CHAPTER IV

(α, β, θ)-SANDWICH SETS

4.1 (α, β, θ)-Sandwich Sets of Idempotents on Regular Γ-semigroups

In this chapter, we construct a new set on a regular Γ-semigroup S, say (α, β, θ)-sandwich set in a regular Γ-semigroup, denote by $S_\theta^{(\alpha,\beta)}(e, f)$ where $\theta \in \Gamma$, e is an α-idempotent and f is a β-idempotent.

For a Γ-semigroup $S, \alpha, \beta, \theta \in \Gamma$ and $e \in E_\alpha(S), f \in E_\beta(S)$, we define a set $S_\theta^{(\alpha,\beta)}(e, f)$ by

$$S_\theta^{(\alpha,\beta)}(e, f) := \{g \in V_\beta^\alpha(e\theta f) \cap E_\theta(S) \mid g\alpha e = f\beta g = g\}.$$

Then $S_\theta^{(\alpha,\beta)}(e, f)$ may be an empty set. If $S_\theta^{(\alpha,\beta)}(e, f) \neq \emptyset$ then $S_\theta^{(\alpha,\beta)}(e, f)$ is called an (α, β, θ)**-sandwich set**.

The next proposition insure that $S_\theta^{(\alpha,\beta)}(e, f) \neq \emptyset$ if S is a regular Γ-semigroup.

Proposition 4.1.1. *Let S be a regular Γ-semigroup, $\alpha, \beta, \theta \in \Gamma$ and $e \in E_\alpha(S), f \in E_\beta(S)$. Then the set $S_\theta^{(\alpha,\beta)}(e, f)$ defined by*

$$S_\theta^{(\alpha,\beta)}(e, f) := \{g \in V_\beta^\alpha(e\theta f) \cap E_\theta(S) \mid g\alpha e = f\beta g = g\}$$

is non-empty.

Proof. Let $x \in V_\beta^\alpha(e\theta f)$ and $g = f\beta x\alpha e$. Then

$$g\alpha e = (f\beta x\alpha e)\alpha e = f\beta x\alpha e = g$$

and

$$f\beta g = f\beta(f\beta x\alpha e) = f\beta x\alpha e = g.$$

Since $x \in V_\beta^\alpha(e\theta f)$, we have $x = x\alpha e\theta f\beta x$ and $e\theta f = e\theta f\beta x\alpha e\theta f$. Now,

$$g\theta g = (f\beta x\alpha e)\theta(f\beta x\alpha e) = f\beta x\alpha e = g,$$

$$g\alpha(e\theta f)\beta g = g\theta g = g,$$

and

$$(e\theta f)\beta g\alpha(e\theta f) = (e\theta f)\beta g\theta f = e\theta g\theta f = e\theta f\beta x\alpha e\theta f = e\theta f.$$

Then $g \in S_\theta^{(\alpha,\beta)}(e,f)$. This is complete the proof. □

Proposition 4.1.2. *Let S be a regular Γ-semigroup, $\alpha, \beta, \theta \in \Gamma$ and $e \in E_\alpha(S)$, $f \in E_\beta(S)$. Then*

$$S_\theta^{(\alpha,\beta)}(e,f) = \{g \in E_\theta(S) \mid g\alpha e = f\beta g = g \text{ and } e\theta g\theta f = e\theta f\}.$$

Proof. Set $A := \{g \in E_\theta(S) \mid g\alpha e = f\beta g = g \text{ and } e\theta g\theta f = e\theta f\}$. We will prove that $A = S_\theta^{(\alpha,\beta)}(e,f)$. Let $g \in A$. Then $g\alpha(e\theta f)\beta g = g\theta g = g$ and

$$(e\theta f)\beta g\alpha(e\theta f) = e\theta g\alpha e\theta f = e\theta g\theta f = e\theta f,$$

so $g \in V_\beta^\alpha(e\theta f)$. It implies that $g \in S_\theta^{(\alpha,\beta)}(e,f)$.

