Chapter 2
Basic Concepts and Preliminaries

Our purpose in this section is to study in a manner the main elementary facts about metric space. A metric space is nothing more than a nonempty set equipped with a concept of distance which is suitable for the treatment of convergence sequences in the set and continuous function defined on the set.

2.1 Metric Spaces and Basic Concept

Definition 2.1.1. ([28]) A *metric space* is a pair (X, d), where X is a set and d is a metric on X(or distance function on X),that is, a real valued function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(1) $d(x, y) \geq 0$

(2) $d(x, y) = 0$ if and only if $x = y$

(3) $d(x, y) = d(y, x)$ (symmetry)

(4) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

A *metric space* consist of two object; a nonempty set X and a metric d on X. The element of X are called the *point* of the metric (X, d).

Remark. In a metric space X, we have, for $x, y, z \in X$,

$$|d(x, y) - d(x, z)| \leq d(y, z).$$

Example 2.1.2. ([5])

(1) $X = \mathbb{R}$; $d(x, y) = |x - y|$, $\forall x, y \in \mathbb{R}$, where $|.|$ denotes the absolute value, is a metric (a distance) on $\mathbb{R}$;

(2) $X = \mathbb{R}^n$; $d(x, y) = [\sum_{i=1}^{n}(x_i - y_i)^2]^{1/2}$, for all $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, is a metric on $\mathbb{R}^n$, called the *euclidean metric.* The next two mappings:

$$\delta(x, y) = \sum_{i=1}^{n} |x_i - y_i|, \quad x, y \in \mathbb{R}^n,$$

$$\rho(x, y) = \max_{1 \leq 1 \leq n} |x_i - y_i|, \quad x, y \in \mathbb{R}^n,$$

are also metrics on $\mathbb{R}^n$

(3) Let $X = \{f : [a,b] \to \mathbb{R} |$ f is continuous$\}$. We define $d : X \times X \to \mathbb{R}^+$ by

$$d(f,g) = \max_{x\in[a,b]} |f(x) - g(x)|, \quad \text{for all } f,g \in X.$$

Then d is a metric on X (called the *Chebyshev metric*); the metric space (X,d) is usually denoted by $C[a,b]$;

(4) Let X be as (3) and $\delta : X \times X \to \mathbb{R}^+$ be given by

$$\delta(f,g) = \max_{x\in[a,b]} (|f(x) - g(x)|e^{-\tau|x-x_0|}),$$

for all $f,g \in X$ where $\tau > 0$ is a constant and $x_0 \in [a,b]$ is fixed. Then δ is a metric on X, called the *Bielecki metric*, and the metric space (X,δ) is usually denoted by $B[a,b]$.

Definition 2.1.3. ([5]) Let (X,d) be a metric space. The topology having as basis the family of all open balls, $B(x,r)$, $x \in X$, $r > 0$, is called the *topology induced* by the metric d

Definition 2.1.4. ([5]) Two metrics d_1 and d_2 defined on the set X are called *equivalent* if they induce the same topology on X.

Remark 2.1.5. ([5])

(1) Two metrics d_1 and d_2 are metrically equivalent if there exist two constants $m > 0, M > 0$ such that

$$md_1(x,y) \leq d_2(x,y) \leq Md_1(x,y), \quad \text{for all } x,y \in X;$$

(2) In Example 2.1.2, the metrics d, δ and ρ from (2) are equivalent; the metrics d from (3) and ρ from (4) are equivalent as well.

Definition 2.1.6. ([28]) A sequence $\{x_n\}$ in a metric space $X = (X,d)$ is said to be *convergent* if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0$$

x is called the *limit* of $\{x_n\}$ and we write

$$\lim_{n\to\infty} x_n = x \quad \text{or, simple, } x_n \to x$$

we say that $\{x_n\}$ *converges to* x. If $\{x_n\}$ is not convergent, it is said to be *divergent*.

For any sequence $\{x_n\}$ in X, we can consider a subsequence $\{x_{n_k}\} = \{x_{n_1}, x_{n_2}, x_{n_3}, ...\}$ of $\{x_n\}$, where $n_1 \leq n_2 \leq n_3 \leq ....$
Remark. If $x_n \to x$, then x is the limit of any subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

Definition 2.1.7. ([28])Let X be a metric space with the metric d. A subset C of X is called a *closed set* if $\{x_n\} \subset C$ and $x_n \to x$ imply $x \in C$.

Theorem 2.1.8. ([54]) **(The fundamental properties of closed sets)** Let X be a metric space. Then the following hold:

(1) X and $\emptyset$ are closed sets;

(2) any intersection of closed sets in X is closed, that is,

$$F_\mu \ (\mu \in M) \text{ are closed } \Rightarrow \bigcap_{\mu \in M} F_\mu \text{ is closed;}$$

(3) any finite union of closed sets in X is closed, that is,

$$F_i \ (i = 1, 2, \ldots, m) \text{ are closed } \Rightarrow \bigcup_{i=1}^{m} F_i \text{ is closed;}$$

Theorem 2.1.9. ([34]) Let $\{x_n\}$ be a sequence in $\mathbb{R}$. If every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has a convergent subsequence, then $\{x_n\}$ is convergent.

Definition 2.1.10. ([55]) Let X and Y be metric spaces and let f be a mapping of X into Y. Then f is said to be *continuous* at x_0 in X if

$$x_n \to x_0 \Rightarrow f(x_n) \to f(x_0).$$

A mapping f of X into Y is said to be *continuous* if it is *continuous* at each x in X, that is

$$x_n \to x \Rightarrow f(x_n) \to f(x).$$

Theorem 2.1.11. ([55])Let X and Y be metric spaces and let f be a mapping of X into Y. Then for $x_0 \in X$, f continuous at x_0 if and only if, for any $\varepsilon > 0$ there exist $\delta > 0$ such that

$$d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon.$$

Theorem 2.1.12. ([55])Let X and Y be metric spaces and let f be a mapping of X into Y. Then f is uniformly continuous on X if and only if, for any $\varepsilon > 0$ there exist $\delta > 0$ such that for $x, y \in X$,

$$d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon.$$

Definition 2.1.13. ([28])A sequence (x_n) in a metric space $X = (X, d)$ is said to be *Cauchy* if for every $\varepsilon > 0$ there is an $N(\varepsilon) \in N$ such that $d(x_m, x_n) < \varepsilon$ for every $m, n \geq N(\varepsilon)$.

Theorem 2.1.14. ([28]) Every convergent sequence in a metric space is a Cauchy sequence.

Theorem 2.1.15. ([28]) Let X be a metric space and let $\{x_n\}$ be a Cauchy sequence of X. If $\{x_n\}$ contain its convergent subsequence, then $\{x_n\}$ is convergent.

Definition 2.1.16. ([28]) A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges.

Remark. The real line is a complete metric space.

