Chapter 1
Introduction

1.1 The Background of Fixed Point Theory

Let X be a nonempty set and $T : X \to X$ a selfmap. We say that $x \in X$ is a fixed point of T if

$$Tx = x$$

and denote by $F(T)$ the set of all fixed points of T.

Example 1.1.1. ([5])

(1) If $X = \mathbb{R}$ and $T(x) = x^2 + 5x + 4$, then $F(T) = \{-2\}$;

(2) If $X = \mathbb{R}$ and $T(x) = x^2 - x$, then $F(T) = \{0, 2\}$;

(3) If $X = \mathbb{R}$ and $T(x) = x + 2$, then $F(T) = \emptyset$;

(4) If $X = \mathbb{R}$ and $T(x) = x$, then $F(T) = \mathbb{R}$.

Let X be any set and and $T : X \to X$ a selfmap. For any given $x \in X$, we define $T^n(x)$ inductively by $T^0(x) = x$ and $T^{n+1}(x) = T(T^n(x))$; we call $T^n(x)$ the *iterate of x under T*. In order to simplify the notations we will often use Tx instead of $T(x)$.

The mapping $T^n (n \geq 1)$ is called the n^{th} *iterate* of T. For any $x_0 \in X$, the sequence $\{x_n\}_{n\geq 0}$ given by

$$x_n = Tx_{n-1} = T^n x_0, \quad n = 1, 2, \ldots$$

is called *the sequence of successive approximations with the initial value x_0*. It is also known as the *Picard iteration* starting at x_0.

For a given selfmap the following properties obviously hold:

(1) $F(T) \subset F(T^n)$, for each $n \in \mathbb{N}$;

(2) $F(T^n) = \{x\}$, for some $n \in \mathbb{N} \Rightarrow F(T) = \{x\}$.

The reverse of (2) is not true, in general, as shown by the next example.

Example 1.1.2. ([5]) Let $T : \{1, 2, 3\} \to \{1, 2, 3\}$, $T(1) = 3$, $T(2) = 2$ and $T(3) = 1$. Then $F(T^2) = \{1, 2, 3\}$ but $F(T) = \{2\}$.

Recent developments in fixed point theory reflect that the algorithmic construction for the approximation of fixed point problems are vigorously purposed and analyzed for various classes of mappings in different spaces. Since, most of the problems from various disciplines of sciences are nonlinear in nature. On the other hand, numerous problems in physics, optimization, and economics reduces to find the fixed points of a certain operator.

The fixed point theory is concerned with finding conditions on the structure that the set X must be endowed as well as on the properties of the operator T, in order to obtain results on;

1. the existence and uniqueness of fixed points;
2. the structure of the fixed point sets;
3. the approximation of fixed points.

However, from a practical point of view, it is important not only to know the fixed point exists (and, possible, is unique), but also to be able to construct that fixed points.

Theorem 1.1.3. (The Banach contraction principle) *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction, that is, there exists $\alpha \in [0, 1)$ such that, for all $x, y \in X$,*

$$d(Tx, Ty) \leq \alpha d(x, y).$$

Then T has a unique fixed point. Moreover, for each $x \in X$, the sequence $\{T^n x\}$ converges strongly to this fixed point.

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. A mapping T of C into itself is called *nonexpansive* if, for all $x, y \in C$

$$\|Tx - Ty\| \leq \|x - y\|$$

Example 1.1.4. *Let $C = [0, 1]$ and $T : C \to C$, $Tx = 1 - x$, for all $x \in C$. Then T is nonexpansive, T has unique fixed point, $F(T) = \{\frac{1}{2}\}$.*

Many researchers are interested in obtaining (additional) condition on T and X as general as possible, and which should guarantee the (strong) convergence of the Picard iteration to a fixed point of T. Moreover, if the Picard iteration converges to a fixed point of T, they will be interested in evaluating the error estimate (or alternatively, the rate of convergence) of the method, that is, in obtaining a stopping criterion for the sequence of successive approximations. However, the Picard iteration may not converge even in the weak topology.

