Chapter 1

Introduction

Graph theory is a field in mathematics with many applications in real life. Graphs are used in describing social networks, in statistical and business modeling, astronomy and many more. There are many directions in which one can study graphs. For us, we are interested in computing general formulas for the determinant of the adjacency matrix of graphs. We study several different families of graphs in the work, in particular, cycles, tree, book-graphs, and some special cases of graphs of graphs.

Let G be a finite simple graph with the vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge list set $E(G)$. The adjacency matrix of the graph G is the $n \times n$ matrix $A(G) = (a_{ij})_{n \times n}$ where

$$a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in E(G); \\ 0, & \text{otherwise.} \end{cases}$$

A graph is said to be singular if its adjacency matrices is singular and a graph is nonsingular or non-singular, if its adjacency matrices is nonsingular. For short, we say the determinant of graph instead of the determinant of the adjacency matrix of that graph.

Further we provide some brief overview of the results obtained in the direction of computing determinants of graphs.

In 1962, Harary (cf.[5]) found a formula to calculate the determinant of graphs by using their sesquivalent spanning subgraphs. Let $A(G)$ be the adjacency matrix of a graph G. Then

$$\det(A(G)) = \sum (-1)^{r(\Gamma)} 2^{s(\Gamma)}$$

where the summation is over all spanning elementary subgraphs of Γ.

In 1996, S.V. Gervacio [14] showed that complement of tree with diameter less than 5 has nonsingular adjacency matrix.

Let T be a tree with diameter 4 and central vertex x_0. Then $\overline{T}$ is nonsingular if and only if $T\backslash x_0$ has at most one component P_2.

In 1996, H.M. Rara [22] gave a reduction procedure for calculating the determinant of the adjacency matrix of some types of graphs and described the singularity of square planar grids. Let G be the graph obtained by joining the vertex x of the graph G_1 to the vertex y of the graph G_2 by and edge. Then $\det A(G) = \det A(G_1)\det A(G_2) - \det A(G_1\backslash x)\det A(G_2\backslash y)$.

In 2002, M. Doob [10] found circulant graphs with $\det(-A(G)) = -\det(G)$: codeterminants with the complete graph K_n.

As mentioned above, the main purpose of this research is to find formulas for the determinant of the adjacency matrices of some special types of graphs and to give characterization for non-singularity of those graphs.

The thesis is organized as follows : Chapter 2 deals with some preliminaries, basic definitions and some useful results that are used throughout this thesis. Chapter 3 gives characterizations for non-singularity of cycles and trees. In Chapter 4, we study two kinds of binary operations of graphs: in particular, a point union of two disjoint graphs, i.e. the resulting graph contains the two graphs joined at a single vertex. We call the second operation covered in the chapter - graph of graphs operation (see section 4.1 for a formal definition and more detailed explanation). With the help of the operation we produce several different types of graphs, namely path of paths, path of cycles, cycle of paths and cycle of cycles and we compute the determinants for each type.

In Chapter 5 a double edge union of two disjoint graphs is considered, i.e. an operation between two disjoint graphs were the resulting graph consists of the two disjoint graphs connected with a pair of edges. We compute the determinant of double edge union of cycles using a general recursive formula for the double edge union of arbitrary graphs. The latter is developed in section 5.1 with the help of special application of the Laplace expansion theorem (see section 5.1 for more details). In Chapter 6, we study yet another kind of binary operation of graphs, so called - the box product of graphs and we find a formula for the determinant of $P_m \square G$, where P_m is a path and G an arbitrary graph.

The conclusion of this research and open problems listed is in Chapter 7.