Chapter 3

Determinant of Cycles and Trees

We divide this chapter into three sections. In section 3.1, we study the eigenvalues of the adjacency matrix of circulant graphs. We prove that every adjacency matrix of complete graph is nonsingular, and characterize nonsingular cycles. In section 3.2, we provide the definition of sesquivalent graphs and then use the Harary's formula (see Proposition 2.3.8) to characterize nonsingular tree. In section 3.3, we provide the difinition of a spacial type of graphs called book graphs and characterize the nonsingular book graphs.

3.1 Nonsingularity of Complete Graphs and Cycles

Let G be a circulant matrix and let $[0, a_2, ..., a_n]$ be the first row of the adjacency matrix of a circulant graph G. Then the eigenvalues of G are given by the following formula;

$$\lambda_r = \sum_{j=2}^{n} a_{1j}\omega^{(j-1)r}, r = 0, 1, ..., n-1 \tag{3.1.1}$$

where $\omega = e^{2\pi i/n}$.

Lemma 3.1.1. *For* $n \geq 2$,

$$\det(A(K_n)) = (-1)^{n-1}(n-1).$$

Proof. Let K_n be a complete graph where $n \geq 2$.

$$A(K_n) = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & 1 \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}.$$

Thus by definition of 2.3.3, $A(K_n)$ is a circulant matrix and by Lemma 2.3.5,

$$\lambda_0 = \sum_{j=2}^{n} a_{1j}\omega^{(j-1)0} = a_{12} + a_{13} + a_{14} + \dots + a_{1n} = 1 + 1 + \dots + 1 = n - 1,$$

$$\lambda_1 = \sum_{j=2}^{n} a_{1j}\omega^{(j-1)1} = \omega + \omega^2 + \omega^3 + \dots + \omega^{(n-1)},$$

$$\lambda_2 = \sum_{j=2}^{n} a_{1j}\omega^{(j-1)2} = \omega^2 + \omega^4 + \omega^6 + \dots + \omega^{(n-1)2},$$

$$\vdots$$

$$\lambda_{n-1} = \sum_{j=2}^{n} a_{1j}\omega^{(j-1)(n-1)} = \omega^{n-1} + \omega^{2(n-1)} + \omega^{3(n-1)} + \dots + \omega^{(n-1)(n-1)}.$$

Hence $\lambda_r = \omega^r + \omega^{2r} + \omega^{3r} + \dots + \omega^{(n-1)r}$ where $r = 1, 2, \dots, n-1$.

Since $1 + \omega^r + \omega^{2r} + \dots + \omega^{(n-1)r} = 0$ then $\omega^r + \omega^{2r} + \dots + \omega^{(n-1)r} = -1$.

Thus $\lambda_1 = \lambda_2 = \dots = \lambda_{n-1} = -1$.

By Theorem 2.2.11, $\det(A(K_n)) = \lambda_0 \cdot \lambda_1 \cdots \lambda_{n-1}$

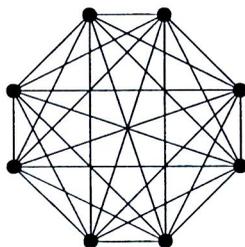$$= (n - 1)(-1)(-1)\cdots(-1)$$

$$= (-1)^{n-1}(n - 1).$$

Therefore, $\det(A(K_n)) = (-1)^{n-1}(n - 1)$. ∎

Corollary 3.1.2. *For all $n \geq 2$, $A(K_n)$ is nonsingular.*

Proof. By Lemma 3.1.1, $\det(A(K_n)) = (-1)^{n-1}(n - 1)$ where $n \geq 2$. Then $\det(A(K_n)) \neq 0$. Therefore, $A(K_n)$ is nonsingular. ∎

Example 3.1.3. Find the determinant of K_8.

K_8 :

$$A(K_8) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Then $\det(A(K_8)) = (-1)^{8-1}(8-1) = -7$.

