Chapter 5

Determinant of Graphs jointed by two edges

In this chapter we find a formula for the determinant of the adjacency matrix of graphs jointed by two edges and apply the formula to find the determinant of cycles jointed by two edges.

5.1 Introduction

Let G and H be two distinct simple graphs and let $G \asymp H$ denote the graph that is obtained by joining G with H by two additional edges (see Figure 1 below). We develop a procedure that allows us to compute the determinant of the connected graph $G \asymp H$, where as usual, under determinant of a graph we understand the determinant of the adjacency matrix of the graph.

The choice of the pair of vertices on each of the graphs G and H at which the connection is established is arbitrary and clearly the determinant of the resulting connected graph is a function of that choice and subsequently of the properties of those vertices. Without loss of generality let us denote the connecting vertices of the graph G by $m-1$ and m and of the graph H by $m+1$ and $m+2$.

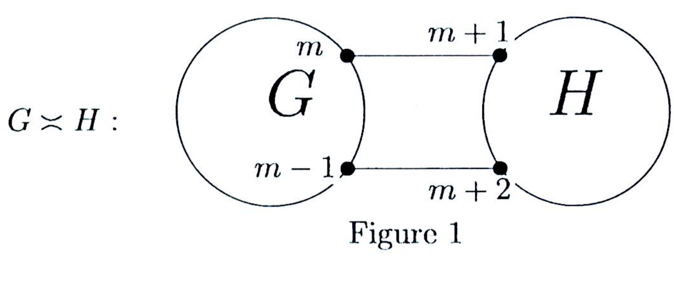

Figure 1

To achieve the main goal of the work we employ a formula for computing the determinant of an $n \times n$ matrix A called *Laplace expansion formula*. Before we formally state it, let us introduce some notation.

Let $\mathbf{r} = (r_1, r_2, \ldots, r_k)$ and $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ be ordered $k-$tuples of row indices and column indices respectively, for a square n by n matrix A, where $1 \leq k < n$,

$1 \leq r_1 < r_2 < \cdots < r_k \leq n$ and $1 \leq c_1 < c_2 < \cdots < c_k \leq n$. We denote the submatrix obtained by *selecting* the rows indicated in $\mathbf{r}$ and the columns indicated in $\mathbf{c}$ by $S(A; \mathbf{r}, \mathbf{c})$. We denote the submatrix obtained by *deleting* the rows indicated in $\mathbf{r}$ and columns indicated in $\mathbf{c}$ by $S^*(A; \mathbf{r}, \mathbf{c})$. If $\mathbf{r} = (r_i)$ and $\mathbf{c} = (c_j)$where $1 \leq i, j \leq n$, then we write $S(A; \mathbf{r}, \mathbf{c})$ as a_{ij} and $S^*(A; \mathbf{r}, \mathbf{c})$ as $M(A)_{ij}$. Observe, that a_{ij} is a single element matrix (a number) and $M(A)_{ij}$ is a submatrix of A obtained by deleting the i-th row and the j-th column. Note that some place we are use the symbol of determinant of an $n \times n$ by $|A|$. To better demonstrate the process let us consider the following example:

Example 5.1.1. Let $A = \begin{pmatrix} 1 & 5 & 0 & -2 \\ 1 & 0 & 1 & 7 \\ 2 & 3 & 0 & 5 \\ 1 & 1 & 6 & 2 \end{pmatrix}$ and $\mathbf{r} = (1, 3, 4)$, $\mathbf{c} = (1, 2, 3)$. Then

$$S(A; \mathbf{r}, \mathbf{c}) = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 3 & 0 \\ 1 & 1 & 6 \end{pmatrix} \text{ and } S^*(A; \mathbf{r}, \mathbf{c}) = \begin{pmatrix} 7 \end{pmatrix}$$

