

Chapter 6

Determinant of Box Product of Some Graphs

In this section we study the determinant of $P_m \square G_n$ and study in this case G_n is C_n .

6.1 Determinant of $P_m \square G_n$

Theorem 6.1.1. *Let $q_n(x)$ be characteristic polynomial of $A(G_n)$ and $\lambda_{i,m}$ is all eigenvalue of $A(P_m)$. Then*

$$\det A(P_m \square G_n) = \prod_{i=1}^m q_n(\lambda_{m,i})$$

Proof. Let $A(P_m \square G_n)$ be the adjacency matrix of $P_m \square G_n$. We can relabel the vertices of this graph so that, as a block matrix, we have

$$A(P_m \square G_n) = \begin{pmatrix} A(G_n) & I_n & & \cdots & & 0 \\ I_n & A(G_n) & I_n & & & \\ & \ddots & \ddots & \ddots & & \\ & & I_n & A(G_n) & I_n & \vdots \\ \vdots & & & \ddots & \ddots & \ddots \\ & & & & I_n & A(G_n) & I_n \\ 0 & \cdots & & & I_n & A(G_n) \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. If we swap row i and $i + n$ of $A(P_m \square G_n)$, for $1 \leq i \leq nm - n$, we will obtain the matrix

$$A^*(P_m \square G_n) = \begin{pmatrix} I_n & A(G_n) & I_n & & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \\ \vdots & & I_n & A(G_n) & I_n & \vdots \\ & & & \ddots & \ddots & \ddots \\ & & & & I_n & A(G_n) & I_n \\ 0 & \cdots & & & I_n & A(G_n) \\ A(G_n) & I_n & & \cdots & & 0 \end{pmatrix}$$

Since we performed $mn - n$ row swaps, we have

$$\det(A(P_m \square G_n)) = (-1)^{mn-n} \det(A^*(P_m \square G_n)).$$

We can focus on the bottom block-row of $A^*(P_m \square G_n)$, which is $[A(G_n), I_n, 0, \dots, 0]$. We can repeatedly perform the block operation to this bottom block-row by subtracting a suitable multiple of another block-row.

If we let $B(0) = I_n$ and $B(1) = -A(G_n)$, then the last block-row of $A^*(P_m \square G_n)$ is $[-B(1), B(0), 0, \dots, 0]$. Then, we recursively define $B(k)$, for $k \geq 2$, by letting

$$B(k) = -A(G_n)B(k-1) - B(k-2).$$

Note that, inductively, it is easy to see that each $B(k)$ is a polynomial of degree k in $A(G_n)$, showing that $B(k)$ and $A(G_n)$. Now, we multiply the first block-row of $A^*(P_m \square G_n)$ by $B(1)$, resulting in $[B(1), A(G_n)B(1), B(1), \dots, 0]$, which we add to the bottom block-row of $A^*(P_m \square G_n)$, obtaining in

$$[0, A(G_n)B(1) + B(0), B(1), \dots, 0] = [0, -B(2), B(1), \dots, 0].$$

We now add this to the product of $B(2)$ with the second block-row of $A^*(P_m \square G_n)$, obtaining

$$[0, 0, A(G_n)B(2) + B(1), B(2), 0, \dots, 0] = [0, 0, -B(3), B(2), \dots, 0]$$

Suppose that after the $(k-1)$ -th iteration of this process we obtain the block-row $[0, \dots, 0, -B(k), B(k-1), 0, \dots, 0]$. Then after iterating this process once more, we obtain

$$[0, \dots, 0, 0, A(G_n)B(k) + B(k-1), B(k), 0, \dots, 0] = [0, \dots, 0, 0, -B(k+1), B(k), 0, \dots, 0].$$

Inductively, it follows that the $(m-2)$ th iteration of this process will result in $[0, \dots, 0, -B(m-1), B(m-2)]$, and thus the $(m-1)$ th iteration will result in $[0, \dots, 0, -B(m)]$.

This finally results in the matrix

$$A^{**}(P_m \square G_n) = \begin{pmatrix} I_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ & & I_n \\ 0 & \cdots & -B(m) \end{pmatrix}.$$

Thus,

$$\begin{aligned}
\det(A) &= (-1)^{mn-n} \det(A^*(P_m \square G_n)) \\
&= (-1)^{mn-n} \det(A^{**}(P_m \square G_n)) \\
&= (-1)^{mn-n} \det(-B(m)) \\
&= (-1)^{mn-n} (-1)^n \det(B(m)) \\
&= (-1)^{mn} \det(B(m)),
\end{aligned}$$

since $B(m)$ is an $n \times n$ matrix. To evaluate $B(m)$, we must look at characteristic polynomials.

Recall that we defined $q_k(x)$ to be the characteristic polynomial of $A(P_k)$, so $q_k(x) = \det(A(P_k) - xI_k)$. Note that $A(P_1) = (0)$ and $A(P_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, implying that $q_1(x) = -x$ and $q_2(x) = x^2 - 1$. Further, since

$$(A(P_k) - xI_k) = \begin{pmatrix} -x & 1 & 0 & \cdots & 0 \\ 1 & -x & 1 & 0 & \cdots \\ & \ddots & -x & \ddots & \\ \vdots & & 1 & -x & -1 & \vdots \\ & & & \ddots & \ddots & \ddots \\ & & & & 1 & -x & 1 \\ 0 & & \cdots & & 1 & -x \end{pmatrix}$$

if we expand by minors along the top row, it is not difficult to see that

$$\det(A(P_k) - xI_k) = -x \det(A(P_{k-1}) - xI_{k-1}) - \det(A(P_{k-2}) - xI_{k-2}),$$

implying that

$$q_k(x) = -xq_{k-1}(x) - q_{k-2}(x)$$

for $k \geq 3$. Further, we can define $q_0 = 1$, and this show that

$$q_2(x) = -xq_1(x) - q_0(x)$$

holds as well.

