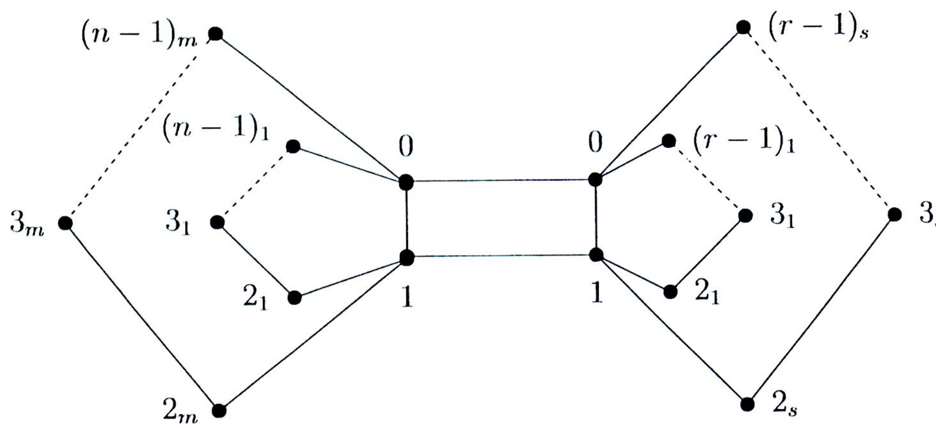
Chapter 7

Open Problem and Conclusion

For the open problem we may use our results in chapter 5 to determine the adjacency of the following graph,

$B_n(m) \asymp B_r(s)$



In this study, we obtain the following results:

7.1 Let $C_n$be a cycle of length n where $n \geq 2$. Then $\det(A(C_n)) = 0$ if and only if $4 \mid n$.

7.2 Let $n \geq 3$. Then

$$\det(A(C_n)) = \begin{cases} 0 & \text{if n} = 4\text{k} \text{ for some } k \in \mathbb{Z}^+; \\ -4 & \text{if n} = 4\text{k} + 2 \text{ for some } k \in \mathbb{Z}^+; \\ 2 & \text{otherwise.} \end{cases}$$

7.3 If a tree T has a sesquivalent spanning subgraph, then $|V(T)|$ must be even, that is the tree T contains an even number of vertices.

7.4 Let P be a sesquivalent spanning subgraph of the tree T. Then $\{x, x_i\} \in E(P)$ if and only if $|T_{x_i}|$ is odd.

7.5 A tree T has a sesquivalent spanning subgraph if and only if for each $x \in V(T)$ there is exactly one $x_i \in N(x)$ such that $|T_{x_i}|$ is odd.

7.6 Every tree has at most one sesquivalent spanning subgraph.

7.7 Let T be a tree of n vertices. Then

$$\det(A(T)) = \begin{cases} 1 & \text{if T has a sesquivalent spanning subgraph} \\ & \text{and } n = 4k \text{ for some } k \in \mathbb{Z}^+; \\ -1 & \text{if T has a sesquvivalent spanning subgraph} \\ & \text{and } n = 4k + 2 \text{for some } k \in \mathbb{Z}^+ \cup \{0\}; \\ 0 & \text{otherwise.} \end{cases}$$

7.8 If n is an odd number, then $B_n(m)$ has no spanning elementary subgraph for all positive integers $m > 2$.

7.9 Let n be odd and m be positive integer such that $m > 2$. Then

$$\det A(B_n(m)) = 0.$$

7.10 Let m, n be natural numbers where n is even and $m \geq 2$. Then

$$\det A(B_n(m)) = \begin{cases} (-1)^{\frac{mn-2m+2}{2}}(m-1)^2 & \frac{n}{2} \text{ is even;} \\ (-1)^{\frac{mn-2m+2}{2}}(m+1)^2 & \frac{n}{2} \text{ is odd.} \end{cases}$$

7.11 Let $m > 2$. Then the graph $B_n(m)$ is nonsingular if and only if n is even.

7.12 If G and H are (vertex) disjoint graphs, then

$$\det A(G \cup H) = \det A(G) \cdot \det A(H).$$

7.13 Let G and H be graphs such that $V(G) \cap V(H) = \{x\}$. Then

$$\det A(G \cup H) = \det A(G \backslash x) \cdot \det A(H) + \det A(H \backslash x) \cdot \det A(G).$$