Conversely, let $h \in S_\theta^{(\alpha,\beta)}(e,f)$. Then $h = h\alpha e = f\beta h$. Since $h \in V_\beta^\alpha(e\theta f)$, we now observe that $h = h\alpha(e\theta f)\beta h$ and $(e\theta f)\beta h\alpha(e\theta f) = e\theta f$. Thus

$$e\theta h\theta f = e\theta h\alpha e\theta f = (e\theta f)\beta h\alpha(e\theta f) = e\theta f,$$

which implies that $h \in A$. Therefore $A = S_\theta^{(\alpha,\beta)}(e,f)$. □

The next results give a connection with Green's equivalence that:

Proposition 4.1.3. *Let e be an α-idempotent, f be a β-idempotent, g be a γ-idempotent and h be a δ-idempotent in a regular semigroup S.*

(1) *If $e\mathcal{L}f$ then $S_\theta^{(\alpha,\gamma)}(e,g) = S_\theta^{(\beta,\gamma)}(f,g)$ for all $\theta \in \Gamma$.*

(2) *If $e\mathcal{R}f$ then $S_\theta^{(\gamma,\alpha)}(g,e) = S_\theta^{(\gamma,\beta)}(g,f)$ for all $\theta \in \Gamma$.*

(3) *If $e\mathcal{L}f$ and $g\mathcal{R}h$ then $S_\theta^{(\alpha,\gamma)}(e,g) = S_\theta^{(\beta,\delta)}(f,h)$ for all $\theta \in \Gamma$.*

Proof. (1) Suppose that $e\mathcal{L}f$, $\theta \in \Gamma$. Let $x \in S_\theta^{(\alpha,\gamma)}(e,g)$. Then $x\alpha e = x = x\gamma g$ and $e\theta x\theta g = e\theta g$. By Lemma 2.2.4, $e\beta f = e$ and $f\alpha e = f$.

We now have that $x = x\alpha e = x\alpha e\beta f = x\beta f$ and $f\theta x\theta g = f\alpha e\theta x\theta g = f\alpha e\theta g = f\theta g$ then $S_\theta^{(\alpha,\gamma)}(e,g) \subseteq S_\theta^{(\beta,\gamma)}(f,g)$. Similarly, we can show that $S_\theta^{(\beta,\gamma)}(f,g) \subseteq S_\theta^{(\alpha,\gamma)}(e,g)$. Hence $S_\theta^{(\alpha,\gamma)}(e,g) = S_\theta^{(\beta,\gamma)}(f,g)$.

(2) By symmetry (1), we have $S_\theta^{(\gamma,\alpha)}(g,e) = S_\theta^{(\gamma,\beta)}(g,f)$.

(3) Suppose that $e\mathcal{L}f$ and $g\mathcal{R}h$. Let $x \in S_\theta^{(\alpha,\gamma)}(e,g)$ for all $\theta \in \Gamma$. Then $x = x\alpha e = g\gamma x$ and $e\theta x\theta g = e\theta g$. By Lemma 2.2.4, we have $e = e\beta f, f = f\alpha e, g = h\delta g$ and $h = g\gamma h$. Then

$$x\beta f = x\alpha e\beta f = x\alpha e = x,$$

$$h\delta x = h\delta g\gamma x = g\gamma x = x$$

and

$$f\theta x\theta h = (f\alpha e)\theta x\theta(g\gamma h) = f\alpha(e\theta x\theta g)\gamma h = f\alpha e\theta g\gamma h = f\theta h.$$

Thus $x \in S_\theta^{(\beta,\delta)}(f,h)$, it implies that $S_\theta^{(\alpha,\gamma)}(e,g) \subseteq S_\theta^{(\beta,\delta)}(f,h)$.

Conversely, let $y \in S_\theta^{(\beta,\delta)}(f,h)$. Then $y = y\beta f = h\delta y$ and $f\theta y\theta h = f\theta h$. Consider

$$y\alpha e = y\beta f\alpha e = y\beta f = y,$$

$$g\gamma y = g\gamma h\delta y = h\delta y = y$$

and

$$e\theta y\theta g = e\beta f\theta y\theta h\delta g = e\beta f\theta h\delta g = e\theta g.$$

Thus $y \in S_\theta^{(\alpha,\gamma)}(e,g)$ which implies that $S_\theta^{(\beta,\delta)}(f,h) \subseteq S_\theta^{(\alpha,\gamma)}(e,g)$.