Then, consider our $B(k)$ defined above. We have $B(0) = I_n = q_0(A(G_n))$ and $B(1) = -A(G_n) = q_1(A(G_n))$; this, combined with the fact that $B(k)$ and $q_k(x)$ satisfy the same recursion relation for $k \geq 2$, implies that $B(k) = q_k(A(G_n))$. Now, since each of the $A(P_k)$ is symmetric, all roots of $q_k(x)$ are real; label them as $\lambda_{k,1}, \lambda_{k,2}, \dots, \lambda_{k,k}$ with $\lambda_{k,1} \geq \lambda_{k,2} \geq \dots \geq \lambda_{k,k}$. Then, it follows that

$$q_k(x) = (\lambda_{k,1} - x)(\lambda_{k,2} - x) \cdots (\lambda_{k,k} - x),$$

so

$$\begin{aligned} \det A(P_m \square G_n) &= (-1)^{mn} \det(q_m(A(G_n))) \\ &= (-1)^{mn} \det\left(\prod_{i=1}^m (\lambda_{m,i} I_n - A(G_n))\right) \\ &= (-1)^{mn} \det\left((-1)^m \prod_{i=1}^m (A(G_n) - \lambda_{m,i} I_n)\right) \\ &= (-1)^{mn} (-1)^{mn} \prod_{i=1}^m \det(A(G_n) - \lambda_{m,i} I_n) \\ &= \prod_{i=1}^m q_n(\lambda_{m,i}). \end{aligned}$$

Noting that $G_n \square P_m$ is isomorphic to $P_m \square G_n$, we immediately obtain the result that $\det A(P_m \square G_n) = \prod_{i=1}^n q_m(\lambda_{n,i})$, implying that

$$\prod_{i=1}^m q_n(\lambda_{m,i}) = \prod_{i=1}^n q_m(\lambda_{n,i})$$

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Example 6.1.2. Find the determinant of $P_2 \square C_4$

$$\begin{aligned} \det A(P_2 \square C_4) &= \prod_{i=1}^2 q_4(\lambda_{2,i}) \\ &= \prod_{i=1}^2 (-\lambda_{i,2}^4 + 4\lambda_{i,2}^2) \\ &= (-\lambda_{2,1}^4 + 4\lambda_{2,1}^2)(-\lambda_{2,2}^4 + 4\lambda_{2,2}^2) \\ &= (-(-1)^4 + 4(-1)^2)(-(1)^4 + 4(1)^2) \\ &= (3)(3) = 9 \end{aligned}$$

6.2 Determinant of $P_m \square C_n$

From [10], M. Doob find the determinant of $P_m \square P_n$ such that has result follow this :

Theorem 6.2.1. *Suppose $n, m \geq 1$. Then, if A is the adjacency matrix of the graph $P_n \square P_m$,*

$$\det(A) = \begin{cases} 0, & \text{if } \gcd(n+1, m+1) \neq 1; \\ (-1)^{\frac{mn}{2}}, & \text{if } \gcd(n+1, m+1) = 1. \end{cases}$$

In the same way we replace G_n by C_n , so $\det A(P_m \square C_n) = \prod_{i=1}^m q_n(\lambda_{m,i})$.

Lemma 6.2.2. [2] *Let $n \geq 1$. If $A(P_m)$ is the adjacency matrix of a path P_m of m vertices, then its eigenvalues are the numbers $\lambda_i = 2 \cos(\frac{\pi}{m+1})i$ where $i = 1, 2, 3, \dots, m$.*

Lemma 6.2.3. [26] *Let $n \geq 3$. If $A(C_n)$ is the adjacency matrix of a cycle C_n of n vertices, then its eigenvalues are the numbers $\lambda_j = 2 \cos(\frac{2\pi}{n})j$ where $j = 1, 2, 3, \dots, n$.*

Lemma 6.2.4. [10] *Let $q_n(x)$ be the characteristic polynomial of C_n . Then*

$$q_n(x) = 2 \cos(n \arccos \frac{\lambda}{2}) - 2.$$

Theorem 6.2.5. *Let A be the adjacency matrix of $P_m \square C_n$ such that $n \geq 3$ and $m \geq 1$. Then*

$$\det A(P_m \square C_n) = 2^m \prod_{i=1}^m (\cos \frac{n\pi i}{m+1} - 1).$$

Proof. By Theorem 6.1.1, we have :

$$\begin{aligned} \det A(P_m \square C_n) &= \prod_{i=1}^m q_n(\lambda_{m,i}) \\ &= \prod_{i=1}^m [2 \cos(n \arccos \frac{\lambda_{m,i}}{2}) - 2] \\ &= \prod_{i=1}^m [2 \cos(n \arccos \frac{(2 \cos \frac{\pi}{m+1})^i}{2}) - 2] \\ &= \prod_{i=1}^m [2 \cos(\frac{n\pi i}{m+1}) - 2] \end{aligned}$$

$$\begin{aligned} &= \prod_{i=1}^m 2[\cos(\frac{n\pi i}{m+1}) - 1] \\ &= 2^m \prod_{i=1}^m (\cos(\frac{n\pi i}{m+1}) - 1). \quad \blacksquare \end{aligned}$$