Lemma 3.1.4. *For $n \geq 2$ if $A(C_n)$ is the adjacency matrix of a cycle C_n, then the eigenvalues are $\lambda_r = 2\cos(\frac{2\pi r}{n})$ where $r = 0, 1, 2, ..., n-1$.*

Proof. Since $A(C_n)$ is circulant matrix, by Lemma 2.3.5,

$$\lambda_0 = \sum_{j=2}^{n} a_{1j}\omega^{(j-1)0} = a_{12} + a_{1n},$$

$$\lambda_1 = \sum_{j=2}^{n} a_{1j}\omega^{(j-1)1} = a_{12}\omega + a_{1n}\omega^{(n-1)},$$

$$\lambda_2 = \sum_{j=2}^{n} a_{1j}\omega^{(j-1)2} = a_{12}\omega^2 + a_{1n}\omega^{(n-1)2},$$

$$\vdots$$

$$\lambda_{n-1} = \sum_{j=2}^{n} a_{1j}\omega^{(j-1)(n-1)} = a_{12}\omega^{n-1} + a_{1n}\omega^{(n-1)(n-1)}, where\ \omega = e^{2\pi i/n}$$

So $\lambda_r = \omega^r + \omega^{r(n-1)}$ for all $r \in \{0, 1, 2, ..., n-1\}$.

Next, consider

$$\omega^r = (\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n})^r$$

$$= (\cos\frac{2\pi r}{n} + i\sin\frac{2\pi r}{n})$$

and

$$\begin{aligned}
\omega^{r(n-1)} &= (\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n})^{r(n-1)} \\
&= (\cos\frac{2\pi r(n-1)}{n} + i\sin\frac{2\pi r(n-1)}{n}) \\
&= (\cos(2\pi r - \frac{2\pi r}{n}) + i\sin(2\pi r - \frac{2\pi r}{n})) \\
&= \cos\frac{2\pi r}{n} - i\sin\frac{2\pi r}{n}.
\end{aligned}$$

Then $\quad \lambda_r = \omega^r + \omega^{r(n-1)}$

$$= 2\cos\frac{2\pi r}{n}.$$

Therefore $\lambda_r = 2\cos\dfrac{2\pi r}{n}$ for all $r \in \{0,1,2,...,n-1\}$. ∎

Lemma 3.1.5. *Let C_n be a cycle of order n where $n \geq 3$. Then $\det(A(C_n)) = 0$ if and only if $4 \mid n$.*

Proof. Let C_n be a cycle of order n. Assume that $\det(A(C_n)) = 0$. By Lemma 3.1.4, $A(C_n)$ has eigenvalues $\lambda_r = 2\cos\dfrac{2\pi r}{n}$ for all $r \in \{0,1,2,...,n-1\}$.

Since $\det(A(C_n)) = 0$, there exists $\lambda_r = 0$ for some $r \in \{0,1,2,...,n-1\}$. Consider r such that $\lambda_r = 0$, we have $2\cos(\dfrac{2\pi r}{n}) = 0$. Then $\dfrac{2\pi r}{n} = \dfrac{\pi}{2}m$ where m is an odd number, hence $mn = 4r$. Therefore, $4|n$.

Conversely, assume that $4|n$. Let $n = 4k$ for some $k \in \mathbb{Z}^+$ and choose $r = k$. Thus $\lambda_k = 2\cos\dfrac{2\pi k}{4k} = 2\cos\dfrac{\pi}{2} = 0$. Therefore, $\det(A(C_n)) = 0$. ∎

Example 3.1.6.

$C_6 :$
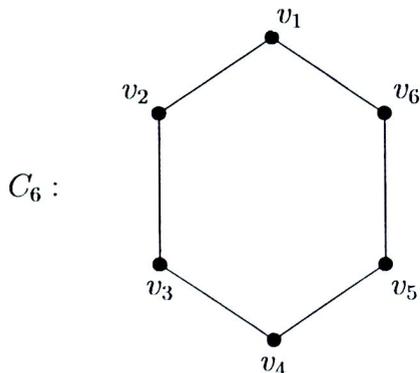
$$A(C_6) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

