we obtain

$$\|T_n x_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore, we have

$$\begin{aligned} \|T x_n - x_n\| &\leq \|T x_n - T_n x_n\| + \|T_n x_n - x_n\| \\ &\leq \sup \{ \|T z - T_n z\| : z \in \{x_n\} \} + \|T_n x_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

By Lemma 2.7.9, there exists $p \in F$ such that

$$\limsup_{n \rightarrow \infty} \langle f(p) - p, j(x_n - p) \rangle \leq 0. \quad (3.2.6)$$

Finally, we show that $x_n \rightarrow p$. From Lemma 2.7.2(b) and (3.2.1), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) T_n x_n - p\|^2 \\ &= \|(1 - \alpha_n - \beta_n)(T_n x_n - p) + \beta_n(x_n - p) + \alpha_n(f(x_n) - p)\|^2 \\ &\leq \|(1 - \alpha_n - \beta_n)(T_n x_n - p) + \beta_n(x_n - p)\|^2 \\ &\quad + 2\alpha_n \langle f(x_n) - p, j(x_{n+1} - p) \rangle \\ &\leq [(1 - \alpha_n - \beta_n)\|T_n x_n - p\| + \beta_n\|x_n - p\|]^2 \\ &\quad + 2\alpha_n \langle f(x_n) - p, j(x_{n+1} - p) \rangle \\ &\leq [(1 - \alpha_n - \beta_n)(\|T_n x_n - T_n p\| + \|T_n p - p\|) + \beta_n\|x_n - p\|]^2 \\ &\quad + 2\alpha_n \langle f(x_n) - p, j(x_{n+1} - p) \rangle \\ &\leq [(1 - \alpha_n - \beta_n)\|x_n - p\| + \beta_n\|x_n - p\|]^2 \\ &\quad + 2\alpha_n \langle f(x_n) - p, j(x_{n+1} - p) \rangle \\ &= (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \langle f(x_n) - p, j(x_{n+1} - p) \rangle \\ &= (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \langle f(x_n) - f(p), j(x_{n+1} - p) \rangle \\ &\quad + 2\alpha_n \langle f(p) - p, j(x_{n+1} - p) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \alpha \|x_n - p\| \|x_{n+1} - p\| \\ &\quad + 2\alpha_n \langle f(p) - p, j(x_{n+1} - p) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - p\|^2 + \alpha_n \alpha \{ \|x_n - p\|^2 + \|x_{n+1} - p\|^2 \} \\ &\quad + 2\alpha_n \langle f(p) - p, j(x_{n+1} - p) \rangle, \end{aligned} \quad (3.2.7)$$

which implies

$$\|x_{n+1} - p\|^2 \leq \frac{(1 - \alpha_n)^2 + \alpha_n \alpha}{1 - \alpha_n \alpha} \|x_n - p\|^2 + \frac{2\alpha_n}{1 - \alpha_n \alpha} \langle f(p) - p, j(x_{n+1} - p) \rangle. \quad (3.2.8)$$

By condition (C2), without loss of generality we may assume that $\alpha_n < \min\{\frac{1}{2(1-\alpha)}, \frac{1}{2\alpha}\}$ for all $n \in \mathbb{N}$. Then $1 - \alpha_n\alpha > \frac{1}{2}$, and $2(1 - \alpha)\alpha_n < 1$ for all $n \in \mathbb{N}$. By (3.2.8), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \frac{1 - 2\alpha_n + \alpha_n\alpha}{1 - \alpha_n\alpha} \|x_n - p\|^2 + \frac{\alpha_n^2}{1 - \alpha_n\alpha} \|x_n - p\|^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha_n\alpha} \langle f(p) - p, j(x_{n+1} - p) \rangle \\
&\leq 1 - \frac{2\alpha_n(1 - \alpha)}{1 - \alpha_n\alpha} \|x_n - p\|^2 \\
&\quad + 2\alpha_n \{ \alpha_n \|x_n - p\|^2 + 2 \langle f(p) - p, j(x_{n+1} - p) \rangle \} \\
&\leq (1 - 2(1 - \alpha)\alpha_n) \|x_n - p\|^2 \\
&\quad + 2\alpha_n \{ \alpha_n M_1 + 2 \langle f(p) - p, j(x_{n+1} - p) \rangle \}
\end{aligned} \tag{3.2.9}$$

where $M_1 \geq 0$ is constant such that $\sup_{n \geq 0} \|x_n - p\| \leq M_1$. Take $\gamma_n = 2(1 - \alpha)\alpha_n$, $\delta_n = 2\alpha_n \{ \alpha_n M_1 + 2 \langle f(p) - p, j(x_{n+1} - p) \rangle \}$. It is easily seen that $\sum_{n=1}^{\infty} \gamma_n = \infty$. By (C2) and (3.2.6), we have

$$\limsup_{n \rightarrow \infty} \delta_n / \gamma_n = \limsup_{n \rightarrow \infty} \frac{1}{1 - \alpha} (\alpha_n M_1 + 2 \langle f(p) - p, j(x_{n+1} - p) \rangle) \leq 0.$$

It follows by (3.2.9) and Lemma 2.7.1 that $\lim_{n \rightarrow \infty} \|x_n - p\| = 0$. Consequently, we have $\{x_n\}$ converges strongly to a common fixed point $p \in \bigcap_{n=0}^{\infty} F(T_n)$, which completes the proof. \square

Let W_n be the W -mapping of C into itself generated by T_n, T_{n-1}, \dots, T_1 and $\alpha_n, \alpha_{n-1}, \dots, \alpha_1$, where $\{T_n\}$ is a sequence of nonexpansive mappings of C into itself and $\{\alpha_n\} \subset [0, b]$ for some $0 < b < 1$. It show in Remark 3.1 of [39] that $\{W_n\}$ satisfies *AKTT-condition*. So, the following result is directly obtained by Theorem 3.2.3.