Hence $S_\theta^{(\alpha,\gamma)}(e,g) = S_\theta^{(\beta,\delta)}(f,h)$. □

Proposition 4.1.4. *Let S be a regular Γ-semigroup. For $\alpha,\beta \in \Gamma, e \in E_\alpha(S), f \in E_\beta(S)$. Then the following conditions hold.*

(1) *If $e\mathcal{L}f$ then $S_\beta^{(\alpha,\beta)}(e,f) = \{f\}$.*

(2) *If $e\mathcal{R}f$ then $S_\alpha^{(\alpha,\beta)}(e,f) = \{e\}$.*

Proof. (1) Let $e\mathcal{L}f$. By Lemma 2.2.4, $e = e\beta f$ and $f = f\alpha e$. Thus $e\beta f\beta f = e\beta f$, so $f \in S_{\beta}^{(\alpha,\beta)}(e,f)$. Let $x \in S_{\beta}^{(\alpha,\beta)}(e,f)$. Then

$$x = f\beta x = f\beta x\alpha e = f\beta x\alpha e\beta f = f\alpha e\beta x\beta f = f\alpha e\beta f = f\beta f = f.$$

Hence $S_{\beta}^{(\alpha,\beta)}(e,f) = \{f\}$.

(2) The proof is similar to the proof of (1). □

Proposition 4.1.5. *Let S be a regular Γ-semigroup and $a,b \in S, \alpha,\beta,\theta \in \Gamma, e \in L_a, f \in R_b$. Then $a\theta b = (a\theta h)\theta(h\theta b)$ and $a\theta h\mathcal{L}h\mathcal{R}h\theta b$ for any $h \in S_{\theta}^{(\alpha,\beta)}(e,f)$.*

Proof. Let $h \in S_{\theta}^{(\alpha,\beta)}(e,f)$ and $e \in L_a, f \in R_b$. By Lemma 2.2.3 and 2.2.4, we have 4 cases.

Case 1. $e = a$ and $f = b$. Then

$$a\theta b = e\theta f = e\theta h\theta f = e\theta h\theta h\theta f = (a\theta h)\theta(h\theta b)$$

and

$$h = h\theta h = h\alpha e\theta h = h\alpha(a\theta h)$$

which implies that $a\theta h\mathcal{L}h$. It follows that $h = h\theta f\beta h = (h\theta b)\beta h$. Hence $h\theta b\mathcal{R}h$.

Case 2. $e = a$ and $f = b\delta x$ for some $x \in S, \delta \in \Gamma$. Then

$$a\theta b = e\theta f\beta b = e\theta h\theta f\beta b = a\theta h\theta h\theta b$$

and

$$h = h\theta h = h\theta f\beta h = (h\theta b)\delta x\beta h$$

which implies that $h\theta b\mathcal{R}h$. It follows that $h = h\alpha e\theta h = h\alpha a\theta h$. Thus $a\theta h\mathcal{L}h$. Therefore $a\theta h\mathcal{L}h\mathcal{R}h\theta b$.

Case 3. $e = c\gamma a$ and $f = b$ for some $c \in S, \gamma \in \Gamma$. Then $a\theta b = a\alpha e\theta f = a\alpha e\theta h\theta f = a\theta h\theta h\theta b$ and $h = h\theta h = h\alpha e\theta h = h\alpha c\gamma a\theta h$ which implies that $a\theta h\mathcal{L}h$. Clearly, $h\theta b\mathcal{R}h$, Thus $a\theta h\mathcal{L}h\mathcal{R}h\theta b$.