Theorem 5.1.2. (Laplace expansion formula) [1] *Let A be a $n \times n$ matrix and let $\mathbf{r} = (r_1, r_2, \ldots, r_k)$ be $k-$tuples of row indices, where $1 \leq k < n$ and $1 \leq r_1 < r_2 < \cdots < r_k \leq n$. Then*

$$\det A = (-1)^{\sigma(\mathbf{r})} \sum_{\mathbf{c}} (-1)^{\sigma(\mathbf{c})} |S(A; \mathbf{r}, \mathbf{c})| |S^*(A; \mathbf{r}, \mathbf{c})| \tag{5.1.1}$$

where $\sigma(\mathbf{r}) = r_1 + r_2 + \cdots + r_k$, $\sigma(\mathbf{c}) = c_1 + c_2 \cdots + c_k$, and the summation is over all $k-$tuples $\mathbf{c} = (c_1, c_2, \ldots c_k)$ for which $1 \leq c_1 < c_2 < \cdots < c_k \leq n$.

Example 5.1.3. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$ and $\mathbf{r} = (1, 2)$. Then $\sigma(\mathbf{r}) = 3$ and $\mathbf{c} = (c_1, c_2)$ for which $1 \leq c_1 < c_2 \leq 4$. Hence,

$$\det A = (-1)^3 \sum_{(c_1, c_2)} (-1)^{(c_1 + c_2)} |S(A; \mathbf{r}, (c_1, c_2))| |S^*(A; \mathbf{r}, (c_1, c_2))|$$

$$= |S(A; \mathbf{r}, (1, 2))| |S^*(A; \mathbf{r}, (1, 2))| - |S(A; \mathbf{r}, (1, 3))| |S^*(A; \mathbf{r}, (1, 3))|$$

$$+ |S(A;\mathbf{r},(1,4))||S^*(A;\mathbf{r},(1,4))| + |S(A;\mathbf{r},(2,3))||S^*(A;r,(2,3))|$$

$$- |S(A;\mathbf{r},(2,4))||S^*(A;\mathbf{r},(2,4))| + |S(A;\mathbf{r},(3,4))||S^*(A;\mathbf{r},(3,4))|$$

$$= \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \det\begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} - \det\begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix} \cdot \det\begin{pmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{pmatrix}$$

$$+ \det\begin{pmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{pmatrix} \cdot \det\begin{pmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{pmatrix} + \det\begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix} \cdot \det\begin{pmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{pmatrix}$$

$$- \det\begin{pmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{pmatrix} \cdot \det\begin{pmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{pmatrix} + \det\begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix} \cdot \det\begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}.$$

5.2 Determinant of graphs joined by two edges

Now we are ready to address our main goal - the determinant of $G \asymp H$. Let G and H be two distinct simple graphs of order greater than one and let us formally denote the vertex sets of the two graphs as $V(G) = \{1, 2, \ldots, m-1, m\}$ and $V(H) = \{m+1, m+2, \ldots, m+n\}$, with G and H connected by the edges $\{m, m+1\}$ and $\{m-1, m+2\}$. (see Figure 1 in Section 1). Clearly, the adjacency matrix of the resulting (connected) graph $G \asymp H$ has the following form:

$$\det A(G \asymp H) = \begin{pmatrix} & & & & & 0 & 0 & \ldots & 0 & 0 \\ & & & & & \vdots & \vdots & \vdots & & \vdots \\ & & A(G) & & & 0 & 1 & \ldots & 0 & 0 \\ & & & & & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 & & & & & \\ 0 & 0 & \ldots & 1 & 0 & & A(H) & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & & & \\ 0 & 0 & \ldots & 0 & 0 & & & & & \end{pmatrix}.$$

To compute the determinant of $A = A(G \asymp H)$ we are going to use the Laplace formula (5.1.1) with minors of dimension $m \times m$ (see Theorem 5.1.2).

Let us fix $\mathbf{r} = (1, 2, ..., m)$ and observe that any choice of $\mathbf{c} = (c_1, c_2, \ldots, c_m)$ that is not of the form $\mathbf{c} = (1, , \ldots, m-2, c_{m-1}, c_m)$ would yield a submatrix $S^*(A; \mathbf{r}, \mathbf{c})$ with a determinant 0. This is so by the definition of S^* (See Section 1) and the fact that all elements of A below the row m in columns from 1 to $m-2$ are 0 (recall that G and H are disjointed).