By Lemma 3.1.4,

$$\begin{aligned} \lambda_0 &= 2\cos 0 = 2 \\ \lambda_1 &= 2\cos\left(\frac{\pi}{3}\right) = 1 \\ \lambda_2 &= 2\cos\left(\frac{2\pi}{3}\right) = -1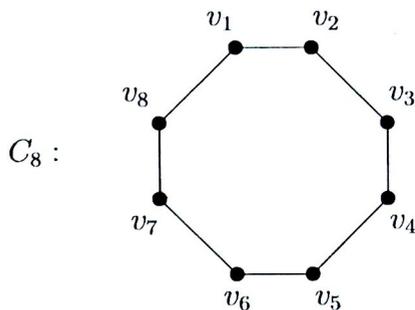 \\ \lambda_3 &= 2\cos(\pi) = -2 \\ \lambda_4 &= 2\cos\left(\frac{4\pi}{3}\right) = -1 \\ \lambda_5 &= 2\cos\left(\frac{5\pi}{3}\right) = 1. \end{aligned}$$

Thus

$$\begin{aligned} \det(A(C_6)) &= \lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_6 \\ &= (2)(1)(-1)(-2)(-1)(1)(1) \\ &= -4. \end{aligned}$$

Example 3.1.7.

C_8 :

$$\det(A(C_8)) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

By Lemma 3.1.5, $\det(A(C_8)) = 0$.

Theorem 3.1.8. *Let $n \geq 3$. Then*

$$\det(A(C_n)) = \begin{cases} 0 & \text{if } n = 4k \text{ for some } k \in \mathbb{Z}^+; \\ -4 & \text{if } n = 4k + 2 \text{ for some } k \in \mathbb{Z}^+; \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Case *I*: If $n = 4k$, then we use Lemma 3.1.5.

Case *II*: Let $n = 4k + 2$ for some $k \in \mathbb{Z}^+$. Then there are exactly three sesquivalent spanning subgraphs of C_n which are
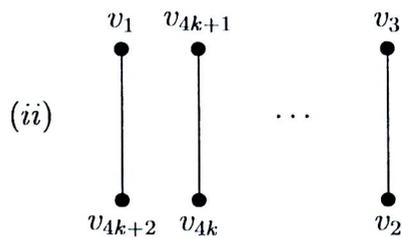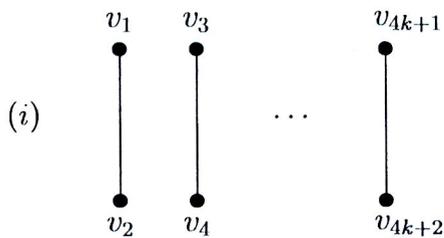
(*iii*) the graph C_n itself which is the only subgraph containing a cycle.

For the graphs (*i*) and (*ii*), we have $c = 2k + 1, r = 2k + 1$ and $s = 0$. For the graph (*iii*), we have $c = 1, r = 4k + 1$ and $s = 1$. Therefore, we have $\det(A(C_{4k+2})) = 2[(-1)^{2k+1}2^0] + (-1)^{4k+1}2^1 = -4$.

Case III : If $n = 4k + 1$ or $4k + 3$, then C_n is the only sesquivalent spanning subgraph of C_n. Thus, $c = 1, r = 4k(\text{or } 4k + 2)$, and $s = 1$. Therefore, we have $\det(A(C_{4k+1})) = (-1)^{4k}2^1 = 2 = \det(A(C_{4k+3}))$. ∎

As an immediate consequence of the above theorem and Theorem 3.1.1 we obtain the following formula which is difficult to prove by means of purely analytic methods:

Corollary 3.1.9. *Let* $n \geq 1$. *Then*

$$2^n \prod_{r=0}^{n-1} \cos(\frac{2\pi r}{n}) = \begin{cases} 0 & \text{if } n = 4k \text{ for some } k \in \mathbb{Z}^+; \\ -4 & \text{if } n = 4k + 2 \text{ for some } k \in \mathbb{Z}^+; \\ 2 & \text{otherwise.} \end{cases}$$

Example 3.1.10. For $n = 3$, we get the easy to chick equality:

$$\cos 0 \cos \frac{2\pi}{3} \cos \frac{4\pi}{3} = \frac{1}{4}.$$

But for $n = 5$ for example we get an interesting and much harder to check equality:

$$\cos 0 \cos \frac{2\pi}{5} \cos \frac{4\pi}{5} \cos \frac{6\pi}{5} \cos \frac{8\pi}{5} = \frac{1}{16}.$$

3.2   Characterization of Non-singular Trees

Let T be a tree. As usual, we denote by $V(T)$ the set of its vertices and by $E(T)$ the set of its edges. Since we want to apply Proposition 2.2.5 to determine $\det(A(T))$, we need to identify all sesquivalent spanning subgraphs of the tree T. It turns out that it is not so common that a tree contains such subgraphs.