Corollary 3.2.4. *Let C be a nonempty closed convex subset of a real strictly convex and reflexive Banach space E which has a uniformly Gâteaux differentiable norm. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings from C into itself such that the common fixed point set $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $f : C \rightarrow C$ be a contractive mapping with a contractive constant $\alpha \in (0, 1)$. For any give $x_0 \in C$, let $\{x_n\}$ be the iteration sequence defined by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) W_n x_n, \quad n \geq 1, \tag{3.2.10}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\beta_n\}$ is a sequence in $[0, 1)$. Suppose the following conditions are satisfied:

- (C1) $\lim_{n \rightarrow \infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
- (C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$.

Then the sequence $\{x_n\}$ defined by (3.2.10) converges strongly to some common fixed point $p \in F$.

Theorem 3.2.5. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm and C be a nonempty closed convex subset of E . Let $\{T_n\}_{n=1}^{\infty}$ be a countable family of nonexpansive mappings from C into itself such that the common fixed point set $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, and let $f : C \rightarrow C$ be a contractive mapping with a contractive constant $\alpha \in (0, 1)$. For any given $x_0 \in C$, let $\{x_n\}$ be the iteration sequence defined by (3.2.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1)$. Suppose that $\{T_n\}$ satisfies condition (B) and the following conditions are satisfied:*

$$(C1) \quad \lim_{n \rightarrow \infty} \alpha_n = 0 \text{ and } \sum_{n=0}^{\infty} \alpha_n = \infty;$$

$$(C2) \quad \lim_{n \rightarrow \infty} \beta_n = 0.$$

Then $\{x_n\}$ converges strongly to a common fixed point $\{T_n\}$.

Proof. By Lemma 3.2.1, we obtain that $\{x_n\}$ is bounded, and so are the sequences $\{f(x_n)\}$ and $\{T_n(x_n)\}$. Let A be a bounded subset of C containing $\{x_n\}$. By condition (B), there exists a nonexpansive mapping T of C into itself such that $\lim_{n \rightarrow \infty} \sup_{x \in A} \|T(T_n x) - T_n x\| = 0$ and $F(T) = F$. By (3.2.1), we have

$$\begin{aligned} \|x_{n+1} - T x_{n+1}\| &\leq \|x_{n+1} - T_n x_n\| + \|T_n x_n - T(T_n x_n)\| + \|T(T_n x_n) - T x_{n+1}\| \\ &\leq 2\|x_{n+1} - T_n x_n\| + \sup_{x \in A} \|T(T_n x) - T_n x\| \\ &= 2\|\alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)T_n x_n - T_n x_n\| \\ &\quad + \sup_{x \in A} \|T(T_n x) - T_n x\| \\ &= 2\|\alpha_n(f(x_n) - T_n x_n) + \beta_n(T_n x_n - x_n)\| + \sup_{x \in A} \|T(T_n x) - T_n x\| \\ &\leq 2\alpha_n \|f(x_n) - T_n x_n\| + 2\beta_n \|T_n x_n - x_n\| + \sup_{x \in A} \|T(T_n x) - T_n x\|. \end{aligned}$$

By condition (B), (C1) and (C2), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0.$$

By Lemma 2.7.9, there exists $p \in F(T) = F$ such that

$$\limsup_{n \rightarrow \infty} \langle f(p) - p, j(x_n - p) \rangle \leq 0. \quad (3.2.11)$$

By using the same arguments and techniques as those of Theorem 3.2.3, we can show that $\{x_n\}$ converges strongly to a point $p \in F$. This completes the proof. \square

If $\beta_n = 0$ and f is a constant in Theorem 3.2.5, we get the results of Song and Zheng [48].

Corollary 3.2.6. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm and K be a nonempty closed convex subset of E . Suppose that $\{T_n\}_{n=1}^{\infty}$ is a countable family of nonexpansive mappings from K into itself such that the common fixed point set $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and the condition (B). Assume that $\{x_n\}$ is a Halpern type iteration defined by for $x_1, u \in K$,*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T_n x_n, \quad n \geq 1, \quad (3.2.12)$$

where $\{\alpha_n\}$ is a sequence in $[0, 1)$. Suppose the following conditions are satisfied:

$$(C1) \quad \lim_{n \rightarrow \infty} \alpha_n = 0;$$

$$(C2) \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to some common fixed point $p \in F$.

The following result is obtained directly by Theorem 3.2.5 and Example 2.7.6

Theorem 3.2.7. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Suppose that $A : D(A) \subset E \rightarrow 2^E$ is an accretive operator with $A^{-1}0 \neq \emptyset$ and K is a nonempty closed convex subset of E such that $\overline{D(A)} \subset K \subset \bigcap_{r>0} R(I + rA)$. Let $\{J_{r_n}\}_{n=1}^{\infty}$ be a countable family of nonexpansive mappings from C into itself such that the common fixed point set $F := \bigcap_{n=1}^{\infty} F(J_{r_n}) \neq \emptyset$, and let $f : C \rightarrow C$ be a contractive mapping with a contractive constant $\alpha \in (0, 1)$. Assume that $\{x_n\}$ is a sequence defined by for $x_1 \in K$,*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) J_{r_n} x_n, \quad n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying (C1) and (C2) of Theorem 3.2.5. If $0 \in R(A)$ and $\{r_n\}$ is a sequence of positive real number such that $\lim_{n \rightarrow \infty} r_n = +\infty$, then $\{x_n\}$ converges strongly to a point $p \in \bigcap_{n=1}^{\infty} F(J_{r_n}) = A^{-1}0$.