Construction of fixed point iteration processes of nonlinear mappings is an important subject in the theory of nonlinear mappings, and finds application in a number of applied areas. Now, fixed point iteration processes for approximating fixed point of nonexpansive mappings, quasi-nonexpansive mappings, asymptotically quasi-nonexpansive mappings and generalized asymptotically quasi-nonexpansive mappings in various space have been studied by many mathematicians.

Let X be a real Banach space, and let C be a nonempty, closed and convex subset of X. Let $T : C \to C$ be a nonlinear mapping. Three classical iteration processes are often used to approximate a fixed point of T.

Halpern's iteration

The first one was introduced by Halpern [16] which is defined as follows: $x_0 \in C$,

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.1.1}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$.

Mann's iteration

The second iteration process was known as Mann's iteration process [31] which is defined by $x_0 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.1.2}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$.

Ishikawa's iteration

The third iteration process was known as Ishikawa's iteration process [20] which is defined by $x_0 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T(\beta_n x_n + (1 - \beta_n)Tx_n), \quad n \geq 0, \tag{1.1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$.

If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.1.2) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm; see Reich [42]). As a matter of fact, process (1.1.2) may fail to converge while process (1.1.3) can still converge for a Lipschitz pseudo-contraction in a Hilbert space; see [11] and [20]. However, in general, process (1.1.2) and (1.1.3) have only weak convergence even in a Hilbert space.

In 2000, Moudafi [35] introduced viscosity approximation method for a nonexpansive mapping as follows:

$$\begin{aligned} x_0 &= x \in C, \ \textit{arbitrarily}; \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0 \end{aligned} \tag{1.1.4}$$

where $f : C \to C$ is a contraction and $\{\alpha_n\} \subset [0, 1]$. He proved that if E is a real Hilbert space and $\{\alpha_n\}$ satisfies some certain control conditions, then the sequence $\{x_n\}$ defined by (1.1.4) converges strongly to a fixed point of T which is the unique solution to the variational inequality $\langle (I - f)\bar{x}, x - \bar{x} \rangle \geq 0$ for all $x \in F(T)$.

In 2004, Xu [65] extended the results of Moudafi [35] for the iterative scheme (2.3.3) to a Banach space setting.

In 2005, Kim and Xu [25] introduced the following iterative method:

$$\begin{aligned} x_0 &= x \in C, \ \textit{arbitrarily}; \\ y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \\ x_{n+1} &= \beta_n u + (1 - \beta_n)y_n, \quad n \geq 0 \end{aligned} \tag{1.1.5}$$

where $u \in C$ is arbitrary (but fixed) and $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. They proved strong convergence of iterative scheme (1.1.5) in the framework of uniformly smooth Banach space.

In 2008, Yao, et al. [66] proposed a modified iteration by combining the Mann iteration and the viscosity approximation method introduced by Moudafi [35]. They defined the iterative scheme as follows:

$$\begin{aligned} x_0 &= x \in C, \ \textit{arbitrarily}; \\ y_n &= \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0, \\ x_{n+1} &= \beta_n f(x_n) + (1-\beta_n)y_n, \quad n \geq 0 \end{aligned} \tag{1.1.6}$$

where $f : C \to C$ is a contraction and $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. They proved under certain different control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that the sequence $\{x_n\}$ generated by (1.1.6) converges strongly to a fixed point of T.

In 2008, Jung [22] introduced a new composite iterative scheme for nonexpansive mapping T as follows:

$$\begin{aligned} x_0 &= x \in C, \ \textit{arbitrarily}; \\ y_n &= \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 0 \\ x_{n+1} &= (1-\beta_n)y_n + \beta_n T y_n, \quad n \geq 0 \end{aligned} \tag{1.1.7}$$

where $f : C \to C$ is a contraction and $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. He proved strong convergence of the sequence $\{x_n\}$ defined by (1.1.7) under the suitable conditions of the control parameters $\{\alpha_n\}$ and $\{\beta_n\}$ and the asymptotic regularity on $\{x_n\}$ in a reflexive Banach space with a uniformly Gâteaux differentiable norm together with the assumption that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings.