7.14 Let $P_n(P_m)$ be a path, of the order n, of paths P_m, where $n \geq 1$ and $m \geq 2$. Then

$$\det A(P_n(P_m)) = \begin{cases} 0 & \text{if } nm \text{ is odd;} \\ (-1)^{\frac{nm}{2}} & \text{if } nm \text{ is even.} \end{cases}$$

7.15 Let $n \geq 3$ and $m \geq 1$. Then

$$\det A(C_n(P_m)) = \begin{cases} (-1)^{\frac{nm}{2}} & \text{if } m \text{ is even;} \\ 2(-1)^{\frac{nm-1}{2}} & \text{if } n \text{ and } m \text{ is odd;} \\ 4(-1)^{\frac{nm-n+2}{2}} & \text{if } m \text{ is odd and } n \equiv 2(mod4); \\ 0 & \text{if } m \text{ is odd and } n \equiv 0(mod4). \end{cases}$$

7.16 Let F be a path, of the order n, of cycles C_m, where $n \geq 1$ and $m \geq 3$. Then

$$\det A(F) = \begin{cases} 0 & \text{if m} \equiv 0(\text{mod } 4); \\ (-4)^n & \text{if m} \equiv 2(\text{mod } 4); \\ n+1 & \text{otherwise.} \end{cases}$$

7.17 Let $n, m \geq 3$. Then

$$\det A(C_n(C_m)) = \begin{cases} (-4)^n & \text{if } m \equiv 2(mod4); \\ 4 & \text{if } m \equiv 1(mod4) \text{ and } n \text{ is odd;} \\ 0 & \text{otherwise.} \end{cases}$$

7.18 Let G and H be two disjoint graphs of order $m > 1$ and $n > 1$ respectively. Denote the vertices of G by $V(G) = \{1, 2, \ldots, m\}$ and the vertices of H by $V(H) = \{m+1, m+2, \ldots, m+n\}$ and let $G \asymp H$ be the joint of the two graphs by the edges $\{m, m+1\}$ and $\{m-1, m+2\}$ (See Figure 1). The the determinant of $A(G \asymp H)$ can be computed as follows:

$$\det A(G \asymp H) = \det A(G) \det A(H) - \det A(G \setminus \{m\}) \det A(H \setminus \{m+1\})$$

$$- 2 \det A^{\nabla}(G) \det A^{\Delta}(H) - \det A(G \setminus \{m-1\}) \det A(H \setminus \{m+2\})$$

$$+ \det A(G \setminus \{m-1, m\}) \det A(H \setminus \{m+1, m+2\}).$$

7.18 Let C_m and C_n be cycle graphs with m and n vertices, respectively. Then for the determinant of the graph $C_m \asymp C_n$ we have

$$\det A(C_m \asymp C_n) = \begin{cases} 1 & \text{if } m \equiv 0(mod\ 4) \text{ and } n \equiv 0(mod\ 4); \\ -1 & \text{if } m \equiv 0(mod\ 4) \text{ and } n \equiv 2(mod\ 4); \\ & \text{or } m \equiv 2(mod\ 4) \text{ and } n \equiv 0(mod\ 4); \\ -4 & \text{if } m \equiv 1(mod\ 4) \text{ and } n \equiv 2(mod\ 4); \\ & \text{or } m \equiv 2(mod\ 4) \text{ and } n \equiv 1(mod\ 4); \\ 9 & \text{if } m \equiv 2(mod\ 4) \text{ and } n \equiv 2(mod\ 4); \\ 0 & \text{otherwise.} \end{cases}$$

7.19 The adjacency matrix of the graph $K_m \asymp K_n$ obtained by joining two complete graphs K_m and K_n, $m, n \geq 2$ by two edges is always singular.

7.20 Let A be the adjacency matrix of $P_m \Box C_n$ such that $n \geq 3$ and $m \geq 1$. Then

$$\det(A) = 2^m \prod_{i=0}^{m-1} (cos\frac{n\pi i}{m+1} - 1).$$