Furthermore, with a similar argument, if the remaining two spots for $\mathbf{c}$, namely c_{m-1} and c_m, are different than $m-1, m, m+1$ or $m+2$ the determinant of the submatrix $S(A; \mathbf{r}, \mathbf{c})$ would equal 0, because it would contain a column of elements from rows 1 to m and some column to the right of column $m+2$, but they are all zeros.

Thus in formula (5.1.1) only the six summands that correspond to $\mathbf{c} = (1, 2, \ldots, m-2, m-1, m)$, $\mathbf{c} = (1, 2, \ldots, m-2, m-1, m+1)$, $\mathbf{c} = (1, 2, \ldots, m-2, m-1, m+2)$, $\mathbf{c} = (1, 2, \ldots, m-2, m, m+1)$, $\mathbf{c} = (1, 2, \ldots, m-2, m, m+2)$, $\mathbf{c} = (1, 2, \ldots, m-2, m+1, m+2)$ would yield a non-zero result.

For the case $\mathbf{c} = (1, 2, \ldots, m-1, m)$, we have $S(A; \mathbf{r}, \mathbf{c}) = A(G)$, $S^*(A; \mathbf{r}, \mathbf{c}) = A(H)$ and so

$$(-1)^{\sigma(\mathbf{c})}|S(A;\mathbf{r},\mathbf{c})||S^*(A;\mathbf{r},\mathbf{c})| = (-1)^{s_m} \det A(G) \cdot \det A(H) \tag{5.2.1}$$

where $s_m = 1 + 2 + \cdots + m$.

We consider the remaining five cases in the next series of lemmas. We adopt the following notation - for a graph G with a vertex $x \in V(G)$, we denote by $G \backslash x$ the subgraph of G that is obtained by removing from G the vertex x and all edges that are incident to x.

Lemma 5.2.1. *Let* $A = A(G \asymp H)$, $\mathbf{r} = (1, 2, \ldots, m)$ *and* $\mathbf{c} = (1, 2, \ldots, m-2, m-1, m+1)$. *Then*

$$(-1)^{\sigma(\mathbf{c})}|S(A;\mathbf{r},\mathbf{c})||S^*(A;\mathbf{r},\mathbf{c})| = (-1)^{s_m+1} \det A(G\backslash m) \cdot \det A(H\backslash m+1),$$

where $s_m = 1 + 2 + \cdots + m$.

Proof. Clearly, $\sigma(\mathbf{c}) = 1 + 2 + \cdots + (m-1) + (m+1) = s_m + 1$.

Next we compute $|S(A;\mathbf{r},\mathbf{c})|$:

$$|S(A;\mathbf{r},\mathbf{c})| = \det \begin{pmatrix} & & & & 0 \\ & & & & 0 \\ & & A(G\backslash m) & & \vdots \\ & & & & 0 \\ a_{m1} & a_{m2} & \cdots & a_{m(m-1)} & 1 \end{pmatrix}$$

$$= \det A(G\backslash m)$$

and

$$|S^*(A;\mathbf{r},\mathbf{c})| = \det \begin{pmatrix} 1 & a_{(m+1)(m+2)} & a_{(m+1)(m+3)} & \cdots & a_{(m+1)(m+n)} \\ 0 & & & & \\ \vdots & & A(H\backslash m+1) & & \\ 0 & & & & \\ 0 & & & & \end{pmatrix}$$

$$= \det A(H\backslash m+1).$$

So $(-1)^{\sigma(\mathbf{c})}|S(A;\mathbf{r},\mathbf{c})||S^*(A;\mathbf{r},\mathbf{c})| \;=\; (-1)^{s_m+1}\det A(G\backslash m)\cdot\det A(H\backslash m+1)$. ■

Lemma 5.2.2. *Let* $A = A(G \asymp H)$, $\mathbf{r} = (1,2,\ldots,m)$ *and* $\mathbf{c} = (1,2,\ldots,m-2,m,m+2)$. *Then*

$$(-1)^{\sigma(\mathbf{c})}|S(A;\mathbf{r},\mathbf{c})||S^*(A;\mathbf{r},\mathbf{c})| \;=\; (-1)^{s_m+3}\det A(G\backslash m-1)\cdot\det A(H\backslash m+2),$$

where $s_m = 1+2+\cdots+m$.

