

# Chapter 2

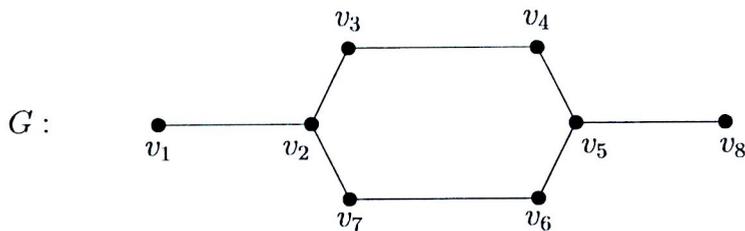
## Preliminaries

In this chapter we provide some important basic definitions and concepts that are used in this research work. Although the details are included in some cases, many of the well known fundamental results for graphs are stated without proof.

### 2.1 Basic Concepts on Graphs

**Definition 2.1.1.** A graph  $G$  consists of a nonempty set of elements, called *vertices*, and a list of unordered pairs of these elements, called *edges*. The set of vertices of the graph  $G$  is called the vertex set of  $G$ , denoted by  $V(G)$ , and the list of edges is called the edge list of  $G$ , denoted by  $E(G)$ . If  $v$  and  $w$  are vertices of  $G$ , then an edge of the form  $\{v, w\}$  or  $\{w, v\}$  is said to join  $v$  and  $w$ .

**Example 2.1.2.**



$$V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$$

$$\text{and } E(G) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_6, v_7\}, \{v_7, v_2\}, \{v_4, v_6\}, \{v_5, v_8\}\}$$

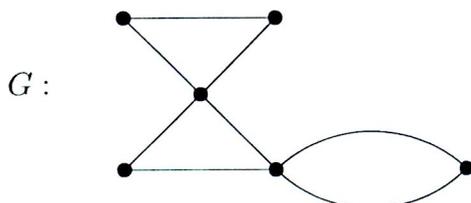
**Definition 2.1.3.** Two or more edges joining the same pair of vertices are called multiple edges, and an edge joining a vertex to itself is called a *loop*. A graph with no loops and no multiple edges is called a *simple graph*.

**Definition 2.1.4.** (1) A *walk* of length  $n$  on a graph is a sequence  $v_0, e_1, v_1, e_2, \dots, v_n$ , where  $v_i$  are vertices while  $e_i$  are edges of the graph such that vertices and edges adjacent in the sequence are incident.

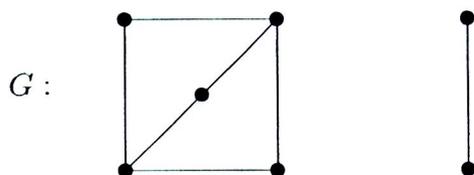
(2) A walk  $v_0, e_1, v_1, e_2, \dots, v_n$ , is said to connect  $v_0$  and  $v_n$ .

(3) A graph is *connected* if every two vertices can be connected by a walk.

**Example 2.1.5.** Nonsimple connected graph

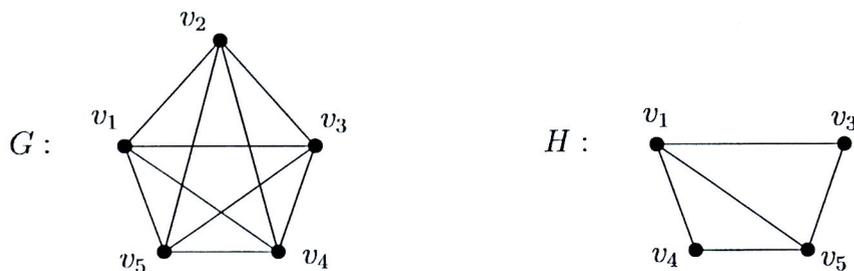


**Example 2.1.6.** Disconnected simple graph



**Definition 2.1.7.** A graph  $H$  is called a *subgraph* of graph  $G$  if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ . If  $u$  and  $v$  are vertices of  $G$ , and  $\{u, v\}$  belong to  $E(G)$ , we say that  $u$  and  $v$  are adjacent.