Theorem 2.1.17. ([55]) Let X be a complete metric space and let Y be a subspace of X. Then

$$Y \text{ is complete} \Leftrightarrow Y \text{ is closed.}$$

Theorem 2.1.18. ([34]) Let $\{x_n\}$ be a sequence in $\mathbb{R}$. If every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has a convergent subsequence converging to the same limit, then $\{x_n\}$ is convergent.

Definition 2.1.19. ([34]) Let X be a metric space and A be any nonempty subset of X. For each x in X, the *distance* $d(x, A)$ from x to A is $\inf\{d(x, y)|\ y \in A\}$.

Definition 2.1.20. ([55]) Let X and Y be metric spaces and let f be a mapping of X into Y. Then f is called a *contraction* or a contractive mapping if there exist a real number r with $0 \leq r < 1$ such that

$$d(f(x), f(y) \leq rd(x, y), \text{ for all } x, y \in X.$$

Definition 2.1.21. ([5]) Let X, d be a metric space. A mapping $T : X \to X$ is called

- Lipschitzian (or L-Lipschitzian) if there exists $L > 0$ such that

  $$d(Tx, Ty) \leq Ld(x, y), \text{for all } x, y \in X;$$

- (strict) contraction (or a-contraction) if T is a-Lipschitzian, with $a \in [0, 1)$;
- nonexpansive if T is 1-Lipschitzian;
- contractive if $d(Tx, Ty) < d(x, y)$, for all $x, y \in X$, $x \neq y$;
- isometry if $d(Tx, Ty) = d(x, y)$, for all $x, y \in X$.

Example 2.1.22. ([5])

(1) $T : \mathbb{R} \to \mathbb{R}$ $T(x) = x/2 + 3$, $x \in \mathbb{R}$, is a strict contraction and $F(T) = \{6\}$;

(2) The function $T : [1/2, 2] \to [1/2, 2]$, $Tx = 1/x$, is 4-Lipschitzian with $F(T) = 1$, while the function T in Example 1.1.1, (3)-(4) are all isometrics;

(3) $T : [1, +\infty] \to [1, +\infty]$, $Tx = x + \frac{1}{x}$ is contractive and $F(T) = \emptyset$.

2.2 Banach Spaces and Hilbert spaces

Our purpose in this section is to study definitions and fundamental theorems relating to Banach spaces and Hilbert spaces.

Definition 2.2.1. ([55]) Let X be a nonempty set, and assume that each pair of elements x and y in X can be combined by a process called *addition* to yield and element z in X de noted by $z = x+y$. Assume also that this operation of addition satisfies the following conditions (V1)-(V4):
(V1) $(x + y) + z = x + (y + z)$;
(V2) $x + y = y + x$;
(V3) there exist a unique element in X, denoted by 0 and called the *zero element*, or the *origin*, such that $x + 0 = x$ for all $x \in X$;
(V4) to each $x \in X$ there corresponds a unique element in X, denoted by $-x$ and called the *negative* of x, such that $x + (-x) = 0$.

We also assume that each scalar $\alpha \in \mathbb{R}$ and each x in X can be combined by a process called *scalar multiplication* to yield an element y in X denoted by $y = \alpha x$ satisfies (V5)-(V8):
(V5) $\alpha(\beta x) = (\alpha\beta)x$;
(V6) $1 \cdot x = x$;
(V7) $(\alpha + \beta)x = \alpha x + \beta x$;
(V8) $\alpha(x + y) = \alpha x + \alpha y$.

The algebraic system X defined by these operations and axioms is called a *linear space*. A linear space is often called a *vector space*, and its elements are spoken of as *vectors*.

Remark. Since we admit the real numbers as scalars, a linear space is also called a *real linear space*.

Definition 2.2.2. ([34]) Let X be a linear space (or vector space). A *norm* on X is a real-valued function $\|\cdot\|$ on X such that the following conditions are satisfied by all members x and y of X and each scalar α:

(1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,

(2) $\|\alpha x\| = |\alpha|\|x\|$,

(3) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

The ordered pair $(X, \|\cdot\|)$ is called a *normed space* or *normed vector space* or *normed linear space*.

Definition 2.2.3. ([34]) Let X be normed space. The *metric induced by the norm* of X is the metric d on X defined by the formula $d(x, y) = \|x-y\|$ for all $x, y \in X$. The *norm topology* of X is the topology obtained from this metric.

Definition 2.2.4. ([28]) Let x be an element and $\{x_n\}$ a sequence in a normed space X. Then $\{x_n\}$ converges strongly to x, written by $x_n \to x$, if $\lim_{n\to\infty} \|x_n-x\| = 0$.

Theorem 2.2.5. ([55]) The norm is a continuous function, and addition and scalar multiplication are jointly continuous:

(1) $x_n \to x \Rightarrow \|x_n\| \to \|x\|$;

(2) $x_n \to x$ and $y_n \to y \Rightarrow x_n + y_n \to x + y$;

(3) $\alpha_n \to \alpha$ and $x_n \to x \Rightarrow \alpha_n x_n \to \alpha x$.

Definition 2.2.6. ([28]) Let C be a subset of a Banach space X. Then the set C is called *convex* if for $x, y \in C$ and $t \in (0,1)$, imply $tx + (1-t)y \in C$.

Definition 2.2.7. ([34]) A Banach norm or complete norm is a norm that induces a complete metric. A normed space is a *Banach space* or *B-space* or *complete normed space* if its norm is a Banach norm.

Definition 2.2.8. ([28]) Let x be an element and $\{x_n\}$ a sequence in a normed space X. Then $\{x_n\}$ *converges weakly* to x written by $x_n \rightharpoonup x$, if $f(x_n) \to f(x)$ wherever $f \in X^*$.

Define the mapping $\varphi : X \to X^{**}$ by $\varphi(x) = g_x$, $x \in X$, where $g_x : X^* \to \mathbb{R}$ is defined by $g_x(f) = f(x)$, $\forall f \in X^*$. Then φ is called the *natural embedding mapping* from X into X^{**} and has the following properties:

(1) φ is linear: $\varphi(\alpha x + \beta y) = \varphi\alpha(x) + \varphi\beta(y)$ for all $x, y \in X$ and for all $\alpha, \beta \in \mathbb{F}$;

(2) $\varphi(x)$ is isometry: $\|\varphi(x)\| = \|x\|$ for all $x \in X$.

Definition 2.2.9. ([1]) A normed space X is said to be *reflexive* if the natural embedding mapping $\varphi : X \to X^{**}$ is onto. In this case, we write $X \cong X^{**}$.

Theorem 2.2.10. ([1], [34]) A normed space X is reflexive if and only if each of its bounded sequence has a weakly convergent subsequence.

Remark 2.2.11. ([1])

- Every finite-dimensional Banach space is reflexive.
- Every Hilbert space is reflexive.
- ℓ_p and L_p for $1 < p < \infty$ are reflexive Banach spaces.
- ℓ_1, ℓ_∞, L_1 and L_∞ are not reflexive.
- c and c_0 are not reflexive.