Proof. Clearly, $\sigma(\mathbf{c}) = 1+2+\cdots+(m-2)+m+(m+2) = s_m+3$. Further, we have

$$|S(A;\mathbf{r},\mathbf{c})| = \det\begin{pmatrix} & & & a_{1m} & 0 \\ & A(G\backslash\{m-1,m\}) & & a_{2m} & \vdots \\ & & & \vdots & 0 \\ a_{(m-1)1} & \cdots & a_{(m-1)(m-2)} & a_{(m-1)m} & 1 \\ a_{m1} & \cdots & a_{m(m-2)} & a_{mm} & 0 \end{pmatrix}$$

$$= -\det\begin{pmatrix} & & & a_{1m} \\ & A(G\backslash\{m-1,m\}) & & a_{2m} \\ & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix}$$

$$= -\det A(G\backslash m-1)$$

and similarly

$$|S^*(A;\mathbf{r},\mathbf{c})| = \det\begin{pmatrix} 0 & a_{(m+1)(m+1)} & a_{(m+1)(m+3)} & \cdots & a_{(m+1)(m+1)} \\ 1 & a_{(m+2)(m+1)} & a_{(m+2)(m+1)} & \cdots & a_{(m+2)(m+n)} \\ 0 & a_{(m+3)(m+1)} & & & \\ \vdots & \vdots & & A(H\backslash\{m+1,m+2\}) & \\ 0 & a_{(m+n)(m+1)} & & & \end{pmatrix}$$

$$= -\det\begin{pmatrix} a_{(m+1)(m+1)} & a_{(m+1)(m+3)} & \cdots & a_{(m+1)(m+n)} \\ a_{(m+3)(m+1)} & & & \\ \vdots & & A(H\backslash\{m+1,m+2\}) & \\ a_{(m+n)(m+1)} & & & \end{pmatrix}$$

$$= -\det A(H\backslash m+2).$$

So $(-1)^{\sigma(\mathbf{c})}|S(A;\mathbf{r},\mathbf{c})||S^*(A;\mathbf{r},\mathbf{c})| \ = \ (-1)^{s_m+3}\det A(G\backslash m-1)\cdot\det A(H\backslash m+2)$.

■

For the fourth possible choice of **c** we have

Lemma 5.2.3. *Let* $A = A(G \asymp H)$, $\mathbf{r} = (1,2,\ldots,m)$ *and* $\mathbf{c} = (1,2,\ldots,m-2,m+1,m+2)$.

Then

$$(-1)^{\sigma(\mathbf{c})}|S(A;\mathbf{r},\mathbf{c})||S^*(A;\mathbf{r},\mathbf{c})| \ = \ (-1)^{s_m+4}\det A(G\backslash\{m-1,m\})\cdot \det A(H\backslash\{m+1,m+2\})$$

where $s_m = 1+2+\cdots+m$.

Proof. Straightforward $\sigma(\mathbf{c}) = 1+2+\cdots+(m-2)+(m+1)+(m+2) = s_m+4$. Further,

$$|S(A;\mathbf{r},\mathbf{c})| = \det\begin{pmatrix} & & & 0 & 0 \\ & A(G\backslash\{m-1,m\}) & & \vdots & \vdots \\ & & & 0 & 0 \\ a_{(m-1)1} & \cdots & a_{(m-1)(m-2)} & 0 & 1 \\ a_{m1} & \cdots & a_{m(m-2)} & 1 & 0 \end{pmatrix}$$