Case 4. $e = c\gamma a$ and $f = b\delta x$ for some $c,x \in S, \gamma,\delta \in \Gamma$. The proof is similar to the proof Case 2 and 3. □

Proposition 4.1.6. *Suppose that S is a regular Γ-semigroup. Then for all $\alpha, \beta, \theta \in \Gamma, e \in E_\alpha(S), f \in E_\beta(S)$ and $g \in S_\theta^{(\alpha,\beta)}(e,f)$ we have $e\theta f$ is (α,β)-inverse of g.*

Proof. Let $g \in S_\theta^{(\alpha,\beta)}(e,f)$. Then

$$e\theta f = e\theta g\theta f = e\theta f\beta g\theta f = e\theta f\beta g\alpha e\theta f$$

and

$$g = g\theta g = g\alpha e\theta f\beta g.$$

Thus $e\theta f \in V_\alpha^\beta(g)$. □

Lemma 4.1.7. *Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma, e \in E_\alpha(S), f \in E_\beta(S)$.*

(1) *If $e\beta f$ is regular then $S_\beta^{(\alpha,\beta)}(e,f) \neq \emptyset$.*

(2) *If $e\alpha f$ is regular then $S_\alpha^{(\alpha,\beta)}(e,f) \neq \emptyset$.*

Proof. (1) Let $e\beta f$ be regular. Then there exist $\gamma, \delta \in \Gamma, x \in V_\gamma^\delta(e\beta f)$. Claim that $f\gamma x\delta e \in E_\beta(S)$. Consider $f\gamma x\delta e = f\gamma(x\delta e\beta f\gamma x)\delta e = (f\gamma x\delta e)\beta(f\gamma x\delta e) \in E_\beta(S)$. Clearly, $f\gamma x\delta e \in S_\beta^{(\alpha,\beta)}(e,f)$. Thus $S_\beta^{(\alpha,\beta)}(e,f) \neq \emptyset$.

(2) The proof of this is similar to the proof of (1). □

In one direction we have the following result.

Theorem 4.1.8. *Let a and b be elements in a regular Γ-semigroup S. Let $\alpha, \beta, \gamma, \delta \in \Gamma, a' \in V_\alpha^\beta(a), b' \in V_\gamma^\delta(b)$ and $g \in S_\theta^{(\alpha,\delta)}(a'\beta a, b\gamma b')$. Then*

(1) *$b'\delta g\alpha a' \in V_\gamma^\beta(a\theta b)$.*

(2) *$b'\delta g \in V_\gamma^\theta(g\theta b)$.*

(3) *$g\alpha a' \in V_\theta^\beta(a\theta g)$.*

(4) *$a\theta g\theta b = a\theta b$.*

Proof. Suppose that $g \in S_\theta^{(\alpha,\delta)}(a'\beta a, b\gamma b')$.

(1) Then

$$
\begin{aligned}
(a\theta b)\gamma(b'\delta g\alpha a')\beta(a\theta b) &= a\theta b\gamma b'\delta g\theta b \\
&= a\theta g\theta b \\
&= (a\alpha a'\beta a)\theta g\theta(b\gamma b'\delta b) \\
&= a\alpha a'\beta a\theta b\gamma b'\theta b \\
&= a\theta b
\end{aligned}
$$

and

$$
\begin{aligned}
b'\delta g\alpha a'\beta a\theta b\gamma b'\delta g\alpha a' &= b'\delta g\theta g\alpha a' \\
&= b'\delta g\alpha a'.
\end{aligned}
$$

We now observe that $b'\delta g\alpha a' \in V_{\gamma}^{\beta}(a\theta b)$.

(2) Consider

$$(b'\delta g)\theta(g\theta b)\gamma(b'\delta g) = b'\delta g\theta g\theta g = b'\delta g$$

and

$$(g\theta b)\gamma(b'\delta g)\theta(g\theta b) = g\theta g\theta g\theta b = g\theta b.$$

Thus $b'\delta g \in V_{\gamma}^{\theta}(g\theta b)$.

(3) Consider

$$(g\alpha a')\beta(a\theta g)\theta(g\alpha a') = g\theta g\alpha a' = g\alpha a'$$

and

$$(a\theta g)\theta(g\alpha a')\beta(a\theta g) = a\theta g\theta g\theta g = a\theta g.$$

Thus $g\alpha a' \in V_{\theta}^{\beta}(a\theta g)$.