Definition 2.2.12. ([1]) A Banach space X is said to be *strictly convex* if

$$\|x\| = \|y\| = 1 \text{ and } x \neq y \text{ imply } \left\|\frac{x+y}{2}\right\| < 1.$$

Lemma 2.2.13. ([1], [50]) A linear normed space X is called *strictly convex* if $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|(1-\lambda)x + \lambda y\| = 1$ for a $\lambda \in (0,1)$ holds if and only if $x = y$.

Example 2.2.14. ([1]) Let $X = \mathbb{R}^n$, $n \geq 2$ with norm $\|\cdot\|_2$ defined by $\|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$, $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. Then X is strictly convex.

Example 2.2.15. ([1]) Let $X = \mathbb{R}^n$, $n \geq 2$ with norm $\|\cdot\|_1$ defined by $\|x\|_1 = \sum_{i=1}^n |x_i|$, $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. Then X is not strictly convex.

Example 2.2.16. ([1]) Let $X = \mathbb{R}^n$, $n \geq 2$ with norm $\|\cdot\|_\infty$ defined by $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$, $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. Then X is not strictly convex.

Definition 2.2.17. ([1], [50]) A Banach space X is called *uniformly convex* if for any $\varepsilon \in (0, 2]$, there exists a $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta$.

Remark 2.2.18. ([1])

- Every Hilbert space is uniformly convex.
- The Banach spaces ℓ_p and L_p with $(1 < p < \infty)$ are uniformly convex.
- The Banach spaces $\ell_1, \ell_\infty, c, c_0, L_1$ and L_∞ are not uniformly convex.

Theorem 2.2.19. ([50]) Let X be a Banach space. Then the following conditions are equivalent:

(1) X is uniformly convex;

(2) if for any two sequences $\{x_n\}$, $\{y_n\}$ in X,

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

then $\lim_{n\to\infty} \|x_n - y_n\| = 0$;

(3) for any ε with $0 < \varepsilon \leq 2$, there exists $\delta > 0$ depending only on $\varepsilon > 0$ such that

$$\|\frac{x+y}{2}\| \leq 1 - \delta$$

for any $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$.

Theorem 2.2.20. ([50]) Every uniformly convex Banach space is strictly convex and reflexive.

Example 2.2.21. ([1]) Let $X = c_0$ and let $\beta > 0$ with the norm $\|\cdot\|_\beta$ defined by

$$\|x\|_\beta = \|x\|_{c_0} + \beta\left(\sum_{i=1}^{n}\left(\frac{x_i}{i}\right)^2\right)^{\frac{1}{2}}, \quad x = \{x_i\} \in c_0.$$

The space $(c_0, \|\cdot\|_\beta)$ for $\beta > 0$ are strictly convex, but not uniformly convex.

Example 2.2.22. ([1]) Let $X = \mathbb{R}^n$, $n \geq 2$ with the norm $\|\cdot\|_1$ defined by $\|x\|_1 = \sum_{i=1}^n |x_i|$. Then X is reflexive, but not uniformly convex.

Definition 2.2.23. ([50]) Let X be a Banach space and let $S(X) = \{x \in X : \|x\| = 1\}$. Then X is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2.1}$$

exists for each $x, y \in S(X)$. In this case the norm of X is said to be *Gâteaux differentiable*. The space X is said to be *uniformly Gâteaux differentiable* if for each $y \in S(X)$, the limit (2.2.1) is attained uniformly for $x \in S(X)$. The norm of X is said to be *Fréchet differentiable* if for each $x \in S(X)$, the limit (2.2.1) is attained uniformly for $y \in S(X)$. The norm of X is called *uniformly Fréchet differentiable* if the limit (2.2.1) is attained uniformly for $x, y \in S(X)$.

Theorem 2.2.24. ([1]) Every uniformly smooth Banach space is reflexive.

Definition 2.2.25. ([50]) Let X be a Banach space. The multi-value mapping $J : X \to 2^{X^*}$ is called the *duality mapping* if

$$J(x) = \{f \in X^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$$

for all $x \in X$, where $\langle \cdot, \cdot \rangle$ denotes the pairing between X and X^*.

Example 2.2.26. ([2]) Let $1 < p < \infty$. Then

(1) For ℓ^p, $J(x) = \|x\|_{\ell^p}^{2-p} y \in \ell^q$, where $x = \{x_1, x_2, ...\}$ and $y = \{x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, ...\}$ with $\frac{1}{p} + \frac{1}{q} = 1$.

(2) For L^p, $J(x) = \|x\|_{L^p}^{2-p} |x|^{p-2} x \in L^q$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 2.2.27. ([50]) Let X be a Banach space and let $J : X \to 2^{X^*}$ be the duality mapping. Then

(1) for each $x \in X$, $J(x)$ is nonempty, bounded, closed and convex;

(2) $J(0) = \{0\}$;

(3) for each $x \in X$ and a real α, $J(\alpha x) = \alpha J(x)$;

(4) for $x, y \in X$, $f \in J(x)$ and $g \in J(y)$, $\langle x - y, f - g \rangle \geq 0$;

(5) for $x, y \in X$, $g \in J(y)$, $\|x\|^2 - \|y\|^2 \geq 2\langle x - y, g \rangle$.

Proposition 2.2.28. [50] *Let X be a Banach space. Then*

(1) *if X is smooth, then J is single-valued;*

(2) *if X is strictly convex, then J is one-to-one, that is, $x \neq y$ implies $J(x) \cap J(y) = \emptyset$;*

(3) *if X is reflexive, then J is onto, that is, for each $x^* \in X^*$, there exists $x \in X$ such that $x^* \in Jx$;*

(4) *if X has a Fréchet differentiable norm, then J is norm-to-norm continuous;*

(5) *if X is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subsets of X.*

Definition 2.2.29. ([50]) Let X be a linear space and let C be a convex subset of X. A function $F : C \to (-\infty, \infty]$ is *convex* on C if for any $x, y \in C$ and $t \in [0,1]$, then $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$.

Definition 2.2.30. ([15]) Let C be subset of a Banach space X. A mapping $T : C \to C$ is called *closed* with respect to y if for each sequence $\{x_n\}$ in C and each $x \in C$, $x_n \to x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

Definition 2.2.31. ([15]) Let C be subset of a Banach space X. A mapping $T : C \to C$ is called *demiclosed* with respect to y if for each sequence $\{x_n\}$ in C which converges weakly to a point $x \in C$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

Definition 2.2.32. ([15]) Let C be subset of a Banach space X and T is a mapping of C into itself. There exist sequence $\{x_n\}$ in C for which $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We shall call such sequences *approximate fixed point sequences.*

Lemma 2.2.33. ([15]) If C is a nonempty closed and convex subset of a strictly convex Banach space X and if $T : C \to C$ is nonexpansive, then $F(T)$ is closed and convex.