$$= -\det\begin{pmatrix} & & & 0 \\ & A(G\backslash\{m-1,m\}) & & \vdots \\ & & & 0 \\ a_{m1} & \cdots & a_{(m)(m-2)} & 1 \end{pmatrix}$$

$$= -\det A(G\backslash\{m-1,m\})$$

and

$$|S^*(A;\mathbf{r},\mathbf{c})| = \det\begin{pmatrix} 0 & 1 & a_{(m+1)(m+3)} & \cdots & a_{(m+1)(m+n)} \\ 1 & 0 & a_{(m+2)(m+3)} & \cdots & a_{(m+2)(m+n)} \\ 0 & 0 & & & \\ \vdots & \vdots & & A(H\backslash\{m+1,m+2\}) & \\ 0 & 0 & & & \\ 0 & 0 & & & \end{pmatrix}$$

$$= -\det \begin{pmatrix} 1 & a_{(m+1)(m+3)} & \cdots & a_{(m+1)(m+n)} \\ 0 & & & \\ \vdots & & A(H\backslash\{m+1, m+2\}) & \\ 0 & & & \\ 0 & & & \end{pmatrix}$$

$$= -\det A(H\backslash\{m+1, m+2\}).$$

So

$$(-1)^{\sigma(\mathbf{c})}|S(A;\mathbf{r},\mathbf{c})||S^*(A;\mathbf{r},\mathbf{c})| = (-1)^{s_m+4} \det A(G\backslash\{m-1, m\}) \cdot \det A(H\backslash\{m+1, m+2\}).$$

■

For the remaining two cases for the $m-$tuples $\mathbf{c}$ let us introduce the following definitions.

Definition 5.2.4. Let A be a square $p \times p$ matrix.

We denote by A^{∇} the $(p-1) \times (p-1)$ submatrix of A, $S(A;\mathbf{r},\mathbf{c})$, where $\mathbf{r} = (1, 2, \ldots, p-2, p)$ and $\mathbf{c} = (1, 2, \ldots, p-1)$.

We denote by A^{Δ} the $(p-1) \times (p-1)$ submatrix $S(A;\mathbf{r},\mathbf{c})$, where $\mathbf{r} = (2, 3, \ldots, p)$ and $\mathbf{c} = (1, 3, \ldots, p)$.

In the context of our discussion and notations the submatrix $A^{\nabla}(G)$ of the adjacency matrix for the graph G is a $(m-1) \times (m-1)$ matrix which coincides with $A(G\backslash m)$ with the last column and row of the matrix $A(G\backslash m)$ replaced by v and t respectively,

$$\text{where } v = \begin{pmatrix} a_{1(m-1)} \\ a_{2(m-1)} \\ \vdots \\ a_{(m-2)(m-1)} \\ a_{m(m-1)} \end{pmatrix} \text{ and } t = \begin{pmatrix} a_{m1} & a_{m2} & \ldots & a_{m(m-1)} \end{pmatrix}.$$

The matrix $A^{\Delta}(H)$ is a $(n-1) \times (n-1)$ matrix which coincides with the matrix $A(H\backslash m+1)$ with the first column and first row of matrix $A(H\backslash m+1)$ replaced by v

and t respectively, where $v = \begin{pmatrix} a_{(m+2)(m+1)} \\ a_{(m+3)(m+1)} \\ \vdots \\ a_{(m+n)(m+1)} \end{pmatrix}$

and $t = \begin{pmatrix} a_{(m+2)(m+1)} & a_{(m+2)(m+3)} & \cdots & a_{(m+2)(m+n)} \end{pmatrix}$.

Lemma 5.2.5. *Let* $A = A(G \asymp H)$, $\mathbf{r} = (1, 2, \ldots, m)$ *and* $\mathbf{c} = (1, 2, \ldots, m-2, m-1, m+2)$. *Then*

$$(-1)^{\sigma(\mathbf{c})}|S(A;\mathbf{r},\mathbf{c})||S^*(A;\mathbf{r},\mathbf{c})| = (-1)^{s_m+3} \det A^{\nabla}(G) \cdot \det A^{\Delta}(H),$$

where $s_m = 1 + 2 + \cdots + m$.