(4) Consider

$$
\begin{aligned}
a\theta g\theta b &= a\alpha a'\beta a\theta g\theta b\gamma b'\delta b \\
&= a\alpha a'\beta a\theta b\gamma b'\delta b \\
&= a\theta b.
\end{aligned}
$$

Hence $a\theta g\theta b = a\theta b$. □

Proposition 4.1.9. *Let e be an α-idempotent and f be a β-idempotent and $\theta \in \Gamma$. If $g, h \in S_\theta^{(\alpha,\beta)}(e, f)$ then $g = g\theta h\theta g$.*

Proof. Let $g, h \in S_\theta^{(\alpha,\beta)}(e, f)$. Then

$$g\alpha e = g = f\beta g, e\theta g\theta f = e\theta f$$

and

$$h\alpha e = h = f\beta h, e\theta h\theta f = e\theta f.$$

Thus

$$g = (g\alpha e)\theta(f\beta g) = g\alpha(e\theta h\theta f)\beta g = g\theta h\theta g.$$

□

A non-empty subset A of a Γ-semigroup S is called a **sub-Γ-semigroup** if $A\Gamma A \subseteq A$. The next proposition show that an (α, β, θ)-sandwich set is a sub-Γ-semigroup.

Proposition 4.1.10. *Let e be an α-idempotent and f be a β-idempotent in a regular Γ-semigroup S. Then $S_\theta^{(\alpha,\beta)}(e, f)$ is a sub-Γ-semigroup of S for all $\theta \in \Gamma$.*

Proof. Let $\theta \in \Gamma$ and $g, h \in S_\theta^{(\alpha,\beta)}(e, f)$. By Proposition 4.1.9, $g\theta h = g\theta h\theta g\theta h$ which implies that $g\theta h$ is a θ-idempotent. It follows that

$$(g\theta h)\alpha e = g\theta h, \;\; f\beta(g\theta h) = g\theta h$$

and

$$e\theta(g\theta h)\theta f = e\theta g\theta f\beta h\theta f = e\theta f\beta h\theta f = e\theta h\theta f = e\theta f$$

which proves that $g\theta h \in S_\theta^{(\alpha,\beta)}(e, f)$.

Therefore $S_\theta^{(\alpha,\beta)}(e, f)$ is a sub-Γ-semigroup of S. □

The (α, β, θ)-sandwich set admits the important characterization of the set of an (α, β)-inverse element.

Proposition 4.1.11. *Let S be a regular Γ-semigroup and $\alpha, \beta, \theta \in \Gamma$. Then $S_\theta^{(\alpha,\beta)}(e, f) = f\beta V_\beta^\alpha(e\theta f)\alpha e$ for all $e \in E_\alpha(S), f \in E_\beta(S)$.*

Proof. Let $x \in S_\theta^{(\alpha,\beta)}(e, f)$. Then $x\alpha e = x = f\beta x$ and $e\theta x\theta f = e\theta f$. Then $x = f\beta x = f\beta x\alpha e$. Claim that $x \in V_\beta^\alpha(e\theta f)$. Consider

$$x\alpha(e\theta f)\beta x = x\theta x = x$$

and

$$e\theta f\beta x\alpha e\theta f = e\theta x\theta f = e\theta f.$$

Thus $x \in f\beta V_\beta^\alpha(e\theta f)\alpha e$. It follows that $S_\theta^{(\alpha,\beta)}(e, f) \subseteq f\beta V_\beta^\alpha(e\theta f)\alpha e$.

Let $y \in f\beta V_\beta^\alpha(e\theta f)\alpha e$. Then $y = f\beta z\alpha e$ for some $z \in V_\beta^\alpha(e\theta f)$. It follows that

$$y\alpha e = y = f\beta y$$

and

$$e\theta y\theta f = e\theta f\beta z\alpha e\theta f = e\theta f$$

which proves that $y \in S_\theta^{(\alpha,\beta)}(e, f)$. Hence $S_\theta^{(\alpha,\beta)}(e, f) = f\beta V_\beta^\alpha(e\theta f)\alpha e$. □