Theorem 2.2.34. ([15]) Let C be a nonempty closed convex subset of a uniformly convex Banach space X and suppose $T : C \to C$ is nonexpansive. Then the mapping $I - T$ is demiclosed on C.

Theorem 2.2.35. ([15]) Let C be a nonempty closed convex subset of a uniformly convex Banach space X and suppose $T : C \to C$ is nonexpansive. If a sequence $\{x_n\}$ in C converges weakly to p and $\{x_n - Tx_n\}$ converges to 0 as $n \to \infty$, then $p \in F(T)$.

Definition 2.2.36. ([28]) An *inner product space* is a vector space X with an inner product defined on X. A *Hilbert space* is a complete inner product space. Here, an inner product on X is a mapping of $X \times X$ into the scalar field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$; that is, with every pair of vector x and y there is associated a scalar which is written and is called the inner product of x and y, such that for all vectors x, y, z and scalar $\alpha \in \mathbb{F}$ we have:

(1) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \Leftrightarrow x = 0$

(2) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

(3) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

(4) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.

An inner product on X defines a norm on X given by $\|x\| = \sqrt{\langle x, x\rangle}$.

Remark 2.2.37. ([55])

(1) An inner product space is called a *real inner product space* for the case when the scalars are the real numbers and $\langle x, y\rangle$ is a real number. For the case, (3) mean

$$\langle x, y\rangle = \langle y, x\rangle.$$

(2) Using (2), (3) and (4) we obtain that for $x, y \in X$ and $\alpha, \beta \in C$,

$$\langle x, \alpha y + \beta z\rangle = \bar{\alpha}\langle x, y\rangle + \bar{\beta}\langle x, z\rangle.$$

Theorem 2.2.38. ([55])(**The Schwarz inequality**)
If x and y are any two vectors in an inner product space X, then $|\langle x, y\rangle| \leq \|x\|\|y\|$.

Theorem 2.2.39. ([55])(**The parallelogram law**)
If x and y are any two vectors in an inner product space X, then

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Theorem 2.2.40. ([55])(The nearest point theorem)
Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let $x \in H$. Then there exist a unique element $y_0 \in C$ such that

$$d(x, C) = d(x, y_0).$$

Definition 2.2.41. ([55]) Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Then for each point $x \in H$, there corresponds a unique point $x_0 \in C$ such that

$$\|x - x_0\| = d(x, C).$$

We call such a mapping defined by $Px = x_0$, or $P_C x = x_0$, the *metric projection* of H onto C.

Lemma 2.2.42. ([55]) Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$. Then

(i) $y = P_C x$ if and only if $\langle x - y, y - z\rangle \geq 0$ for all $z \in C$,

(ii) $P^2 = P$ and P_C is nonexpansive,

(iii) $\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$ for all $x, y \in H$,

(iv) $\langle x - P_C x, P_C x - y\rangle \geq 0$ for all $x \in H$ and $y \in C$,

(v) $\|P_C x - y\|^2 \leq \|x - y\|^2 - \|P_C x - x\|^2$ for all $x \in H$ and $y \in C$.

Lemma 2.2.43. ([59]) Let C be nonempty closed convex subset of a real Hilbert space H. Let $P_C : H \to C$ be the metric projection of H onto C. Let $\{x_n\}_{n=1}^{\infty}$ be sequence in C and let $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all u in C. Then $\{P_C x_n\}_{n=1}^{\infty}$ converges strongly.

Lemma 2.2.44. ([37]) Let H be a real Hilbert space. Then the following well known results hold:

(i) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$, for all $x, y \in H$ and for all $t \in [0,1]$.

(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$ for all $x, y \in H$.

(iii) If $\{x_n\}_{n=1}^{\infty}$ is a sequence in H which converges weakly to $z \in H$ then

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} (\|x_n - z\|^2 + \|z-y\|^2), \quad \forall y \in H.$$

Lemma 2.2.45. ([55]) Let $\{x_n\}$ be a Cauchy sequence of an inner product space H such that $x_n \rightharpoonup x$. Then $x_n \to x$.

2.3 Fixed Point of Nonexpansive Mappings

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. A mapping T of C into itself is called *nonexpansive* if, for all $x, y \in C$

$$\|Tx - Ty\| \leq \|x - y\|.$$

The set of all fixed points of T is denoted by $F(T)$, that is $F(T) = \{x \in C|\ x = Tx\}$ and a self-mapping $f : C \to C$ is a *contraction* on C if there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x-y\|$ for all $x, y \in C$. We use Π_C to denote the collection of all contractions on C, that is $\Pi_C = \{f : C \to C|f$ is a contraction with a constant $\alpha\}$. An operator A is *strongly positive* if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x^2\| \ \ \forall x \in H.$$

In 1966 Petryshyn [40] studies the set of fixed point of T nonempty and strongly convergence and of the Krasnoselsij iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \qquad n = 1,\ 2,\ \ldots \tag{2.3.1}$$

in Hilbert space, where $\lambda \in (0,1)$ and T be nonexpansive and demicompact operator.

In 1967, Browder and Petryshyn [8] prove unique fixed point of nonexpansive mapping T of C into itself.

In 1974, Senter and Dotson [44] studied the convergence of the Mann iteration scheme defined by $x_1 \in C$,

$$x_{n+1} = \alpha_n T x_n + (1-\alpha_n)x_n, \ \forall n \geq 1, \tag{2.3.2}$$

in a uniformly convex Banach space, where $\{\alpha_n\}$ is a sequence satisfying $0 < a \leq \alpha_n \leq b < 1\ \forall n \geq 1$ and T is a nonexpansive (or a quasi-nonexpansive) mapping.

In 1993, Tan and Xu [60] proved weak convergence of the Ishikawa iteration scheme defined by $x_1 \in C$,

$$x_{n+1} = \alpha_n T\big(\beta_n T x_n + (1-\beta_n)x_n\big) + (1-\alpha_n)x_n, \tag{2.3.3}$$

in uniformly convex Banach space which satisfies Opial's condition, where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences satisfying $0 < a \le \alpha_n, \beta_n \le b < 1$ $\forall n \ge 1$ and T is nonexpansive mapping.

2.4 Fixed Point Theory of Nonspreading Mappings

Throughout this section, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. In a Hilbert space, it is known that

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \tag{2.4.1}$$

for all $x, y \in H$ and for all $t \in \mathbb{R}$. see, for instance, [51]. Further, in a Hilbert space, we have that

$$2\langle x-y, z-w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2, \tag{2.4.2}$$

for all $x, y, z, w \in H$.