Proof. Clearly, $\sigma(\mathbf{c}) = 1 + 2 + \cdots + (m-1) + (m+2) = s_m + 2$. Further, for the determinants of $S(A;\mathbf{r},\mathbf{c})$ and $S^*(A;\mathbf{r},\mathbf{c})$ we get

$$|S(A;\mathbf{r},\mathbf{c})| = \det \begin{pmatrix} & & & a_{1(m-1)} & 0 \\ & A(G\backslash\{m-1,m\}) & & a_{2(m-1)} & \vdots \\ & & & \vdots & 0 \\ a_{(m-1)1} & a_{(m-1)2} & \cdots & a_{(m-1)(m-1)} & 1 \\ a_{m1} & a_{m2} & \cdots & a_{m(m-1)} & 0 \end{pmatrix}$$

$$= -\det \begin{pmatrix} & & & a_{1(m-1)} \\ & A(G\backslash\{m-1,m\}) & & a_{2(m-1)} \\ & & & \vdots \\ & & & a_{(m-2)(m-1)} \\ a_{m1} & a_{m2} & \cdots & a_{m(m-1)} \end{pmatrix}$$

$$= -\det A^{\nabla}(G)$$

and

$$|S^*(A;\mathbf{r},\mathbf{c})| = \det \begin{pmatrix} 1 & a_{(m+1)(m+1)} & a_{(m+1)(m+3)} & \cdots & a_{(m+1)(m+n)} \\ 0 & a_{(m+2)(m+1)} & a_{(m+2)(m+3)} & \cdots & a_{(m+2)(m+n)} \\ 0 & a_{(m+3)(m+1)} & & & \\ \vdots & \vdots & & A(H\backslash\{m+1,m+2\}) & \\ 0 & a_{(m+n)(m+1)} & & & \end{pmatrix}$$

$$= \det \begin{pmatrix} a_{(m+2)(m+1)} & a_{(m+2)(m+3)} & \cdots & a_{(m+2)(m+n)} \\ a_{(m+3)(m+1)} & & & \\ \vdots & & A(H\backslash\{m+1, m+2\}) & \\ a_{(m+n)(m+1)} & & & \end{pmatrix}$$

$$= \det A^{\Delta}(H).$$

So, $(-1)^{\sigma(\mathbf{c})}|S(A;\mathbf{r},\mathbf{c})||S^*(A;\mathbf{r},\mathbf{c})| = (-1)^{s_m+3} \det A^{\nabla}(G) \cdot \det A^{\Delta}(H)$. ■

Finally, we have

Lemma 5.2.6. *Let* $A = A(G \asymp H)$, $\mathbf{r} = (1, 2, \ldots, m)$ *and* $\mathbf{c} = (1, 2, \ldots, m-2, m, m+1)$. *Then*

$$(-1)^{\sigma(\mathbf{c})}|S(A;\mathbf{r},\mathbf{c})||S^*(A;\mathbf{r},\mathbf{c})| = (-1)^{s_m+3} \det A^{\nabla}(G) \cdot \det A^{\Delta}(H),$$

where $s_m = 1 + 2 + \cdots + m$.

Proof. The result follows directly by mimicking the proof of Lemma 5.2.5, combined with the observation that $S(A;\mathbf{r},\mathbf{c}) = (G^{\nabla})^T$, $S^*(A;\mathbf{r},\mathbf{c}) = (H^{\Delta})^T$ and the fact that the determinant of a matrix is the same as the determinant of its transpose. ■

Combining the results of the discussion so far we obtain the central result for the section.