**Example 2.1.8.** Consider the following graphs  $G$  and  $H$ ,



Then

$$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$$

$$E(G) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \\ \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}\}$$

and

$$V(H) = \{v_1, v_3, v_4, v_5\}$$

$$E(H) = \{\{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_3, v_5\}, \{v_4, v_5\}\}.$$

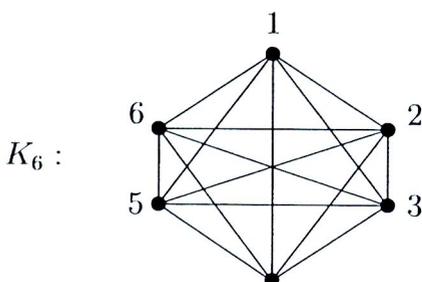
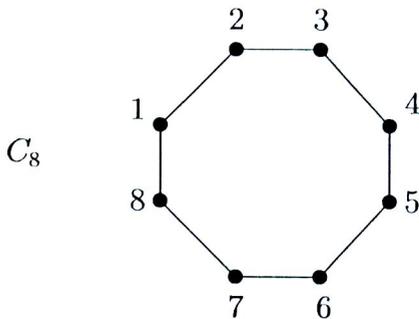
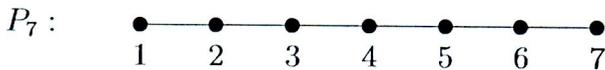
Therefore,  $H$  is a subgraph of the graph  $G$ .

**Definition 2.1.9.** For each positive integer  $n \geq 1$ , a *path* of length  $n - 1$ , denoted by  $P_n$ , is a graph with the vertex set  $V(P_n) = \{v_1, v_2, \dots, v_n\}$  and edge set  $E(P_n) = \{\{v_i, v_{i+1}\} \subseteq V(P_n) \mid i = 1, 2, \dots, n - 1\}$ .

**Definition 2.1.10.** For each positive integer  $n \geq 3$ , a *cycle* of order  $n$ , denoted by  $C_n$ , is a graph with the vertex set  $V(C_n) = \{v_1, v_2, \dots, v_n\}$  and edge set  $E(C_n) = \{\{v_i, v_{i+1}\} \subseteq V(P_n) \mid i = 1, 2, \dots, n - 1\} \cup \{\{v_1, v_n\}\}$

**Definition 2.1.11.** For each positive integer  $n \geq 1$ , a *complete graph* of  $n$  points, denoted by  $K_n$ , is a graph with the vertex set  $V(K_n) = \{v_1, v_2, \dots, v_n\}$  and edge set  $E(K_n) = \{\{v_i, v_j\} \mid i, j = 1, 2, \dots, n \text{ and } i \neq j\}$ .

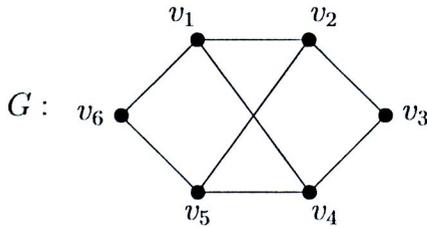
**Example 2.1.12.** Examples of path, cycle and complete graph.



**Definition 2.1.13.** Let  $G$  be graph and  $v \in V(G)$ . The number of all vertices which are adjacent to the vertex  $v$  is called the *degree* of  $v$ , denote by  $deg_G(v)$ .

**Definition 2.1.14.** A graph  $G$  is said to be *regular* of degree  $k$  (or  $k$ -regular) if each of its vertices has degree  $k$ .

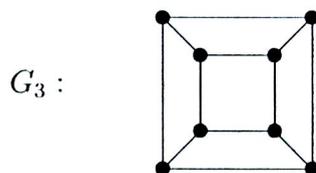
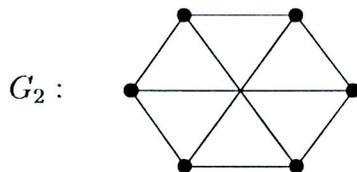
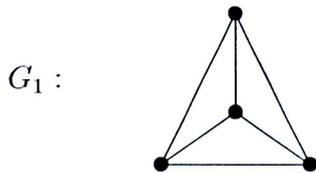
**Example 2.1.15.** Consider the degree of each vertices of the following graph.