Let C is a nonempty closed convex subset of H and let T be a mapping of C into itself. We denoted by $F(T)$ the set of all fixed points of T that is, $F(T) = \{z \in C : Tz = z\}$. Let $T : C \to C$ be a nonexpansive mapping. We know that if C is a nonempty bounded closed convex subset of H, then $F(T)$ is nonempty. We can prove from (2.4.1) that $F(T)$ is closed and convex. A mapping $F : C \to C$ is *firmly nonexpansive* if

$$\|Fx - Fy\|^2 \le \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$. It is know that P_C is a firmly nonexpansive mapping of H onto C. A mapping $T : C \to C$ is *nonspreading* if

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \tag{2.4.3}$$

for all $x, y \in C$. Iemoto and Takahashi [18] proved that $T : C \to C$ is nonspreading if and only if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \tag{2.4.4}$$

for all $x, y \in C$. A mapping $T : C \to C$ is called *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - u\| \le \|x - u\|$ for all $x \in C$ and $u \in F(T)$. Observe that if T is nonspreading and $F(T) \neq \emptyset$, then T is quasi-nonexpansive. Further, we know that the set of fixed points of a quasi-nonexpansive mapping is closed and convex; see [21]. Then we can define the metric projection of H onto $F(T)$.

Let us give two examples of nonspreading mappings. The following example is due to Kohsaka and Takahashi ([27]). For the sake of completeness, we give the proof.

Example 2.4.1. ([27]) Let C be a nonempty closed convex subset of H and let F be a firmly nonexpansive of C into itself. Then F is nonspreading.

Proof. Putting $z = Fx$ and $w = Fy$ in 2.4.2, we have that

$$2\langle x - y, Fx - Fy\rangle = \|x - Fy\|^2 + \|y - Fx\|^2 - \|x - Fx\|^2 - \|y - Fy\|^2,$$

for all $x, y \in C$. So, we get

$$\begin{aligned} \|Fx - Fy\|^2 &\leq \langle x - y, Fx - Fy\rangle \\ &= \frac{1}{2}(\|x - Fy\|^2 + \|y - Fx\|^2 - \|x - Fx\|^2 - \|y - Fy\|^2) \end{aligned}$$

and hence

$$2\|Fx - Fy\|^2 + \|x - Fx\|^2 + \|y - Fy\|^2 \leq \|x - Fy\|^2 + \|y - Fx\|^2.$$

This implies that

$$2\|Fx - Fy\|^2 \leq \|x - Fy\|^2 + \|y - Fx\|^2.$$

So, F is nonspreading. □

The following is an example of nonspreading mappings which is not nonexpansive; see Igarashi, Takahashi and Tanaka ([19]).

Example 2.4.2. ([19]) Let H be a Hilbert space. Set $E = \{x \in H : \|x\| \leq 1\}$, $D = \{x \in H : \|x\| \leq 2\}$ and $C = \{x \in H : \|x\| \leq 3\}$. Define a mapping $S : C \to C$ as follows:

$$Sx = \begin{cases} 0, & \text{if } x \in D, \\ P_E(x), & \text{if } x \in C \backslash D, \end{cases}$$

where P_E is the metric projection of H onto E. Then S is not nonexpansive but nonspreading.

Theorem 2.4.3. ([26]) Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let S be a nonspreading mapping of C into itself. Then the following are equivalent:

- There exists $x \in C$ such that $\{S^n x\}$ is bounded;
- $F(S)$ is nonempty.

Theorem 2.4.4. ([26]) Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let S be a nonspreading mapping of C into itself. Then $F(S)$ is closed and convex.

Lemma 2.4.5. ([51]) Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $\sum_{n=1}^{\infty} \alpha_n\beta_n < \infty$, then $\liminf_{n\to\infty} \beta_n = 0$.

Lemma 2.4.6. ([60]) Suppose that $\{s_n\}$ and $\{e_n\}$ are sequences of nonnegative real numbers such that $s_{n+1} \le s_n + e_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} e_n < \infty$, the $\lim_{n\to\infty}$ exists.

Theorem 2.4.7. ([18]) Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself and let T be a nonexpansive mapping of C into itself such that $F(S) \cap F(T) \neq \phi$. Define a sequence $\{x_n\}$ as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)[\beta_n S x_n + (1-\beta_n) T x_n], \end{cases} \tag{2.4.5}$$

for all $n \in N$, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. Then, the following hold:

(i) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, then $\{x_n\}$ converges weakly to $v \in F(S)$;

$(ii) If \sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, then $\{x_n\}$ converges weakly to $v \in F(T)$;

(iii) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then $\{x_n\}$ converges weakly to $v \in F(S) \bigcap F(T)$.

In 2010, Kurokawaa and Takahashi ([29]) introduced the following iteration method for nonspreading mapping:

$$\begin{cases} x_1, u \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = \dfrac{1}{n} \sum_{k=0}^{n-1} T^k x_n, \end{cases} \tag{2.4.6}$$

when $\{\alpha_n\} \subset [0,1]$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $p \in F(T)$, where $p = P_{F(T)} u$.

Let H be a real Hilbert space. Following the terminology of Browder-Petryshyn [8], we say that a mapping $T : D(T) \subseteq H :\rightarrow H$ is $k-$*strictly pseudononspreading* if there exists $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x-y\|^2 + k\|x - Tx - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle, \tag{2.4.7}$$

for all $x, y \in D(T)$. Clearly every nonspreading mapping is k-strictly psedononspreading. The following example shows that the class of k-strictly pseudononspreading mapping is more general than the class of nonspreading mappings.

Example 2.4.8. ([37]) Let $\mathbb{R}$ denote the reals with the usual norm. Let $T : \mathbb{R} \to \mathbb{R}$ be define for each $x \in \mathbb{R}$ by

$$Tx = \begin{cases} x, & x \in [-\infty, 0); \\ -2x, & x \in [0, \infty). \end{cases}$$

To see that T is $\frac{1}{3}-$stricly pseudononspreading but is not nonspreading.

Since our class of maps contains the class of nonspreading maps, it also contains the class of firmly nonexpansive maps. Observe that if T is k-strictly pseudononspreading and $F(T) \neq \emptyset$, then for all $x \in D(T)$ and for all $p \in F(T)$ we have

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2. \tag{2.4.8}$$

Thus every k-strictly pseudononspreading map with a nonempty fixed point set $F(T)$ is *demicontractive* (see example ([17], [36]).

Lemma 2.4.9. ([37]) Let C be nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be k-strictly pseudononspreading mapping. If $F(T) \neq \emptyset$, then it is closed and convex.

Lemma 2.4.10. ([37]) Let C be nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be k-strictly pseudononspreading mapping. Then $(I - T)$ is demiclosed at 0.

Theorem 2.4.11. ([37]) Let C be a nonempty closed convex subset of of a real Hilbert space. Let $T : C \to C$ be a k-strictly pseudononspreading mapping with a nonempty fixed point set $F(T)$. Let $\beta \in [k, 1)$ and let $T_\beta := \beta I + (1-\beta)T$. Let $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $u \in C$ and let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in C generated from an arbitrary $x_1 \in C$ by

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \ n \geq 1, \\ z_n = \frac{1}{n}\sum_{k=0}^{n-1} T_\beta^k x_n, \ n \geq 1. \end{cases} \tag{2.4.9}$$

Then $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ converges strongly to $P_{F(T)}u$, where $P_{F(T)} : H \to F(T)$ is the metric projection of H onto $F(T)$.