Theorem 5.2.7. *Let G and H be two disjoint graphs of order $m > 1$ and $n > 1$ respectively. Denote the vertices of G by $V(G) = \{1, 2, \ldots, m\}$ and the vertices of H by $V(H) = \{m+1, m+2, \ldots, m+n\}$ and let $G \asymp H$ be the joint of the two graphs by the edges $\{m, m+1\}$ and $\{m-1, m+2\}$ (See Figure 1). The the determinant of $A(G \asymp H)$ can be computed as follows:*

$$\det A(G \asymp H) = \det A(G) \det A(H) - \det A(G \setminus \{m\}) \det A(H \setminus \{m+1\})$$
$$- 2 \det A^{\nabla}(G) \det A^{\Delta}(H) - \det A(G\setminus\{m-1\}) \det A(H\setminus\{m+2\})$$
$$+ \det A(G \setminus \{m-1, m\}) \det A(H \setminus \{m+1, m+2\}).$$

Proof. Recall that $\mathbf{r} = (1, 2, \ldots, m)$ and so $\sigma(\mathbf{r}) = 1 + 2 \cdots + m = s_m$. Thus, substituting in the Laplace formula (5.1.1) the results from the formula (5.2.1) and the Lemmas 5.2.1 to 5.2.6 we get:

$$
\begin{aligned}
\det A(G \asymp H) &= (-1)^{\sigma(r)} \sum_{\mathbf{c}} (-1)^{\sigma(\mathbf{c})} |S(A;\mathbf{r},\mathbf{c})||S^*(A;\mathbf{r},\mathbf{c})| \\
&= (-1)^{s_m} [(-1)^{s_m} \det A(G) \det A(H) \\
&\quad + (-1)^{s_m+1} \det A(G \setminus \{m\}) \det A(H \setminus \{m+1\}) \\
&\quad + (-1)^{s_m+3} \det A(G \setminus \{m-1\}) \det A(H \setminus \{m+2\}) \\
&\quad + (-1)^{s_m+4} \det A(G \setminus \{m-1, m\}) \det A(H \setminus \{m+1, m+2\}) \\
&\quad + 2\,(-1)^{s_m+3} \det A^{\nabla}(G) \det A^{\Delta}(H)\,].
\end{aligned}
$$

So finally we have

$$
\begin{aligned}
\det A(G \asymp H) &= \det A(G) \det A(H) - \det A(G \setminus \{m\}) \det A(H \setminus \{m+1\}) \\
&\quad - 2 \det A^{\nabla}(G) \det A^{\Delta}(H) - \det A(G \setminus \{m-1\}) \det A(H \setminus \{m+2\}) \\
&\quad + \det A(G \setminus \{m-1, m\}) \det A(H \setminus \{m+1, m+2\}),
\end{aligned}
$$

as needed. ∎

5.3 The determinant of $C_m \asymp C_n$ and $K_m \asymp K_n$

In this section we apply the main result of Section 2 to calculate the determinant of cycles joined by two edges and complete graphs joined by two edges.

Before we implement the formula from Theorem 5.2.7, recall that for the determinant of a cycle C_n we have (see also [4])

$$\det(A(C_n)) = \begin{cases} 0 & \text{if } n \equiv 0(mod\ 4), \\ -4 & \text{if } n \equiv 2(mod\ 4), \\ 2 & \text{otherwise.} \end{cases} \tag{5.3.1}$$

and for the determinant of a path graph P_n we have (see also [4])

$$\det(A(P_n)) = \begin{cases} (-1)^k & \text{if } n = 2k \text{ for some k} \in \mathbb{Z}^+, \\ \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.2}$$

Next, we calculate the determinants of $A^{\nabla}(C_m)$ and $A^{\Delta}(C_m)$, where we transform the adjacency matrix of C_m as stated in Definition 5.2.4.

Lemma 5.3.1. *Let C_m be a cycle graph with m vertices. Then*

$$\det A^{\nabla}(C_m) = \det A^{\Delta}(C_m) = \det A(P_{m-2}) + (-1)^m.$$

Proof. By definition, the matrix $A^{\nabla}(C_m)$ has the form

$$A^{\nabla}(C_m) = \begin{pmatrix} & & & 0 \\ & A(C_m \backslash m-1, m\}) & & \vdots \\ & & & 0 \\ & & & 1 \\ 1 & 0 & \dots\ 0 & 1 \end{pmatrix}.$$

Observe that $A(C_m \backslash \{m-1, m\}) = A(P_{m-2})$. Observe, further, that if we remove the first column from the adjacency matrix of a path graph, the resulting matrix is in a lower triangular form, with ones on the main diagonal and zeros above it.