In 2007, Aoyama, et al.[3] introduced a Halpern type iterative scheme for finding a common fixed point of a countable family of nonexpansive mappings as follows:

$$\begin{aligned} x_0 &= x \in C, \ \textit{arbitrarily}; \\ x_{n+1} &= \alpha_n u + (1-\alpha_n)T_n x_n, \quad n \geq 0 \end{aligned} \tag{1.1.8}$$

where $u \in C$ is arbitrary (but fixed), $\{\alpha_n\} \subset [0,1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings with some conditions. They proved that the sequence $\{x_n\}$ defined by (1.1.8) converges strongly to a common fixed point of $\{T_n\}$.

In 2009, Takahashi [52] modified the iterative scheme (1.1.4) for a countable family of nonexpansive mappings $\{T_n\}$ as follows:

$$\begin{aligned} x_0 &= x \in C, \ \textit{arbitrarily}; \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)T_n x_n, \quad n \geq 0 \end{aligned} \tag{1.1.9}$$

where $f : C \to C$ is a contraction, $\{\alpha_n\} \subset [0,1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings with some conditions. They proved that the sequence $\{x_n\}$ defined by (1.1.9) converges strongly to a common fixed point of $\{T_n\}$.

In 2009, Plubtieng and Wangkeeree [41] modified the iterative scheme (1.1.6) for a countable family of nonexpansive mappings $\{T_n\}$ as follows:

$$\begin{aligned} x_0 &= x \in C, \ \textit{arbitrarily}; \\ y_n &= \alpha_n x_n + (1-\alpha_n)T_n x_n, \quad n \geq 0, \\ x_{n+1} &= \beta_n f(x_n) + (1-\beta_n) y_n, \quad n \geq 0 \end{aligned} \tag{1.1.10}$$

where $f : C \to C$ is a contraction, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings with some conditions. They proved, under certain different control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, that the sequence $\{x_n\}$ defined by (1.1.10) converges strongly to a common fixed point of $\{T_n\}$.

In 2009, Klin-eam, Suatai [24] modified the iterative scheme (1.1.7) for a countable family of nonexpansive mappings $\{T_n\}$ as follows:

$$\begin{aligned} x_0 &= x \in C, \ \textit{arbitrarily}; \\ y_n &= \alpha_n f(x_n) + (1-\alpha_n)T_n x_n, \quad n \geq 0, \\ x_{n+1} &= (1-\beta_n) y_n + \beta_n T_n y_n, \quad n \geq 0 \end{aligned} \tag{1.1.11}$$

where C is a nonempty closed subset of a Banach space, $f : C \to C$ is a contraction, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings with some conditions. They proved, under certain different control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, that the sequence $\{x_n\}$ defined by (1.1.11) converges strongly to a common fixed point of $\{T_n\}$.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings with $F := \bigcap_{n=1}^N F(T_i) \neq \emptyset$. There are many authors introduced iterative method for finding an element of F which is an optimal point for the minimization problem. For $n > N$, T_n is understood as $T_{(n \ mod \ N)}$ with the mod function taking values in $\{1, 2, ..., N\}$. Let u be a fixed element of H.

In 2003, Xu [63] proved that the sequence $\{x_n\}$ generated by

$$x_{n+1} = (1 - \epsilon_n A)T_{n+1}x_n + \epsilon_{n+1}u \tag{1.1.12}$$

converges strongly to the solution of the quadratic minimization problem

$$\min_{x \in F} \frac{1}{2}\langle Ax, x\rangle - \langle x, u\rangle \tag{1.1.13}$$

under suitable hypotheses on ϵ_n and under the additional hypothesis,

$$F = F(T_1T_2...T_N) = F(T_NT_1...T_{N-1}) = ... = F(T_2T_3...T_NT_1).$$

In 1999, Atsushiba and Takahashi[4] defined the mapping W_n as follows:

$$\begin{aligned}
U_{n,0} &= I,\\
U_{n,1} &= \gamma_{n,1}T_1 + (1-\gamma_{n,1})I,\\
U_{n,2} &= \gamma_{n,2}T_2U_{n,1} + (1-\gamma_{n,2})I,\\
U_{n,3} &= \gamma_{n,3}T_3U_{n,2} + (1-\gamma_{n,3})I,\\
&\ \ \vdots\\
U_{n,N-1} &= \gamma_{n,N-1}T_{N-1}U_{n,N-2} + (1-\gamma_{n,N-1})I,\\
W_n &= U_{n,N} = \gamma_{n,N}T_NU_{n,N-1} + (1-\gamma_{n,N})I, \qquad (1.1.14)
\end{aligned}$$

where $\{\gamma_{n,i}\}_i^N \subseteq [0,1]$. This mapping is called the $W-$mapping generated by $T_1, T_2, ..., T_N$ and $\gamma_{n,1}, \gamma_{n,2}, ..., \gamma_{n,N}$.