2.5 Equilibrium Problems in Hilbert Spaces

Equilibrium problems play a central role in numerous disciplines including economics, management science, operations research, and engineering. Numerous algorithms have been developed for the computation of equilibrium points. Variational inequality theory, a powerful computational algorithm, is one of them which has numerous applications in various disciplines of sciences such as mathematical programming, game theory, mechanics and geometry.

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$, respectively. Let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. A mapping $A : C \to H$ is called α*-inverse-strongly monotone*, if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha\|Ax - Ay\|^2, \ \ \forall x, y \in C.$$

Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying

$(A1)$ $F(x, x) = 0 \quad \forall x \in C$;

(A2) F is monotone, i.e. $F(x,y)+F(y,x)\leq 0\ \ \forall x,y\in C$;
(A3) $\forall x,y,z\in C,\ \ \lim_{t\to 0^+}F(tz+(1-t)x,y)\leq F(x,y)$;
(A4) $\forall x\in C, y\mapsto F(x,y)$ is convex and lower semicontinuous.
The equilibrium problem for F is to determine its equilibrium points, i.e. the set

$$EP(F)=\{x\in C: F(x,y)\geq 0,\ \forall y\in C\}. \tag{2.5.1}$$

Lemma 2.5.1. ([6]) Let C be a nonempty closed convex subset of a Hilbert space H and $f: C\times C\to\mathbb{R}$ satisfy $(A1)-(A4)$. Let $r>0$ and $x\in H$. Then, there exists $z\in C$ such that

$$f(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\quad \textit{for all } y\in C. \tag{2.5.2}$$

Lemma 2.5.2. ([14]) Let C be a nonempty closed convex subset of a Hilbert space H and $f: C\times C\to\mathbb{R}$ satisfy $(A1)-(A4)$. For $r>0$ and $x\in H$, define a mapping $T_r: H\to C$ as follows:

$$T_r(x)=\{z\in C: f(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\ \forall y\in C\}\ \ \textit{for all}\ \ x\in H. \tag{2.5.3}$$

Then the following hold:
(1) T_r is single-valued;
(2) T_r is firmly nonexpansive, i.e., for all $x,y\in H$,
$\|T_r(x)-T_r(y)\|^2\leq\langle T_r(x)-T_r(y),x-y\rangle$;
(3) $F(T_r)=EP(f)$;
(4) $EP(f)$ is closed and convex.

Many problems in physics, optimization, and economics are seeking some elements of $EP(F)$, see [6], [14]. Several iterative methods have been proposed to solve the equilibrium problem. In 2005, Combettes and Hirstoaga [14] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

In 2007, Takahashi and Takahashi [57] proved the following theorem:

Theorem 2.5.3. *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying the condition (A1)-(A4) and let T be a nonexpansive mapping of C into H such that $F(T)\bigcap EP(F)\neq\emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1\in H$ and*

$$\begin{cases} F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0, & \forall y\in C,\\ x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)Tu_n, \end{cases} \tag{2.5.4}$$

for all $n\in N$, where $\{\alpha_n\}\subset[0,1]$ and $\{r_n\}\subset(0,1)$ satisfy:
(C1) $\alpha_n\to 0$;
(C2) $\sum_{n=0}^{\infty}\alpha_n=\infty$;
(C3) either $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ or $\lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n}=1$.
and $\liminf_{n\to\infty}r_n>0$ and $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(T)\bigcap EP(F)$, where $z = P_{F(T)\bigcap EP(F)}f(z)$.

In 2008 Ceng, Al-Homidan, Ansari, Yao [9], introduced a general iterative method for finding a common element of $EP(F)$ and $F(T)$. They defined $\{x_n\}$, $\{u_n\}$ in the following way:

$$\begin{cases} x_1 \in H, \text{ arbitrarily;} \\ F(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)Tu_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{2.5.5}$$

where T be an k-strictly pseudo contractive mapping of C into itself for some $0 \leq k < 1$ and $\{\alpha_n\} \in [0,1]$, $\{\lambda_n\} \subset (0,\infty)$. They proved strong and weakly convergence of the scheme (2.5.5) to an element of $F(T)\bigcap EP(F)$ in the framework of a Hilbert space, under some suitable conditions on $\{\alpha_n\}$, $\{\lambda_n\}$ and bifunction F.

In 2008, Takahashi and Takahashi [56] introduced a general iterative method for finding a common element of EP and $F(T)$. They proved the following theorem.

Theorem 2.5.4. *Let C be a closed convex subset of a real Hilbert space and let $F : C \times C \to \mathbb{R}$ be a bifunction sastisfing the condition (A1)-(A4). Let A be an α-inverse strongly monotone mapping of C into H and let T be nonexpansive mappings of C into itself with $F(T)\bigcap EP \neq \emptyset$. Let $u \in C$ and $x_1 \in C$ and let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta)T(a_n u + (1-a_n)z_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{2.5.6}$$

where

$\{a_n\} \in [0,1], \{\beta_n\} \subset [0,1]$ *and* $\{\lambda_n\} \subset [0, 2\alpha]$ *sastisfy the following condition:*

(*i*) $0 < a \leq \lambda_n \leq b < 2\alpha, \; 0 < c \leq \beta_n \leq d < 1$

(*ii*) $\lim_{n\to\infty} |\lambda_n - \lambda_{n+1}| = 0$

(*iii*) $\lim_{n\to\infty} a_n = 0, \; \sum_{n=1}^{\infty} a_n = \infty.$

Then $\{x_n\}$ converges strongly to $z = P_{\bigcap_{i=1}^N F(T)\bigcap EP}u$.

2.6 The Concept of W-Mappings and K-Mappings

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings with $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$. There are many authors introduced iterative method for finding an element of F which is an optimal point for the minimization problem. For $n > N$, T_n is understood as $T_{(n \ mod \ N)}$ with the mod function taking values in $\{1, 2, ..., N\}$. Let u be a fixed element of H.