Lemma 3.2.1. *If a tree T has a sesquivalent spanning subgraph, then $|V(T)|$ must be even, that is the tree T contains an even number of vertices.*

Proof. By definition of tree graph, no cycle is a subgraph of a tree. Thus, if P is a sesquivalent spanning subgraph of a tree T, then P consists of a number of separate (single) edges. Each edge contains two(different) vertices, hence P must contain an even number of vertices. If, additionally, P is a spanning subgraph of T, then $V(T) = V(P)$ and hence $|V(T)|$ is even.

Suppose that $x \in V(T)$ and $N(x) = \{x_1, x_2, \ldots, x_{n_x}\}$ is the set of all neighbors of x in T, Let us define, for each $x_i \in N(x)$, a subset T_{x_i} of the set $V(T)$:

$T_{x_i} = \{y : y = x_i \text{ or there is a path from } y \text{ to } x_i \text{ which does not pass through } x\}.$

$\blacksquare$

Lemma 3.2.2. *Let P be a sesquivalent spanning subgraph of the tree T. Then $\{x, x_i\} \in E(P)$ if and only if $|T_{x_i}|$ is odd.*

Proof. It is easy to notice that all T_{x_i} are disjoint subsets of $V(T)$. Since they are connected subsets of $V(T)$, they may also be regarded as subtrees of the tree T. Let P_i denote the subgraph of P induced on the set T_{x_i}. Now, it is obvious that P_i is a sesquivalent spanning subgraph of the tree T_{x_i} if $\{x, x_i\} \notin E(P)$. If $\{x, x_i\} \in E(P)$, then the subgraph of P induced on the set $T_{x_i} \cup \{x\}$ is a sesquivalent spanning subgraph of the appropriate subgraph of T. Thus, by Lemma 3.2.1, we conclude that $\{x, x_i\} \in E(P)$ if and only if $|T_{x_i}|$ is odd.

If P is a sesquivalent spanning subgraph of the tree T, then there is the only one $x_i \in N(x)$ such that $\{x, x_i\} \in E(P)$. Thus, by lemma3.2.1., if a tree T has a sesquivalent spanning subgraph, then for very $x \in V(T)$ there is exactly on $x_i \in N(x)$ such that

$|T_{x_i}|$ is odd. Let us prove that this is also a sufficient condition for a tree to have a sesquivalent spanning subgraph. ∎

Theorem 3.2.3. *A tree T has a sesquivalent spanning subgraph if and only if for every $x \in V(T)$ there is exactly one $x_i \in N(x)$ such that $|T_{x_i}|$ is odd.*

Proof. One implication follows from the above lemma. To prove the reverse implication, let us assume that T is a tree such that for every $x \in V(T)$ there is exactly one element in $N(x)$, denote this element by $f(x)$, such that $|T_{f(x)}|$ is odd.

$$E(P) = \{\{x, f(x)\} : x \in V(T)\}.$$

To prove that P is a sesquivalent subgraph of T, it suffices to show that $f(f(x)) = x$ for each $x \in V(T)$.

Let $x \in V(T)$ and let $N(x) = \{x_1, x_2, ..., x_{n_x}\}$. We know that $f(x) \in N(x)$. It also means that $x \in N(f(x))$. Thus, it suffices to prove that the set

(1) $\{y : y = x$ or there is a path from y to x which does not pass $f(x)\}$

has an odd number of elements. But any path that joins y with x (and does not pass $f(x)$) must contain an element of $N(x)$ (different from $f(x)$). Thus, the set (1) equals to

(2) $$\bigcup\{T_{x_i} : x_i \neq f(x)\} \cup \{x\}.$$

The sets T_{x_i} are disjoint and each of them contains an even number of elements if $x_i \neq f(x)$ (we recall that among the set T_{x_i} only $T_{f(x)}$ contains an odd number of elements). Moreover, the element x belongs to none of these sets. Therefore, we conclude that the set (2) (and hence (1) as well) contains an even number of elements which was to be proved. ∎