Then,

$$deg_G(v_1) = deg_G(v_2) = deg_G(v_4) = deg_G(v_5) = 3, \quad deg_G(v_3) = deg_G(v_6) = 2.$$

**Example 2.1.16.** The followings graphs are 3-regular graphs.

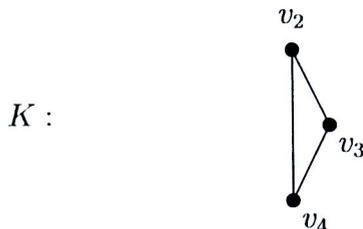
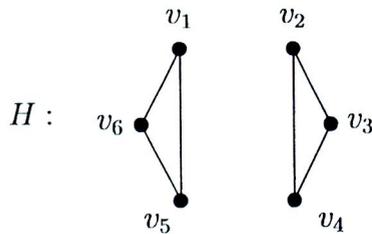
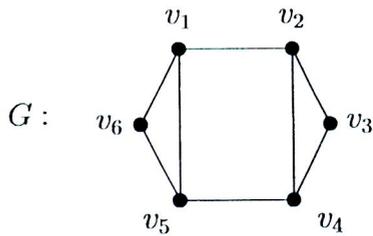


Next definition is more formal way of defining connected graph than the definition 2.1.4.

**Definition 2.1.17.** A graph  $G$  is said to be connected if for each pair  $u, v$  of distinct vertices, there exists a sequence  $a_1, a_2, a_3, \dots, a_n$  in  $V(G)$  such that  $a_1 = u, a_n = v$  and  $\{a_i, a_{i+1}\} \in E(G)$  for all  $i = 1, 2, \dots, n - 1$ .

**Definition 2.1.18.** A subgraph  $H$  of the graph  $G$  is said to be a *component* of the graph  $G$  if  $H$  is a maximal connected subgraph, that is  $H$  is a connected subgraph and for any  $v \in V(G) \setminus V(H), v$  is not adjacent to any vertex in  $V(H)$ .

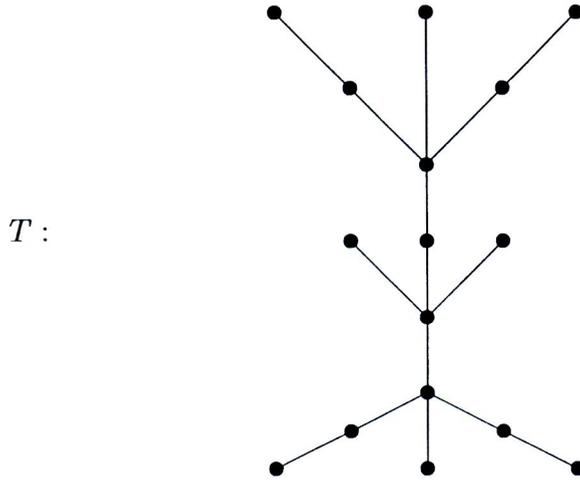
**Example 2.1.19.**



Consider graphs  $G, H, K$  above,  $G$  and  $K$  are connected, but  $H$  is not connected graph, since there is no any sequence  $a_1, a_2, \dots, a_n$  in  $V(H)$  such that  $a_1 = v_1, a_n = v_n$  and  $\{a_i, a_{i+1}\} \in E(H)$  for all  $i = 1, 2, \dots, n - 1$ . Moreover,  $K$  is a component of the graph  $H$  but not a component of the graph  $G$ .

**Definition 2.1.20.** A simple graph  $G$  is a *tree* if  $G$  is connected and contains no cycle as a subgraph.

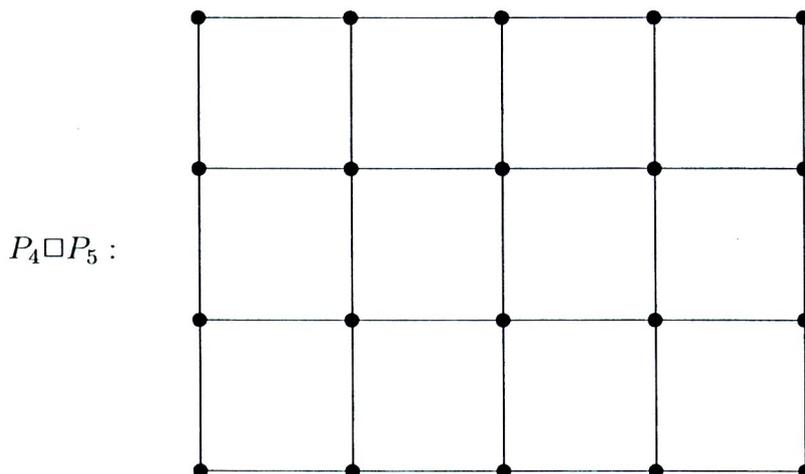
**Example 2.1.21.** Tree



**Definition 2.1.22.** The box product of two graphs  $G_1 = (X_1, E_1)$  and  $G_2 = (X_2, E_2)$  is the graph  $G_1 \square G_2$  where  $V(G_1 \square G_2) = X_1 \times X_2$  and  $\{(x_1, x_2), (y_1, y_2)\} \in E(G_1 \square G_2)$  iff one of the following condition is satisfied:

- (1)  $x_1 = y_1$  and  $\{x_2, y_2\} \in E_2$ ;
- (2)  $x_2 = y_2$  and  $\{x_1, y_1\} \in E_1$ .