In 2003, Xu [64] proved that the sequence $\{x_n\}$ generated by

$$x_{n+1} = (1 - \epsilon_n A)T_{n+1}x_n + \epsilon_n u$$

converges strongly to the solution of the quadratic minimization problem

$$\min_{x \in F} \frac{1}{2}\langle Ax, x\rangle - \langle x, u\rangle$$

under suitable hypotheses on $\{\epsilon_n\}$ and under the additional hypothesis,

$$F = F(T_1T_2...T_N) = F(T_NT_1...T_{N-1}) = ... = F(T_2T_3...T_NT_1). \tag{2.6.1}$$

Note that by using above iteration for finding a common fixed point of a finite family of nonexpansive mappings, the hypothesis (2.6.1) is so strong condition. So there are many researchers try to reduce this condition. To overcome this, Atsushiba and Takahashi [4] introduced a new mapping, called $W-$mapping, for finding a common fixed point of a finite family of those mapping without the condition (2.6.1). After that the concept of $W-$mapping are used by many mathematicians for investigating common fixed point problems and other related problems such as equilibrium and variational inequality problems, see [[12], [13], [38], [43]]. This mapping is defined as follows:

Let $\lambda_1, \lambda_2, \ldots \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, 2, \ldots N$. Let W be a mapping defined by

$$\begin{aligned}
U_1 &= \lambda_1 T_1 + (1 - \lambda_1)I, \\
U_2 &= \lambda_2 T_2 U_1 + (1 - \lambda_2)I, \\
U_3 &= \lambda_3 T_3 U_2 + (1 - \lambda_3)I, \\
&\cdot \\
&\cdot \\
&\cdot \\
U_{N-1} &= \lambda_{N-1} T_N - 1 U_{N-2} + (1 - \lambda_{N-1})I, \\
W &= U_N = \lambda_N T_N U_{N-1} + (1 - \lambda_N)I.
\end{aligned} \tag{2.6.2}$$

This mapping is called the $W-$mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$. Atsushiba and Takahashi [4] proved the following useful lemma used for many theorems in fixed point theory.

Lemma 2.6.1. ([4]) Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N - 1$ and $0 < \lambda_N \leq 1$. Let W be the W-mapping generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$. Then $F(W) = \bigcap_{i=1}^N F(T_i)$.

Definition 2.6.2. *Let C be a nonempty convex subset of real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself, and let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, ..., N$. We define a mapping $K : C \to C$ as follows:*

$$\begin{aligned} U_1 &= \lambda_1 T_1 + (1 - \lambda_1) I, \\ U_2 &= \lambda_2 T_2 U_1 + (1 - \lambda_2) U_1, \\ U_3 &= \lambda_3 T_3 U_2 + (1 - \lambda_3) U_2, \\ &\ \ \vdots \\ U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1}) U_{N-2}, \\ K &= U_N = \lambda_N T_N U_{N-1} + (1 - \lambda_N) U_{N-1}. \end{aligned} \tag{2.6.3}$$

Such a mapping K is called the *K-mapping* generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$.

Lemma 2.6.3. ([23]) Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \leq 1$. Let K be the $K-mapping$ of C into itself generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$. Then $F(K) = \bigcap_{i=1}^N F(T_i)$.

By using the inequalities (2.4) and (2.5) of Lemma 2.11 in [23], we obtain the following result:

Lemma 2.6.4. ([23]) Let H be a Hilbert space, C a closed convex nonempty subset of H, $\{T_i\}_{i=1}^N$ a finite family of nonexpansive mappings from H into itself with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. For every $n \in N$, let K_n be K-mapping generated by $T_1, ..., T_N$ and $\lambda_{n,1}, ..., \lambda_{n,N}$ with $\{\lambda_{n,i}\}_{i=1}^N \subset [a, b]$ where $0 < a \leq b < 1$. Then

$$\|K_{n+1}w_n - K_n w_n\| \leq M \sum_{j=1}^N |\lambda_{n+1,j} - \lambda_{n,j}|.$$

where $M = \sup\{\sum_{j=2}^N (\|T_j U_{n,j-1} w_n\| + \|U_{n,j-1} w_n\|) + \|T_1 w_n\| + \|w_n\|\} < \infty$, and for every bounded sequence $\{w_n\}$ in H.

Lemma 2.6.5. ([23]) Let C be a nonempty closed convex subset of Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and $\{\lambda_{n,i}\}_{i=1}^N$ sequences in [0,1] such that $\lambda_{n,i} \to \lambda_i$, as $n \to \infty$, $(i = 1, 2, ..., N)$. Moreover, for every $n \in \mathbb{N}$, let K and K_n be the K-mappings generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$, and $T_1, T_2, ..., T_N$ and $\lambda_{n,1}, \lambda_{n,2}, ..., \lambda_{n,N}$ respectively. Then, for every bounded sequence $\{x_n\} \subset C$,, we have $\lim_{n\to\infty} \|K_n x_n - K x_n\| = 0$.

Let C be a nonempty closed convex subset of a Banach space X, and $f : C \to C$ be a contractive mapping with a contrative constant $\alpha \in (0, 1)$. Let $\{T_n\}_{n=1}^\infty : C \to C$ be an infinite family of nonexpansive mappings and let $\alpha_1, \alpha_2, ...$ be real numbers such that $0 \leq \alpha_i \leq 1$ for every $i \in \mathbb{N}$. For any $n \in \mathbb{N}$, define a mapping W_n of C into itself as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \alpha_n T_n U_{n,n+1} + (1-\alpha_n)I, \\
U_{n,n-1} &= \alpha_{n-1} T_{n-1} U_{n,n} + (1-\alpha_{n-1})I, \\
&\vdots \\
U_{n,k} &= \alpha_k T_k U_{n,k+1} + (1-\alpha_k)I, \\
U_{n,k-1} &= \alpha_{k-1} T_{k-1} U_{n,k} + (1-\alpha_{k-1})I, \\
&\vdots \\
U_{n,2} &= \alpha_2 T_2 U_{n,3} + (1-\alpha_2)I, \\
W_n &= U_{n,1} = \alpha_1 T_1 U_{n,2} + (1-\alpha_1)I. \qquad (2.6.4)
\end{aligned}$$

Such a mapping W_n is called $W-$mapping generated by $T_n, T_{n-1}, ..., T_1$ and $\alpha_n, \alpha_{n-1}, ..., \alpha_1$; see [46].

Lemma 2.6.6. ([46]) Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $T_1, T_2, ...$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty and $\alpha_1, \alpha_2, ...$ be real number such that $0 < \alpha_i \leq b < 1$ for any $i \in \mathbb{N}$. Then for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.

Using Lemma 2.6.6, one can define mapping W of C into itself as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$$

for every $x \in C$. Such a W is called the $W-$*mapping* generated by $T_1, T_2, ...$ and $\alpha_1, \alpha_2, ...$

Lemma 2.6.7. ([46]) Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $T_1, T_2, ...$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty and $\alpha_1, \alpha_2, ...$ be real number such that $0 < \alpha_i \leq b < 1$ for any $i \in \mathbb{N}$. Then $F(W) = \bigcap_{i=1}^{\infty} F(T_i)$.

2.7 Some Useful Lemmas and Theorems

Lemma 2.7.1. ([65]) Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \quad n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7.2. ([1]) Let X be a Banach space and $J : X \to 2^{X^*}$ the duality mapping. Then we have the following:

(a) $\|x+y\|^2 \geq \|x\|^2 + 2\langle y, j(x)\rangle$ for all $x, y \in X$ where $j(x) \in J(x)$.