In 2000, Takahashi and Shimoji [58] proved that If X is strictly convex Banach space, then $F(W_n) = \bigcap_{i=1}^N F(T_i)$, where $0 < \lambda_{n,i} < 1, \quad i = 1, 2, ..., N$.

In 2007, Shang et al.[45], introduced a composite iteration scheme as follows:

$$\begin{aligned}
x_0 &= x \in C \text{ arbitrarily chosen },\\
y_n &= \beta_n x_n + (1-\beta_n)W_n x_n, \qquad (1.1.15)\\
x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n,
\end{aligned}$$

where $f : C \to C$ is a contraction, and A is a linear bounded operator. They proved that the sequence $\{x_n\}$ converges to a common fixed point of the finite family of nonexpansive mappings, under certain appropriate assumptions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$.

Note that the iterative scheme (1.1.15) is not well-defined because $x_n (n \geq 1)$ may not lie in C, so $W_n x_n$ is not defined. However, if $C = H$, the iterative scheme (1.1.15) is well-defined and Theorem 2.1 [45] is obtained. In the case $C \neq H$, we have to modify the iterative scheme (1.1.15) in order to make it well-defined.

In 2009, Kangtunyakarn and Suantai [23] introduced a new mapping, called K-mapping, for finding a common fixed point of a finite family of nonexpansive mappings. For a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ and sequence $\{\gamma_{n,i}\}_i^N$ in [0,1], the mapping $K_n : C \to C$ is defined as follows:

$$\begin{aligned}
U_{n,1} &= \gamma_{n,1}T_1 + (1-\gamma_{n,1})I,\\
U_{n,2} &= \gamma_{n,2}T_2U_{n,1} + (1-\gamma_{n,2})U_{n,1},\\
U_{n,3} &= \gamma_{n,3}T_3U_{n,2} + (1-\gamma_{n,3})U_{n,2},\\
&\ \ \vdots\\
U_{n,N-1} &= \gamma_{n,N-1}T_{N-1}U_{n,N-2} + (1-\gamma_{n,N-1})U_{n,N-2},\\
K_n &= U_{n,N} = \gamma_{n,N}T_NU_{n,N-1} + (1-\gamma_{n,N})U_{n,N-1}, \qquad (1.1.16)
\end{aligned}$$

The mapping K_n is called the *K-mapping* generated by $T_1, ..., T_N$ and $\gamma_{n,1}, \gamma_{n,2} ..., \gamma_{n,N}$.

They used the $K-mapping$ for finding a common fixed point of a finite family of nonexpansive mappings and the solutions of equilibrum problems.

Let C be a nonempty closed convex subset of a Banach space X, and $f : C \to C$ be a contractive mapping with a contrative constant $\alpha \in (0, 1)$. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings and let $\alpha_1, \alpha_2, ...$ be real numbers such that $0 \leq \alpha_i \leq 1$ for every $i \in \mathbb{N}$. For any $n \in \mathbb{N}$, define a mapping W_n of C into itself as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \alpha_n T_n U_{n,n+1} + (1 - \alpha_n)I, \\
U_{n,n-1} &= \alpha_{n-1} T_{n-1} U_{n,n} + (1 - \alpha_{n-1})I, \\
&\vdots \\
U_{n,k} &= \alpha_k T_k U_{n,k+1} + (1 - \alpha_k)I, \\
U_{n,k-1} &= \alpha_{k-1} T_{k-1} U_{n,k} + (1 - \alpha_{k-1})I, \\
&\vdots \\
U_{n,2} &= \alpha_2 T_2 U_{n,3} + (1 - \alpha_2)I, \\
W_n &= U_{n,1} = \alpha_1 T_1 U_{n,2} + (1 - \alpha_1)I. \qquad (1.1.17)
\end{aligned}$$

Such a mapping W_n is called $W-$mapping generated by $T_n, T_{n-1}, ..., T_1$ and $\alpha_n, \alpha_{n-1}, ..., \alpha_1$; see [46].