**Example 2.1.23.**



## 2.2 Basic Concepts on Linear Algebra

**Definition 2.2.1.** Let  $I_n = \{1, 2, \dots, n\}$ . A permutation of  $I_n$  is any one to one mapping  $\beta$  of  $I_n$  onto itself.

**Definition 2.2.2.** The set of all permutation on  $I_n = \{1, 2, \dots, n\}$  denoted by  $S_n$ , a permutation  $\beta$  in  $S_n$  is said to have an *inversion* whenever  $\beta(i) > \beta(j)$  for  $i < j$ .

**Definition 2.2.3.** A permutation  $\beta$  is called *even* or *odd* according to whether the number of inversions in its result  $\beta = (k_1 k_2 \dots k_n)$  is an even or an odd integer, respectively.

**Example 2.2.4.** The permutation  $\beta = (2143)$  is even because there are two inversions  $\beta(1) = 2 > \beta(2) = 1$  and  $\beta(3) = 4 > \beta(4) = 3$ .

**Definition 2.2.5.** For any permutation  $\beta$  in  $S_n$ , let

$$\text{sgn}(\beta) = \begin{cases} 1 & \text{if } \beta \text{ is even;} \\ -1 & \text{if } \beta \text{ is odd.} \end{cases}$$

**Definition 2.2.6.** Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

the determinant of the  $n \times n$  matrix  $A$  is the number

$$\det(A) = \sum_{\beta \in S_n} \text{sgn}(\beta) a_{1\beta(1)} a_{2\beta(2)} \cdots a_{n\beta(n)}.$$

**Example 2.2.7.** Let  $A = \begin{pmatrix} 5 & 4 \\ 2 & 8 \end{pmatrix}$ .

Then  $S_2 = \{(1), (12)\}$ . By Definition 2.2.6,

$$\begin{aligned} \det(A) &= \text{sgn}(1) a_{11} a_{22} + \text{sgn}(12) a_{12} a_{21} \\ &= (1)(5)(8) + (-1)(4)(2) \\ &= 48 - 8 \\ &= 32 \end{aligned}$$

**Definition 2.2.8.** An *eigenvector* of an  $n \times n$  matrix  $A$  is a nonzero vector  $\mathbf{x}$  such that  $A\mathbf{x} = \lambda\mathbf{x}$  for some scalar  $\lambda$ . A scalar  $\lambda$  is called an *eigenvalue* of  $A$  if there is a nontrivial solution  $\mathbf{x}$  of  $A\mathbf{x} = \lambda\mathbf{x}$ .

**Example 2.2.9.** Let  $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$ ,  $\mathbf{u} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$  and  $\mathbf{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ . Are  $\mathbf{u}$  and  $\mathbf{v}$  eigenvector of  $A$ ?

$$A\mathbf{u} = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -24 \\ 20 \end{pmatrix} = -4 \begin{pmatrix} 6 \\ -5 \end{pmatrix} = -4\mathbf{u}$$

$$A\mathbf{v} = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -9 \\ 11 \end{pmatrix} \neq \lambda \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

Thus  $\mathbf{u}$  is an eigenvector corresponding to an eigenvalue  $-4$ , but  $\mathbf{v}$  is not an eigenvector of  $A$ , because  $A\mathbf{v}$  is not a multiple of  $\mathbf{v}$ .

**Definition 2.2.10.** The scalar equation  $\det(A - \lambda I) = 0$  is called the *characteristic equation* of  $A$  and scalar  $\lambda$  is an eigenvalue of an  $n \times n$  matrix  $A$  if and only if  $\lambda$  is a root of the characteristic equation.

**Theorem 2.2.11.** Let  $L = \{\lambda \mid \lambda \text{ is an eigenvalue of } n \times n \text{ matrix } A\}$ . Then

$$\det(A) = \prod_{\lambda \in L} \lambda.$$

**Example 2.2.12.** Find the characteristic equation of  $A = \begin{pmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ .

Form  $A - \lambda I$  we have:

$$\det(A - \lambda I) = \det \begin{pmatrix} 5 - \lambda & -2 & 6 & -1 \\ 0 & 3 - \lambda & -8 & 0 \\ 0 & 0 & 5 - \lambda & 4 \\ 0 & 0 & 0 & 1 - \lambda \end{pmatrix}$$

$$= (5 - \lambda)(3 - \lambda)(5 - \lambda)(1 - \lambda)$$

$$= \lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0.$$

Thus  $\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$  is characteristic equation of  $A$  and by Theorem 2.2.6, we have  $\det(A) = (5)(3)(5)(1) = 75$ .

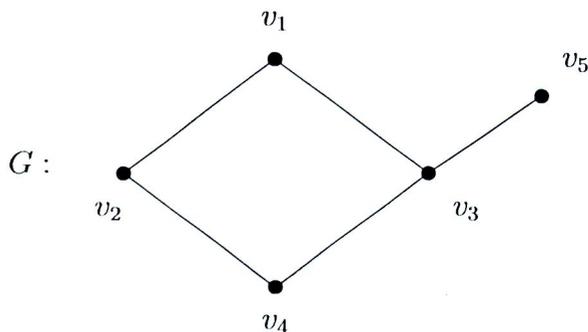
## 2.3 The Adjacency Matrices

**Definition 2.3.1.** The adjacency matrix of a simple graph  $G$  is the  $n \times n$  matrix  $A(G) = (a_{ij})_{n \times n}$  where the entries  $a_{ij}$  are given by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0 & \text{otherwise.} \end{cases}$$

We denote  $\det(A(G))$ , the determinant of the adjacency matrix  $A(G)$ .

**Example 2.3.2.** Consider the following graph,



$$\text{Then } A(G) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Note that, the determinant of this matrix is 0 because the first row and the third row are the same.

**Definition 2.3.3.** (cf. [ 5 ]) An  $n \times n$  matrix  $A$  is said to be a *circulant* matrix if its entries satisfy  $a_{1(j-i+1)}$ , where the subscripts are reduced modulo  $n$  and lie in the set  $\{1, 2, \dots, n\}$ .

**Definition 2.3.4.** A circulant graph is a graph  $G$  whose adjacency matrix  $A(G)$  is a circulant matrix.

In this thesis, the eigenvalues of a graph  $G$  means the eigenvalues of adjacency matrix  $A(G)$ .

**Lemma 2.3.5.** (cf.[5]) Suppose  $[0, a_2, \dots, a_n]$  is the first row of the adjacency matrix of a circulant graph  $G$ . Then the eigenvalues of  $G$  are

$$\lambda_r = \sum_{j=2}^n a_{1j} \omega^{(j-1)r}, r = 0, 1, \dots, n-1$$

where  $\omega = e^{2\pi i/n}$ .

**Definition 2.3.6.** An elementary graph is a graph where each component is 1-regular or 2-regular. In other words, each component is a single edge ( $K_2$ ) or a cycle ( $C_n$ ). A *spanning elementary subgraph* of  $\Gamma$  is an elementary subgraph which contains all vertices of  $\Gamma$ . In some cases we will call a spanning elementary subgraph of  $\Gamma$  a *sesquivalent spanning subgraph* of  $\Gamma$ .

**Definition 2.3.7.** For a general graph  $\Gamma$  with  $n$  vertices,  $m$  edges and  $c$  components, the rank of  $\Gamma$  and the co-rank of  $\Gamma$  are defined respectively,

$$r(\Gamma) = n - c, s(\Gamma) = m - n + c.$$

**Proposition 2.3.8.** (cf.[5]) Let  $A(G)$  be the adjacency matrix of a graph  $G$ . Then

$$\det(A(G)) = \sum (-1)^{r(\Gamma)} 2^{s(\Gamma)}$$

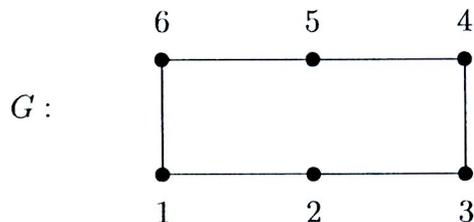
where the summation is over all sesquivalent spanning subgraph of  $\Gamma$ .

