CHAPTER 4

THEORETICAL FRAMEWORK

This chapter is divided into two parts. The first part presents the model introduced by Silos (2007) that is key to this study. It is a portfolio choice model in which agents differ in age, income and wealth, with housing being modeled explicitly. This model can be used to explain the properties of wealth distribution and portfolio composition, and their relationship to macroeconomic shocks.

The second part is aimed to discuss briefly the framework for growth accounting, which is important to the estimation of aggregate productivity shock in the model presented in the first part.

4.1 The model of portfolio choice.

We employ Silos (2007) business cycle model in which agents differ in age, income and wealth. Housing is added to the model with its role of providing shelter services and serving as collateral for loans.

The model economy at each point in time consists of individuals with unit mass, belonging to one of I generations. Death is certain at age I. Therefore, the fraction of individuals of age $i \in I$ is $1/I$. Individuals have zero wealth in the beginning of the first period, work for T years. In each year, they have one unit of time endowment that can be used as working time or leisure.

Individuals' utility in each period depends on non-housing consumption (c), housing service (s), and leisure (l). They maximize their expected lifetime utility: -

$$U(c,s,l) = E\sum_{i=1}^{I} \beta^{i-1} u(c_i, s_i, l_i) \tag{4.1}$$

The function u is increasing, continuous, and strictly concave and β denotes the discount factor. Silos (2007) chooses the utility function of the logarithmic class:

$$u(c_i, s_i, l_i) = \theta \ln(c_i) + (1-\theta)\ln(s_i) + \omega \ln(l_i) \tag{4.2}$$

There assumes to be uncertainty in productivity in both the aggregate and individual level. Households have different realization of the productivity shocks (ξ) to their own labor supply. There is also an aggregate shock (z) that affects the entire economy, implying different wage and rental rates in each period. The aggregate technology used to produce output Y combines capital K and labor N through the function $z \cdot F(K, N)$, which satisfies the standard properties: strictly increasing, strictly concave and homogeneous of degree 1. Let $\pi(z', \xi' \mid z, \xi)$ denote the probability of transiting from state (z, ξ) to state (z', ξ'). For simplicity, the number of states for the aggregate productivity shock is set at two, a recession value ($z = z_b$) and an expansion value ($z = z_g$).

The output can be consumed, invested in business capital (k) or invested in housing capital (h). The technology for producing housing service from housing capital is simply that one unit of housing capital h gives one unit of housing service s. Adjustments to housing capital imply transaction costs, denoted by TC. The business capital and housing capital depreciate at the rate δ_k and δ_h, respectively. The aggregate resource constraint in period t is:

$$Y_t = z_t F(K_t, N_t) = C_t + K_{t+1} - (1-\delta_k)K_t + H_{t+1} - (1-\delta_h)H_t + TC \qquad (4.3)$$

Households observe the aggregate shock z and the idiosyncratic shock ξ before making a decision on consumption, leisure, investment in business capital (for the next period), and investment in housing capital (for the next period). Transfers of housing capital between agents are carried out at the end of the period.

Households are allowed to borrow up to a certain fraction of the holding of housing capital in the current period before depreciation. There assumes to be no contingent claims markets for hedging idiosyncratic productivity shocks across individuals.

Let Φ denote the joint distribution over assets, ages, and productivity shock. Interest rates and wages are denoted by $r(z, \Phi)$ and $w(z, \Phi)$, which clearly show their dependence on the entire distribution of agents. This distribution determines the aggregate capital-labor ratio the following period, which, in turn, determines factor prices.

With the holding of business capital *k*, the holding of housing capital *h*, the idiosyncratic productivity shock *ξ*, the aggregate shock *z*, and the joint distribution over assets, ages, and productivity shocks *Φ* as state variables, the household's problem can be written in recursive form as follows:

$$V_i(k,h,\xi,z,\Phi) = \max_{\{k',h',l,c\}} \quad u(c,s,l) + \beta \sum_{\xi',z'} \pi(z',\xi' | z,\xi) V_{i+1}(k',h',\xi',z',\Phi') \tag{4.4}$$

$$\text{s.t.} \qquad c + k' + h' + \Omega(h',h) \le (1-l)\, w(z,\Phi)\xi\eta_i + (1 + r(z,\Phi) - \delta_k)k + (1-\delta_h)h \tag{4.5}$$