Thus, computing the determinant of $A^{\nabla}(C_m)$ by adding the cofactors expanded on the last row we have

$$\det A^{\nabla}(C_m) = \det A(P_{m-2}) + (-1)^m.$$

Next, by definition $A^{\Delta}(C_m)$ is in the form

$$A^{\Delta}(C_m) = \begin{pmatrix} 1 & 1 \; 0 \; \ldots & 0 \\ 0 & & \\ \vdots & & A(C_m\backslash\{1,2\}) \\ 0 & & \\ 1 & & \end{pmatrix}$$

and computing the cofactors along the first column we get

$$\det A^{\Delta}(C_m) = \det A(P_{m-2}) + (-1)^m.$$ ■

Theorem 5.3.2. *Let C_m and C_n be cycle graphs with m and n vertices, respectively. Then for the determinant of the graph $C_m \asymp C_n$ we have*

$$\det A(C_m \asymp C_n) = \begin{cases} 1 & \text{if } m \equiv 0 (mod\ 4) \text{ and } n \equiv 0 (mod\ 4); \\ -1 & \text{if } m \equiv 0 (mod\ 4) \text{ and } n \equiv 2 (mod\ 4); \\ & \text{or } m \equiv 2 (mod\ 4) \text{ and } n \equiv 0 (mod\ 4); \\ -4 & \text{if } m \equiv 1 (mod\ 4) \text{ and } n \equiv 2 (mod\ 4); \\ & \text{or } m \equiv 2 (mod\ 4) \text{ and } n \equiv 1 (mod\ 4); \\ 9 & \text{if } m \equiv 2 (mod\ 4) \text{ and } n \equiv 2 (mod\ 4); \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Substituting the result of Lemma 5.3.1 into the formula of Theorem 5.2.7 we obtain directly the following result:

$$\begin{aligned} \det A(C_m \asymp C_n) &= \det A(C_m) \cdot \det A(C_n) - 2 \det A(P_{m-1}) \cdot \det A(P_{n-1}) \\ &- 2[(\det A(P_{m-2}) + (-1)^m) \cdot (\det A(P_{n-2}) + (-1)^n)] \\ &+ \det A(P_{m-2}) \cdot \det A(P_{n-2}). \end{aligned}$$

Finally, substituting the results from equations (5.3.1) and (5.3.2) into the result above we obtain the conclusion of the theorem. ■

Now consider $K_m \asymp K_n$, the joined of the complete graphs K_m and K_n, $m, n \geq 2$. Recall that for the determinant of a complete graph K_s we have (see [4])

$$\det A(K_s) = (-1)^{s-1}(s-1) \tag{5.3.3}$$

and observe that

$$\det A^{\nabla}(K_s) = \det A^{\Delta}(K_s) = (-1)^{s-2}, \quad \text{for any} \quad s \geq 2. \tag{5.3.4}$$

Thus, using the results above into the general formula from Theorem 5.2.7, for the determinant of the graph $K_m \asymp K_n$ we have:

$$\begin{aligned}\det A(K_m \asymp K_n) &= \det A(K_m) \cdot \det A(K_n) - 2\det A(K_{m-1}) \cdot \det A(K_{n-1}) \\ &\quad -2[(\det A^{\nabla}(K_m)(\det A^{\Delta}(K_n))] + \det A(K_{m-2}) \cdot \det A(K_{n-2}) = 0.\end{aligned}$$

So we proved the following result:

Theorem 5.3.3. *The adjacency matrix of the graph $K_m \asymp K_n$ obtained by joining two complete graphs K_m and K_n, $m, n \geq 2$ by two edges is always singular.*