Since that right hand side of the above equivalence depends only on the tree T, and it is either fulfilled or is not for any given tree, a sesquivalent spanning subgraph (if it exists) is defined uniquely by Lemma 3.2.2. Thus, we get

Corollary 3.2.4. *Every tree has at most one sesquivalent spanning subgraph.*

Now, we can apply Proposition 2.3.8 to compute the determinant of the adjacency matrix of a tree. We get

Theorem 3.2.5. *Let T be a tree of n vertices. Then*

$$\det(A(T)) = \begin{cases} 1 & \text{if } T \text{ has a sesquivalent spanning subgraph} \\ & \text{and } n = 4k \text{ for some } k \in \mathbb{Z}^+; \\ -1 & \text{if } T \text{ has a sesquvivalent spanning subgraph} \\ & \text{and } n = 4k + 2 \text{for some } k \in \mathbb{Z}^+ \cup \{0\}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose that T has a sesquivalent spanning subgraph P and $n = 4k$. Then the number of components, c of P and the number of edges both are $\dfrac{n}{2} = \dfrac{4k}{2} = 2k$. So $r(P) = n - c = 4k - 2k = 2k$ and $s(P) = 2k - 4k + 2k = 0$. Thus, we get $\det(A(T)) = (-1)^{2k}(2)^0 = 1$.

Similarly, if T has a sesquivalent spanning subgraph P and $n = 4k + 2$, then $c = 2k + 1$ and hence $r(P) = 2k + 1$ and $s(P) = 0$. Thus, we get $\det(A(T)) = (-1)^{2k+1}(2)^0 = -1$ ∎

Combining Corollary 3.2.4 and Theorem 3.2.5 above we have the following characterization for non-singular trees:

Corollary 3.2.6. *A tree T is non-singular if and only if T has a sesquivalent spanning subgraph.*

As direct corollary from Theorem 3.2.5 we can also obtain a well known result for the determinant of a path P_n.

Corollary 3.2.7. *Let P_n be a path of n vertices. Then*

$$\det(A(P_n)) = \begin{cases} (-1)^k & \text{if } n = 2k \text{ for some } k \in \mathbb{Z}^+, \\ \\ 0 & \text{otherwise.} \end{cases}$$

3.3 Characterization of Non - singular Book Graphs

In this section, we find formulas for the determinant of the adjacency matrices of book graphs. We start by providing definition of the object.

For each $i = 1, 2, \ldots, m$, let G_i be a graph which is isomorphic to cycle C_n with the vertex set $V(G_i) = \{0_i, 1_i, 2_i, \ldots, (n-1)_i\}$, and edge set $E(G_i) = \{\{x_i, (x+1)_i\} | x = 0, 1, 2, \ldots, n-1\}$ where $0_i = 0, 1_i = 1$ for all $i = 1, \ldots, m$ and $+$ is addition modulo n. (Note that for all $i \neq j, V(G_i) \cap V(G_j) = \{0, 1\}$).

We call a graph $B_n(m)$ a C_n book graph of $m \geq 2$ pages if the vertex set of $B_n(m)$ is $V(B_n(m)) = \bigcup_{i=1}^{m} V(G_i)$ and the edge set is $E(B_n(m)) = \bigcup_{i=1}^{m} E(G_i)$.

Example 3.3.1. $B_5(3)$
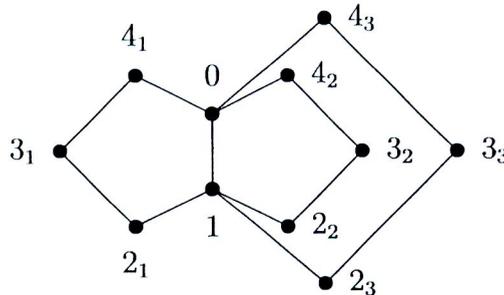
We proceed to compute the determinant of a C_n book graph by implementing the formula from Proposition 2.3.8. Thus in this case we have to compute for which we need to compute the rank and co-rank for every sesquivalent spanning subgraph of the given book graph.

In the case when the pages of a book graph are cycles of odd length it is not difficult to see that such sesquivalent spanning subgraphs do not exist, provided there are more than two pages in the book graph.