In 2009, Yao et al. [66] unified iterative algorithm for finding a common fixed point of an infinite countable family of nonexpansive mappings as follows:

$$\begin{aligned}
x_0 &= x \in C, \ arbitrarily; \\
x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) W_n x_n, \qquad (1.1.18)
\end{aligned}$$

where W_n is the $W-$mapping generated by $T_1, T_2, ..., T_N$ and $\xi_1, \xi_2, ..., \xi_n$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings with some conditions. They proved, under certain control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, that the sequence $\{x_n\}$ defined by (1.1.18) converges strongly to a common fixed point of $\{T_n\}$.

The purpose of this thesis is third fold. Firstly, we construct and study new general iterative method by using the K-mapping for finding a common fixed points of a finite family of nonexpansive mappings in Hilbert spaces and we introduce an iterative method for finding a common fixed points of a countable family of nonexpansive mappings in Banach spaces. Secondly, we construct and study new iteration processes for finding a common element of a nonspreading-type mapping and equilibrium problem in a Hilbert space. Finally, we find sufficient condition for strong convergence theorem of an iterative method for finding a common fixed point of two nonspreading-type mappings in Hilbert spaces.

This thesis is divided into 5 chapters. Chapter 1 is an introduction to the research problems. Chapter 2 deals with some preliminaries and give some useful results that will be used in later chapters. Chapter 3-Chapter 4 are the main results of this research and the conclusion is in Chapter 5. Precisely, in section 3.1, we introduce a new general iterative method by using the $K-$ mapping for finding a common fixed point of a finite family of nonexpansive mappings in the framework of Hilbert spaces. A strong convergence theorem of the proposed iterative method is established under some certain control conditions. Our results improve and extend the results announced by many others. In section 3.2, let C be a nonempty closed convex subset of a real Banach space E. Let $f : C \to C$ be a given contractive mapping and $\{T_n\}_{n=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings such that the common fixed point sets $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For give $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)T_n x_n, \quad n \geq 0,$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$. It is shown that in a uniformly smooth Banach space the sequence $\{x_n\}$ converges strongly to $p \in F$ when $\{T_n\}$ satisfies $AKTT-condition$ and $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy some control conditions, and in a reflexive and strictly convex Banach space with a uniformly Gâteaux differential norm, the sequence $\{x_n\}$ converges strongly to $p \in F$ when $\{T_n\}$ satisfies the condition (B) and $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy some control conditions. Our results improve and extend many results in this area. Finally, we apply our results to solve the problem of finding a zero of an accretive operator in a Banach space. In section 4.1, we prove a strong convergence theorem for a nonspreading-type mappings and equilibrium problem in Hilbert spaces by using an idea of mean convergence. The main result of this paper extend the results obtained by Osilike and Isiogugu (Nonlinear Analysis 74 (2011) 1814-1822) and Kurokawa and Takahashi (Nonlinear Analysis 73 (2010) 1562-1568). Moreover, example and numerical results are also given. In section 4.2, by using the idea of mean convergence, we introduce an iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the fixed points set of a nonspreading-type mappings in Hilbert space. A strong convergence theorem of the proposed iterative scheme is established under some control conditions. The main result of this paper extend the results obtained by Osilike and Isiogugu (Nonlinear Analysis 74 (2011) 1814-1822) and Kurokawa and Takahashi (Nonlinear Analysis 73 (2010) 1562-1568). We also give an example and numerical results are also given. In section, 4.3, by using the idea of mean convergence, we introduce a new iterative scheme for finding a common fixed point of two k-strictly pseudononspreading mapping in Hilbert spaces. A strong convergence theorem of the proposed iteration is obtained. Our main result can be appiled for finding a common fixed point of two nonspreading mappings in Hilbert spaces.