$$k' \ge -(1-\gamma)h, \;\; s = h, \;\; c > 0, \;\; h' > 0, \;\; 0 < l \le 1 \tag{4.6}$$

$$\Phi' = G(z,\Phi) \tag{4.7}$$

Households maximize expected lifetime utility by choosing the level of housing capital and business capital to hold in the next period, leisure, and non-housing consumption. Equation (4.5) is the budget constraint in which the sources of income are capital interest and compensation for labor. In addition to consumption and investment, part of the income is devoted to pay the costs of adjusting the housing capital, represented by the function *Ω*(*h'*, *h*). The first part of equation (4.6) is the borrowing constraint which specifies that households cannot borrow more than a fraction 1-*γ* of the house already owned. Equation (4.7) is the law of motion for the aggregate wealth distribution which households take as given.

A recursive competitive equilibrium for this economy is a sequence of age-dependent value functions $\{V_i\}_{i=1}^{I}$, policy functions $\{k_i', c_i, h_i', l_i\}_{i=1}^{I}$, factor prices *w* and *r* and an aggregate law of motion *G* such that:

1. The policy functions are the solution to the recursive problem defined by equation (4.4) to (4.7) and the value functions satisfy equation (4.4) for an agent of generation *i*.

2. Factor prices satisfy

$$r(z,\Phi) = z \cdot F_K(K(z,\Phi), N(z,\Phi)) \tag{4.8}$$

$$w(z,\Phi) = z \cdot F_N(K(z,\Phi), N(z,\Phi)) \tag{4.9}$$

3. Markets clear

$$K(G(z,\Phi)) = \sum_{i=1}^{I} \frac{1}{I} \int k_i^{'}(k,h,\xi,z,\Phi)\, d\Phi \quad (capital\ market) \tag{4.10}$$

$$N(z,\Phi) = \sum_{i=1}^{I} \frac{\eta_i}{I} \int (1 - l_i(k,h,\xi,z,\Phi))\xi d\Phi \quad (labor\ market) \tag{4.11}$$

$$\begin{aligned}
&\sum_{i=1}^{I} \frac{1}{I} \int c_i(k,h,\xi,z,\Phi) + k_i^{'}(k,h,\xi,z,\Phi) + h_i^{'}(k,h,\xi,z,\Phi) \\
&\quad + \Omega(h_i^{'}(k,h,\xi,z,\Phi),h))d\Phi \\
&\qquad = F(K(z,\Phi), N(z,\Phi)) + (1-\delta_k)K(z,\Phi) \\
&\qquad\quad + (1-\delta_h)H(z,\Phi) \quad (goods\ market)
\end{aligned} \tag{4.12}$$

4. The aggregate law of motion G is generated by the aggregate shock z and the decision rules k' and h'.

Some modification is needed to compute the equilibrium defined above. In this model, households solve a slightly different problem than the one postulated in equation (4.4)-(4.7). In particular, since the income-wealth distribution is an infinite-dimensional object, a ‘partial information’ approach similar to Krusell and Smith (1997 and 1998) where the entire distribution is replaced by a finite number of moments is adopted. This is consistent with the reasoning that households may not be able to keep track of all the state variables specified. In other words, households keep track of some moments of the distribution, instead of the entire distribution itself, and use them to forecast future prices. The final goal of households is to forecast capital and labor, as they suffice to forecast interest rates and wages. One hopes that these forecasts are accurate enough so as to have negligible quantitative implications.

Silos (2007) follows Krusell and Smith's work by letting the household use first-order autoregressive processes (conditional on the current aggregate state) in the logarithm of K as their forecasting instrument. So, the new 'approximate' household's problem becomes:

$$V_i(k,h,\xi,z,K) = \max_{\{k',h',l,c\}} \quad u(c,s,l) + \beta \sum_{\xi' z'} \pi(z',\xi' | z,\xi) V_{i+1}(k',h',\xi',z',K') \quad (4.13)$$

$$\text{s.t.} \qquad c + k' + h' + \Omega(h',h) \le (1-l)\, w(z,K)\xi\eta_i$$

$$+ (1 + r(z,K) - \delta_k)k + (1-\delta_h)h \qquad (4.14)$$

$$k' \ge -(1-\gamma)h, \;\; s = h, \;\; c > 0, \;\; h' > 0, \;\; 0 < l \le 1 \qquad (4.15)$$