**Lemma 2.3.9.** For a graph  $G$  if no sesquivalent spanning subgraph exists, then the  $\det A(G) = 0$ .

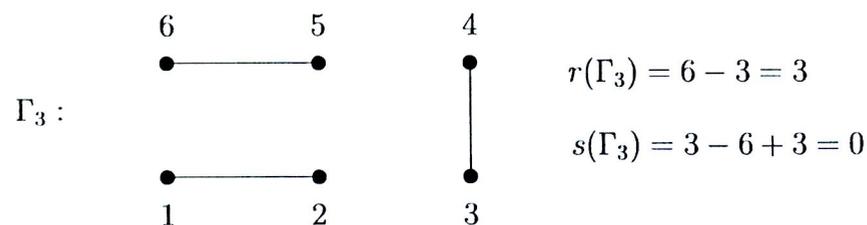
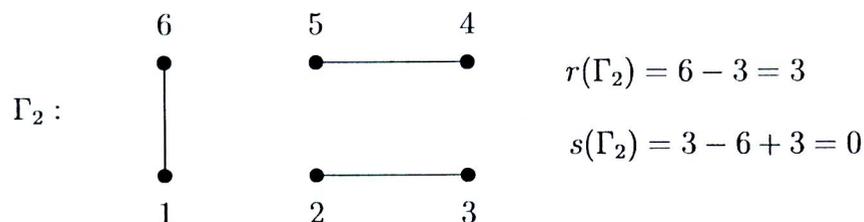
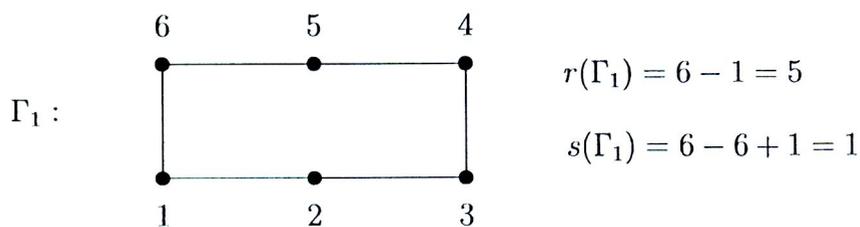
**Proof.** Consider a term  $sgn(\beta) a_{1\beta(1)} a_{2\beta(2)} \cdots a_{n\beta(n)}$  in the expansion of  $\det(A)$ . This term vanishes if, for some  $i \in \{1, 2, \dots, n\}$ ,  $a_{i\beta(i)} = 0$ ; that is, if  $\{v_i, v_{\beta(i)}\}$  is not edge of  $G$ . Thus, if the term corresponding to a permutation  $\beta$  is non-zero, then  $\beta$  can be expressed uniquely as the composition of disjoint circuit of length at least two. Each cycle  $(ij)$  of length two corresponds to the factors  $a_{ij} a_{ji}$ , and signifies a single edge  $v_i, v_j$  in  $G$ . Each cycles  $(pqr \dots t)$  of length greater than two corresponds to the factors  $a_{pq} a_{qr} \dots a_{tp}$ , and signifies a circuit  $\{v_p, v_q, \dots, v_t\}$  in  $G$ . Since a graph  $G$  has no spanning elementary subgraph, there is  $a_{i\beta(i)} = 0$ . Thus  $\det A(G) = 0$ . ■



**Example 2.3.10.** Consider the following graph,



There are altogether three spanning elementary subgraphs of  $G$ .

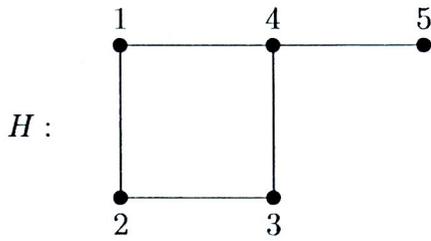


Then

$$\det(A(G)) = (-1)^5(2)^1 + (-1)^3(2)^0 + (-1)^3(2)^0 = -2 - 1 - 1 = -4.$$



**Example 2.3.11.** Consider the following graph,



There is no spanning elementary subgraph of  $H$ . Then  $\det(A(H)) = 0$ .

**Definition 2.3.12.** A graph  $G$  is said to be *singular* if its adjacency matrix is singular, otherwise it is said to be *nonsingular*.

**Theorem 2.3.13.** (cf.[22])  $A(G)$  is singular if and only if at least one of its components is singular.