CHAPTER V

CONCLUSIONS

The following results are all main theorems of this thesis:

1. Let a and b be elements of a regular Γ-semigroup S. Then the following statements are equivalent.

   (1) $a \leqslant b$.

   (2) $a \in b\Gamma S$ and there exist $\alpha, \beta \in \Gamma, a' \in V_\alpha^\beta(a)$ such that $a = a\alpha a'\beta b$.

   (3) There exist $\beta, \gamma \in \Gamma, f \in E_\beta(S), g \in E_\gamma(S)$ such that $a = f\beta b = b\gamma g$.

   (4) $H_a \leqslant H_b$ and for all $\alpha, \delta \in \Gamma, b' \in V_\alpha^\delta(b), a = a\alpha b'\delta a$.

   (5) $H_a \leqslant H_b$ and there exist $\alpha, \delta \in \Gamma, b' \in V_\alpha^\delta(b), a = a\alpha b'\delta a$.

2. The partial order on $E(S)$ of a regular semigroup S is the restriction of the natural partial order on S to $E(S)$.

3. Let S be a regular Γ-semigroup. Then the following statements are equivalent.

   (1) $\leqslant$ is right compatible.

   (2) S satisfies $\mathcal{L}$-majorization for idempotents.

   (3) S satisfies $\mathcal{L}$-majorization.

4. Let S be a regular Γ-semigroup. Then the following statements are equivalent.

   (1) $\leqslant$ is compatible.

   (2) S satisfies $\mathcal{L}$- and $\mathcal{R}$-majorization for idempotents.

   (3) S satisfies $\mathcal{L}$- and $\mathcal{R}$-majorization.

5. Let S be a regular Γ-semigroup and the natural partial order on S be compatible with multiplication. Then

$$\omega := \{(a,b) \in S \times S \mid c \leqslant a \text{ and } c \leqslant b \text{ for some } c \in S\}$$

is a congruence on S.

6. Let S be a regular Γ-semigroup such that ω is a congruence and the natural homomorphism for ω is reflecting the natural partial order. Then ω is the least primitive congruence on S.

7. Let a and b be elements in a regular Γ-semigroup S. Let $\alpha, \beta, \gamma, \delta \in \Gamma, a' \in V_{\alpha}^{\beta}(a), b' \in V_{\gamma}^{\delta}(b)$ and $g \in S_{\theta}^{(\alpha,\delta)}(a'\beta a, b\gamma b')$. Then

(1) $b'\delta g\alpha a' \in V_{\gamma}^{\beta}(a\theta b)$.

(2) $b'\delta g \in V_{\gamma}^{\theta}(g\theta b)$.

(3) $g\alpha a' \in V_{\theta}^{\beta}(a\theta g)$.

(4) $a\theta g\theta b = a\theta b$.

8. Suppose that S is a regular Γ-semigroup and $\alpha, \beta, \theta \in \Gamma$. Let $e \in E_{\alpha}(S), f \in E_{\beta}(S)$. Then the mappings

$$\varphi : x \mapsto (x\theta f, e\theta x), \qquad \text{and } \psi : (y, z) \mapsto y\alpha w\beta z$$

(where $w \in V_{\alpha}^{\beta}(e\theta f)$) are mutually inverse θ-isomorphisms between sub Γ-semigroup $S_{\theta}^{(\alpha,\beta)}(e, f)$ and $S_{\theta}^{(\alpha,\beta)}(e, f)\theta f \times e\theta S_{\theta}^{(\alpha,\beta)}(e, f)$.

9.Let S be a Γ-semigroup and $E(g\Gamma S\Gamma g)$ is a commutative sub Γ-semigroup of S for all $g \in E(S)$. Then the following statements hold.

(1) $\mid S_{\theta}^{(\alpha,\beta)}(e, f) \mid \leqslant 1$ for all $\alpha, \beta, \theta \in \Gamma, e \in E_{\alpha}(S), f \in E_{\beta}(S)$.

(2) If $a, b, x, y \in Reg(S)$ with $a \leqslant x$, $b \leqslant y$ where $Reg(S)$ is a sub Γ-semigroup then $a\theta b \leqslant x\theta y$ for some $\theta \in \Gamma$.