$$\ln K' = a_k(z) + b_k(z)\ln K, \quad z = z_b, z_g \qquad (4.16)$$

$$\ln N = a_n(z) + b_n(z)\ln K, \quad z = z_b, z_g \qquad (4.17)$$

The process of solving this problem can be briefly described as follows. Given values for $a(z) = \{a_k(z), a_n(z)\}$ and $b(z) = \{b_k(z), b_n(z)\}$, agents compute optimal policies. Simulating the economy for a large number of time periods (and for a large number of households) provides a time series for aggregate business capital and aggregate labor. New values for parameters $a(z)$ and $b(z)$ are estimated by least squares. This procedure is repeated as many times as necessary until the values for $a(z)$ and $b(z)$ used by households roughly coincide with the ones obtained from the aggregate time series. If the forecasting model is adequate (as measured by some metric such as the root mean square error (RMSE), correlation between predicted and actual values, etc.), an equilibrium has been found. If not, it is necessary to either increase the number of moments used or change the functional form in (4.16)–(4.17).

4.2 The growth accounting framework

In the model described in Section 4.1, the stochastic process for the aggregate productivity shock needs to be estimated. The deviation around trend of the logarithm of the Solow residual is defined as the aggregate productivity shock. This section will explain briefly the growth accounting framework which will be used in estimating the Solow residual.

As in Tinakorn and Sussangkarn (1998), assume that two inputs – capital *K* and labot *N* – are used in the production of aggregate output *Y*, under the following production function:

$$Y(t) = f(K(t), N(t), t) \tag{4.18}$$

The increase in output *Y* must be attributable to the increase in inputs and/or the change in the production technique. Take the total differentiation of *Y* with respect to time, we get:

$$\frac{dY}{dt} = \frac{\partial f(.)}{\partial K}\frac{dK}{dt} + \frac{\partial f(.)}{\partial N}\frac{dN}{dt} + \frac{\partial f(.)}{\partial t} \tag{4.19}$$

Dividing through with Y and denoting the dot over a variable as differentiation with respect to time, equation (4.19) can be rewritten as:

$$\frac{\dot{Y}}{Y} = \frac{\partial f(.)}{\partial K}\frac{\dot{K}}{Y} + \frac{\partial f(.)}{\partial N}\frac{\dot{N}}{Y} + \frac{\dot{f}(.)}{Y} \tag{4.20}$$

Denoting β_K and β_N the output elasticity with respect to capital and to labor, respectively, we have:

$$\frac{\dot{Y}}{Y} = \beta_K\frac{\dot{K}}{K} + \beta_N\frac{\dot{N}}{N} + \frac{\dot{f}(.)}{Y} \tag{4.21}$$

where $\beta_K = \frac{\partial f(.)}{\partial K}\frac{\dot{K}}{f(.)}$ and $\beta_N = \frac{\partial f(.)}{\partial N}\frac{\dot{N}}{f(.)}$

Rewriting equation (4.21) as

$$\frac{\dot{f}(.)}{f(.)} = \frac{\dot{Y}}{Y} - \beta_K \frac{\dot{K}}{K} - \beta_N \frac{\dot{N}}{N} \tag{4.22}$$

The term $\frac{\dot{f}(.)}{f(.)}$ represents the part of the increase in output that cannot be accounted for by the increase in inputs. In order to calculate this residual, the estimates of the output elasticity β_K and β_N are needed. With the assumption of profit maximization by the producer and the competitive factor market, the marginal value product of each factor is equal to the factor cost. This can be written as:

$$P \cdot MP_K = P \cdot \frac{\partial f(.)}{\partial K} = W_K \tag{4.23}$$

$$P \cdot MP_N = P \cdot \frac{\partial f(.)}{\partial N} = W_N \tag{4.24}$$

This implies,

$$\beta_K = \frac{\partial f(.)}{\partial K}\frac{\dot{K}}{f(.)} = \frac{W_K}{P}\frac{K}{f(.)} = s_K \tag{4.25}$$

$$\beta_N = \frac{\partial f(.)}{\partial N}\frac{\dot{N}}{f(.)} = \frac{W_N}{P}\frac{N}{f(.)} = s_N \tag{4.26}$$

where s_K and s_N is the share of income going to capital and labor, respectively.

That is, we can estimate the elasticity of output with respect to capital and labor by the share of income going to capital and labor, respectively.