Proposition 3.3.2. *If n is an odd number, then $B_n(m)$ has no sesquivalent spanning subgraph for all positive integers $m > 2$.*

Proof. Let m be a positive integer greater than two and n an odd integer. Suppose $B_n(m)$ has sesquivalent spanning subgraph and let us consider the two possible partitioning cases: (Look at the graph below for a motivation)

Case I : $B_n(m)$ is partitioned only into single edges.

If one of the spanning edges for the sesquivalent spanning subgraph of $B_n(m)$ is $\{0, 1\}$ then each page has a (remaining) order of $n - 2$, (since every page contains the edge $\{0, 1\}$). Thus, 2 must divide $n - 2$, so that the single edge partitioning of the pages to be possible. This applies that n is even, a contradiction to the assumption on n.

If the single edge partitioning of $B_n(m)$ does not contain the edge $\{0, 1\}$ then, there is a page that has to span all vertices starting at 0 and finishing at 1, but that is a path of even length and as such does not assume a single edge partitioning.

Hence a partitioning of $B_n(m)$ into single edges is impossible.

Case II : $B_n(m)$ is partitioned into a cycle and single edges.

First observe that no sesquivalent spanning subgraph of $B_n(m)$ can contain more than one cycle. The cycle of the sesquivalent spanning subgraph must be an entire page of the book or two pages "transformed" into one cycle by removing the edge $\{0, 1\}$. Clearly, in the former case, after the cycle (page) is chosen the remaining pages have an order of $n - 2$, that is they are paths of even length and as such do not assume elementary partitioning, and in the latter case the same holds, since there are more than two pages ($m > 2$).

Thus, Case II is also impossible and therefore $B_n(m)$ has no sesquivalent spanning subgraph. ∎

From the result of the Proposition 3.3.2 and Corollary 2.3.9 we have:

**Lemma 3.3.3.** *Let n be odd and m be positive integer such that $m > 2$. Then*

$$\det A(B_n(m)) = 0.$$

To complete the study for the determinant of book graphs let us consider book graphs whose pages are cycles of even order, i.e. for given $B_n(m)$ let n be even and $m \geq 2$.
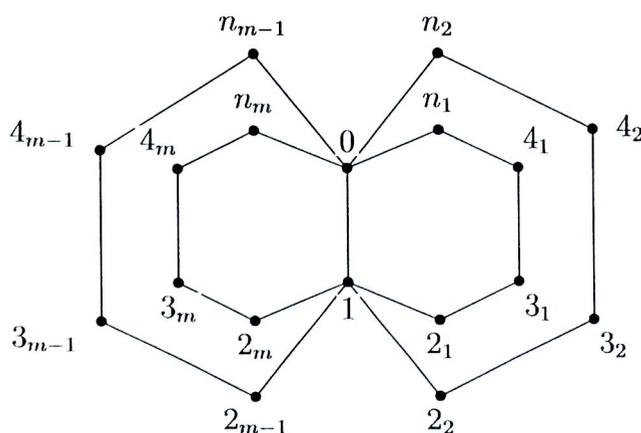
As we discussed in the proof of Proposition 3.3.2 (look at the graph above for notation) the sesquivalent spanning subgraph of $B_n(m)$ are of the following types:

Γ_2 :

Γ_3 :