10. Let S be a regular Γ-semigroup and $\alpha, \beta, \theta \in \Gamma$. Then the following conditions are equivalent.

(1) For any $a \in S$, $S_{\theta}^{(\alpha,\beta)}(a)$ is a right θ-zero semigroup.

(2) If $e \in E_{\alpha}(S), f \in E_{\beta}(S)$ such that $e\mathcal{D}f$ then $S_{\theta}^{(\alpha,\beta)}(e, f)$ is a right θ-zero semigroup.

(3) If $a \in S$ and $x, y \in V_{\alpha}^{\beta}(a\theta a)$ then $(a\alpha x\beta a)\theta(a\alpha y\beta a) = a\alpha y\beta a$.

(4) If $a, x, y \in S$ with $a\theta a = a\theta a\alpha x\beta a\theta a = a\theta a\alpha y\beta a\theta a$ then

$$(a\alpha x\beta a)\theta(a\alpha y\beta a) = (a\alpha y\beta a)\theta(a\alpha y\beta a).$$

(5) If $x, y \in S$, $e \in E_\alpha(S), f \in E_\beta(S)$ such that $e\mathcal{D}f$

$$e\theta x = e\theta y = e\theta f\mathcal{L}x\mathcal{L}y, \text{ and } x, y \leqslant f \text{ then } x = y.$$

11. Let S be a regular Γ-semigroup and $\alpha, \beta, \theta \in \Gamma$. Then the following conditions are equivalent.

(1) For any $a \in S$, $\mid S_\theta^{(\alpha,\beta)}(a) \mid = 1$.

(2) If $a \in S$ and $x, y \in V_\alpha^\beta(a\theta a)$ then $a\alpha x\beta a = a\alpha y\beta a$.

(3) If $a, x, y \in S$ with $a\theta a = a\theta a\alpha x\beta a\theta a = a\theta a\alpha y\beta a\theta a$ then

$$(a\alpha x\beta a)\theta(a\alpha x\beta a) = (a\alpha y\beta a)\theta(a\alpha y\beta a).$$

(4) If $a, x, y \in S$ with $a\theta a = a\theta a\alpha x\beta a\theta a = a\theta a\alpha y\beta a\theta a$ then

$$(a\alpha x\beta a)\theta(a\alpha y\beta a) = (a\alpha y\beta a)\theta(a\alpha x\beta a).$$

(5) If $e \in E_\alpha(S), f \in E_\beta(S)$ such that $e\mathcal{D}f$ then $\mid S_\theta^{(\alpha,\beta)}(e, f) \mid = 1$.

12. Let S be a regular Γ-semigroup. Then the following statements are equivalent.

(1) S is a locally inverse Γ-semigroup.

(2) $\leqslant$ is compatible.

(3) $\mid S_\theta^{(\alpha,\beta)}(e, f) \mid = 1$ for all $\alpha, \beta, \theta \in \Gamma, e \in E_\alpha(S), f \in E_\beta(S)$.

13. Let $\phi : S \to S'$ be a homomorphism of regular Γ-semigroups. Then ϕ reflects natural partial orders of S and S'.

14. Let S be a Γ-semigroup without zero. Then the following conditions are equivalent.

(1) S is completely simple.

(2) S is regular and every idempotent is primitive.

15. A regular Γ-semigroup S without zero is completely simple if and only if the natural partial order on S is the identity relation.

16. Let ρ be a congruence on a regular Γ-semigroup S. Then ρ is strictly compatible if and only if $e\rho$ is a completely simple sub Γ-semigroup of S for all $e \in E(S)$.

17. Let S be a regular Γ-semigroup. Then the following statements are equivalent.

(1) For all $e \in E_\alpha(S), \alpha \in \Gamma$, $(e]$ is directed.

(2) ρ is an equivalence relation.

(3) ρ is congruence.

18. Let S be a regular Γ-semigroup and

$$\rho = \{(x, y) \in S \times S \mid z \leqslant x \text{ and } z \leqslant y \text{ for } \text{ some } z \in S\}.$$

Then the congruence ρ is the finest primitive congruence on S.