(b) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$ for all $x, y \in X$ where $j(x+y) \in J(x+y)$;

Lemma 2.7.3. ([65], Theorem 4.1) *Let E be a uniformly smooth Banach space, C a closed convex subset of E, $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $f \in \prod_C$. Then $\{z_t\}$ defined by $z_t = tf(z_t) + (1-t)T(z_t)$ converges strongly to a point in $F(T)$.*

Let C be a nonempty closed convex subset of a Banach space E. Suppose $\{T_n\}_{n=1}^{\infty}$ is a countable family of nonexpansive mappings from C to itself with $F := \bigcap_{n=0}^{\infty} F(T_n) \neq \emptyset$. $\{T_n\}_{n=1}^{\infty}$ is said to satisfy *condition* (B), if for any bounded subset A of C, there exists a nonexpansive mapping T of C into itself such that

$$\lim_{n\to\infty} \sup_{x\in A} \|T(T_n x) - T_n x\| = 0 \ \ \textit{and} \ \ F(T) = F.$$

$\{T_n\}$ is said to satisfy the $AKTT$ – *condition* [3] if for each bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty.$$

Example 2.7.4. *Let $T_1, T_2, \cdots$ be nonexpansive mappings on C and $\gamma_1, \gamma_2, \cdots$ be real numbers such that $0 < \gamma_i < 1$ for all $i \in \mathbb{N}$. Let W_n be the W-mappings generated by $T_1, T_2, \cdots, T_n$ and $\gamma_1, \gamma_2, \cdots, \gamma_n$. Then $\{W_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition.*

Example 2.7.5. *Let $T_1, T_2, \cdots$ be nonexpansive mappings on C. Define the mapping $V_n : C \to C$ by*

$$V_n x = \sum_{i=1}^{n} \lambda_n^i T_i x, \quad \forall x \in C,$$

where $\{\lambda_n^i\}$ satisfy the conditions:
(1) $\sum_{i=1}^{n} \lambda_n^i = 1$ for each $n \in \mathbb{N}$;
(2) $\lambda^i := \lim_{n\to\infty} \lambda_n^i > 0$ for each $i \in \mathbb{N}$;
(3)$\sum_{n=1}^{\infty} \sum_{i=1}^{n} |\lambda_{n+1}^i - \lambda_n^i| < +\infty$.
Then $\{V_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition.

Some applications for zero of accretive operators.

Let $A : D(A) \subset E \to 2^E$ be an accretive operator (that is, for all $x, y \in D(A)$ there exists $j(x-y) \in J(x-y)$ such that $\langle u - v, j(x-y)\rangle \geq 0$, for $u \in Ax$ and $v \in Ay$) and $A^{-1}0 = \{x \in D(A) : 0 \in Ax\}$. We use J_r and A_r to denote the resolvent and Yosida's approximation of A, respectively. Namely,

$$J_r = (I + rA)^{-1} \ \ \textit{and} \ \ A_r = \frac{I - J_r}{r}, \ \ r > 0.$$

For J_r and A_r, the following is well known (see, [[50], pp. 129-144]):
(i) $A_r x \in AJ_r$ for all $x \in R(I+rA)$ and $A^{-1}0 = F(J_r) = \{x \in D(J_r); J_r x = x\}$;
(ii) $\|A_r x\| \leq |Ax| = \inf\{\|y\|; y \in Ax\}$ for all $x \in D(A) \bigcap R(I + rA)$;
(iii) $J_r : R(I + rA) \to D(A)$ is nonexpansive (i.e. $\|J_r x - J_r y\| \leq \|x - y\|$ for all $x, y \in R(I + rA)$).

The following example can be founded in [48]

Example 2.7.6. ([48] *Let E be a Banach space and $A : D(A) \subset E \to 2^E$ be an accretive operator. Assume that K is a nonempty closed convex subset of E such that $\overline{D(A)} \subset K \subset \bigcap_{r>0} R(I+rA)$. If $0 \in R(A)$ and $r_n > 0$, a real number sequence such that $\lim_{n\to\infty} r_n = +\infty$. Let J_r be the resolvent, namely $J_r = (I+rA)^{-1}$. Then $\{J_{r_n}\}_{n=1}^{\infty}$ satisfies the condition (B).*

Lemma 2.7.7. ([3], Lemma 3.2) *Let E be a Banach space and C be a nonempty closed convex subset of E. Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in C\} < \infty$. Then, for each $y \in C$, $T_n y$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$. Then*

$$\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : z \in C\} = 0.$$

Lemma 2.7.8. ([41]) Let C be a bounded closed convex subset of a smooth Banach space E. Let T be a nonexpansive mapping of C into itself and let f be a contraction mapping of C into itself with coefficient α $(0 < \alpha < 1)$. For each $t \in (0,1)$, let z_t be a unique point of C with $z_t = tf(z_t) + (1-t)T(z_t)$. Let $\{x_n\}$ be a sequence in C. If $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then

$$\limsup_{n\to\infty} \langle f(z_t) - z_t, J(x_n - z_t)\rangle \le \frac{t}{2} \limsup_{n\to\infty} \|z_t - x_n\|^2$$

for all $t \in (0,1)$, where J is the duality mapping of E.

Lemma 2.7.9. ([47], Theorem 2.3) Let C be a nonempty closed convex subset of a reflexive and strictly convex Banach space E with uniformly Gâteaux differentiable norm, and $T : C \to C$ be a nonexpansive mapping with a fixed point. If there exists a bounded sequence $\{x_n\}$ such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then for $f \in \prod_C$, there exists $p \in F(T)$ such that

$$\limsup_{n\to\infty} \langle f(p) - p, j(x_n - p)\rangle \le 0.$$

Lemma 2.7.10. *In a real Hilbert space H, there holds the inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$$

Lemma 2.7.11. ([49]) Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose

$$x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$$

for all integer $n \ge 0$ and

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Lemma 2.7.12. ([32]) Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma}$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.

Lemma 2.7.13. ([32]) Let H be a Hilbert space. Let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $T : C \to C$ be a nonexpansive mapping with a fixed point $x_t \in C$ of the contraction $C \ni x \mapsto t\gamma f(x) + (1 - tA)Tx$. Then x_t converges strongly as $t \to 0$ to a fixed point $\bar{x}$ of T, which solves the variational inequality

$$\langle (A - \gamma f)\bar{x}, z - \bar{x}\rangle \geq 0, \quad z \in F(T). \tag{2.7.1}$$

Lemma 2.7.14. ([31]) **Demi-closedness principle.** Assume that T is nonexpansive self-mapping of closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$. Here, I is identity mapping of H.

Lemma 2.7.15. ([3, 64]) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers satisfying the condition

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\beta_n, \quad n \geq 0,$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are real sequences such that

(i) $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.