Γ_4 :
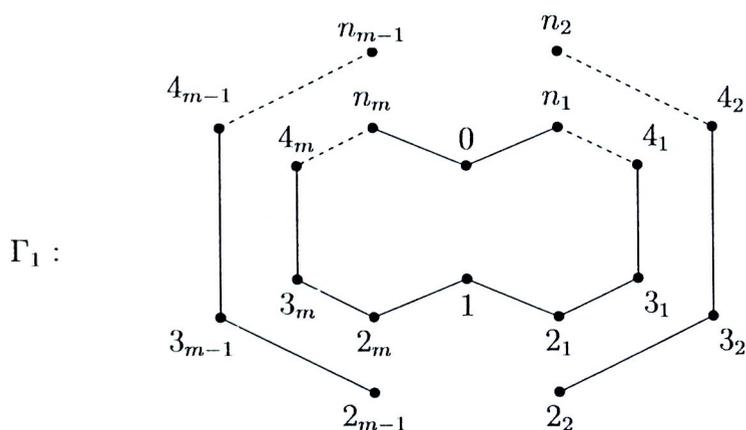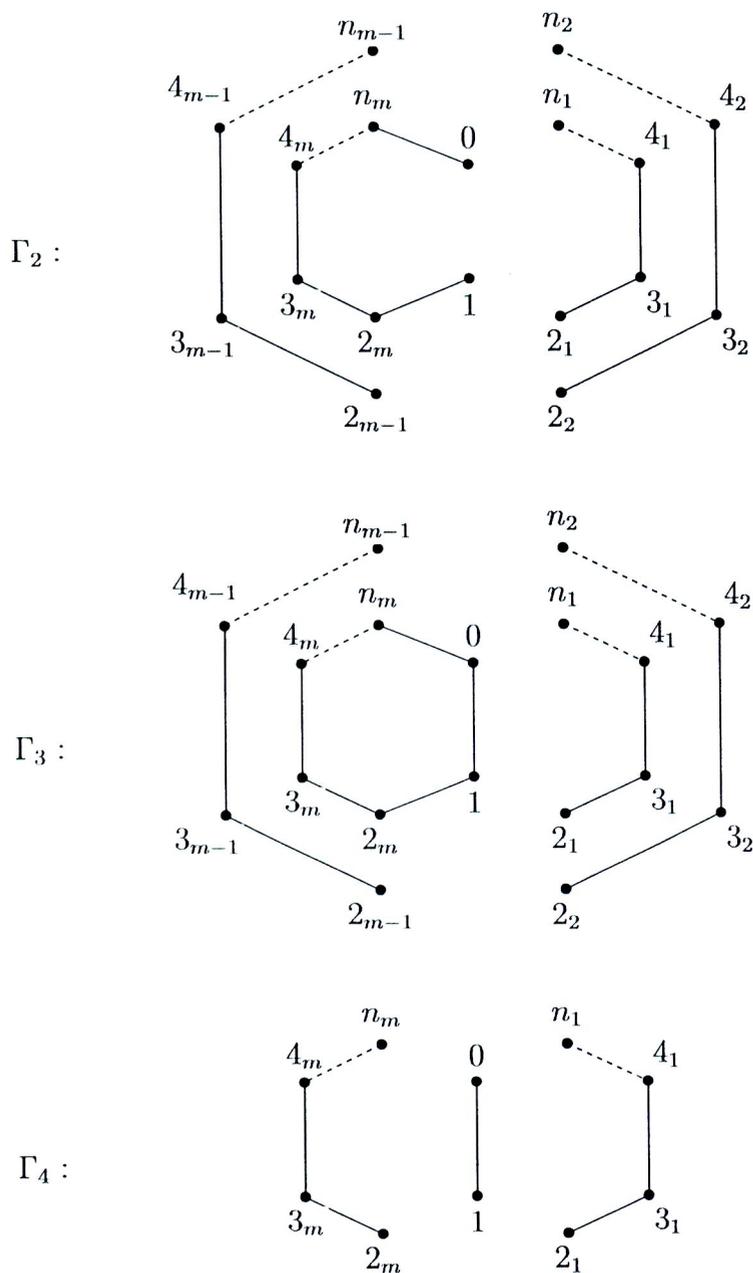
For each of the observed patterns we compute $r(\Gamma)$ and $s(\Gamma)$ as follows:

Γ_1 :

$$r(\Gamma_1) = (n-2)m + 2 - \{\frac{(n-2)(m-2)}{2} + 1\} = \frac{mn - 2m + 2n - 2}{2}$$

$$s(\Gamma_1) = \frac{(n-2)(m-2)}{2} + 2n - 2 - \{(n-2)m + 2\} + \{\frac{(n-2)(m-2)}{2} + 1\}$$

$$= 1$$

$$\Gamma_2: \qquad r(\Gamma_2) = (n-2)m + 2 - \{\frac{(n-2)(m-1)+n}{2}\} = \frac{mn - 2m + 2}{2}$$

$$s(\Gamma_2) = \frac{(n-2)(m-1)+n}{2} - \{(n-2)m+2\} + \{\frac{(n-2)(m-1)+n}{2}\}$$

$$= 0$$

$$\Gamma_3: \qquad r(\Gamma_3) = (n-2)m + 2 - \{\frac{(m-1)(n-2)}{2}+1\} = \frac{mn - 2m + n}{2}$$

$$s(\Gamma_3) = \frac{(n-2)(m-1)}{2} + n - \{(n-2)m+2\} + \{\frac{(m-1)(n-2)}{2}+1\}$$

$$= 1$$

$$\Gamma_4: \qquad r(\Gamma_4) = (n-2)m + 2 - \{\frac{(n-2)m}{2}+1\} = \frac{mn - 2m + 2}{2}$$

$$s(\Gamma_4) = \frac{(n-2)m}{2} + 1 - \{(n-2)m+2\} + \{\frac{mn - 2m + 2}{2}\}$$

$$= 0. \qquad\qquad\qquad \blacksquare$$

Now we are ready to compute the determinant of the book graphs with pages of even order.

Theorem 3.3.4. *Let m, n be natural numbers where n is even and $m \geq 2$. Then*

$$\det A(B_n(m)) = \begin{cases} (-1)^{\frac{mn-2m+2}{2}} (m-1)^2 & \text{if } \frac{n}{2} \text{ is even;} \\[3mm] (-1)^{\frac{mn-2m+2}{2}} (m+1)^2 & \text{if } \frac{n}{2} \text{ is odd.} \end{cases}$$

Proof. Let n be even and $m \geq 2$. By diagrams $(\Gamma_1 - \Gamma_4)$ above we see that for each $B_n(m)$ there are $\binom{m}{2}$ sesquivalent spanning subgraph of type Γ_1, there are m sesquivalent spanning subgraph of type Γ_2, there are m sesquivalent spanning subgraph of type Γ_3 and only one of type Γ_4 .

Then by Proposition 2.3.8 we compute the determinant of $A(B_n(m))$ to be the following:

$$\det(A(B_n(m))) = C_2^m\{(-1)^{\frac{mn-2m+2n-2}{2}}(2)^1\} + m\{(-1)^{\frac{mn-2m+2}{2}}(2)^0\}$$

$$+ m\{(-1)^{\frac{mn-2m+n}{2}}(2)^1\} + \{(-1)^{\frac{mn-2m+2}{2}}(2)^0\}$$

$$= \frac{m!}{(m-2)!2!}\{(-1)^{\frac{mn-2m+2n-2}{2}}2\} + m\{(-1)^{\frac{mn-2m+2}{2}}\}$$

$$+ m\{(2)(-1)^{\frac{mn-2m+n}{2}}\} + (-1)^{\frac{mn-2m+2}{2}}$$

$$= m(m-1)\{(-1)^{\frac{mn-2m-2}{2}}\} + m\{(-1)^{\frac{mn-n+2}{2}}\} + 2m\{(-1)^{\frac{mn-2m+n}{2}}\}$$

$$+ (-1)^{\frac{mn-2m+2}{2}}$$

$$= (-1)^{\frac{mn-2m}{2}}\{m(m-1)(-1)^{-1} + m(-1) + 2m(-1)^{\frac{n}{2}} + (-1)\}$$

$$= (-1)^{\frac{mn-2m}{2}}\{-m^2 + 2m(-1)^{\frac{n}{2}} - 1\}.$$

We see that if $\frac{n}{2}$ is even, then $\det A(B_n(m)) = (-1)^{\frac{mn-2m}{2}}(-m^2 + 2m - 1)$ and if $\frac{n}{2}$ is odd, then $\det A(B_n(m)) = (-1)^{\frac{mn-2m}{2}}(-m^2 - 2m - 1)$. ∎

By combining the results from Corollary 3.3.3 and Theorem 3.3.4, we obtain the following characterization for non-singular book graphs:

Corollary 3.3.5. *Let $m > 2$. Then the graph $B_n(m)$ is nonsingular if and only if n is even.*

For completeness of the discussion observe that the graph $B_n(2)$ is never singular with the determinant of the graph equal:

$$\det A(B_n(2)) = \begin{cases} -1 & \text{if } \frac{n}{2} \text{ is even;} \\\\ -9 & \text{if } \frac{n}{2} \text{ is odd;} \\\\ -2 & \text{if } n \text{ is odd.} \end{cases}$$