Let S and S' be Γ-semigroups and $\theta \in \Gamma$. The mapping $\varphi : S \to S'$ is called a **θ-homomorphism** if $(a\theta b)\varphi = (a\varphi)\theta(b\varphi)$ for all $a, b \in S$. Let φ be a θ-homomorphism of S into S' and let ψ be an θ-homomorphism of S' into S. If $\psi \circ \varphi$ is the identity mapping of S onto itself, and if $\varphi \circ \psi$ is the identity mapping of S' onto itself then φ is a **θ-isomorphism** of S onto S', and ψ is the **inverse θ-isomorphism**. Such θ-isomorphisms ψ and φ are called **mutually inverse θ-isomorphisms**. If there exists a mapping is θ-isomorphism between S and S' then S and S' are called **θ-isomorphic** and we denote by $S \equiv_\theta S'$.

The purpose of this section is to investigate sets of $S_\theta^{(\alpha,\beta)}(e,f)\theta f$ and $e\theta S_\theta^{(\alpha,\beta)}(e,f)$ and mapping between $S_\theta^{(\alpha,\beta)}(e,f)$ and $S_\theta^{(\alpha,\beta)}(e,f)\theta f \times e\theta S_\theta^{(\alpha,\beta)}(e,f)$ are mutually inverse θ-isomorphisms for all $\theta \in \Gamma$.

For a Γ-semigroup S and $e \in E(S)$, we shall give a notation for $(e]$ defined by

$$(e] := \{x \in S \mid x \leqslant e\}.$$

By Proposition 3.2.4, we have that $x \in E(S)$ and by Theorem 3.2.9, we obtain that $x \preccurlyeq e$. The next two lemmas are very important results for the main theorem.

Lemma 4.1.12. *Let S be a regular Γ-semigroup, $\alpha, \beta, \theta \in \Gamma$. For any $e \in E_\alpha(S), f \in E_\beta(S)$ and $x \in V_\alpha^\beta(e\theta f)$, we have*

$$S_\theta^{(\alpha,\beta)}(e,f)\theta f = S_\beta^{(\alpha,\beta)}(x\beta e\theta f, f) = \{q \in S \mid e\theta q = e\theta f \mathcal{L} q \leqslant f\} \subseteq L_{e\theta f} \cap (f].$$

Proof. We show first that $S_\theta^{(\alpha,\beta)}(e,f)\theta f \subseteq S_\beta^{(\alpha,\beta)}(x\beta e\theta f, f)$. Let $y \in S_\theta^{(\alpha,\beta)}(e,f)\theta f$. Then there exists $z \in S_\theta^{(\alpha,\beta)}(e,f)$ such that $y = z\theta f$. Then $z\alpha e = z = f\beta z$ and $e\theta z\theta f = e\theta f$. It follows that

$$y\beta y = (z\theta f)\beta(z\theta f) = z\theta z\theta f = z\theta f = y.$$

Thus $y \in E_\beta(S)$. Claim that $y \in S_\beta^{(\alpha,\beta)}(x\beta e\theta f, f)$. Consider

$$\begin{aligned} y\alpha x\beta e\theta f &= z\theta f\alpha x\beta e\theta f \\ &= z\alpha e\theta f\alpha x\beta e\theta f \\ &= z\alpha e\theta f \\ &= z\theta f \\ &= y, \end{aligned}$$

$$f\beta y = f\beta z\theta f = z\theta f = y$$

and

$$(x\beta e\theta f)\beta y\beta f = x\beta e\theta f\beta z\theta f\beta f = x\beta e\theta z\theta f\beta f = x\beta e\theta f\beta f.$$

Then $y \in S_{\beta}^{(\alpha,\beta)}(x\beta e\theta f, f)$. Therefore $S_{\theta}^{(\alpha,\beta)}(e,f)\theta f \subseteq S_{\beta}^{(\alpha,\beta)}(x\beta e\theta f, f)$.

Set $D := \{q \in S \mid e\theta q = e\theta f \mathcal{L} q \leqslant f\}$.

Next, we prove that $S_{\beta}^{(\alpha,\beta)}(x\beta e\theta f, f) \subseteq D \subseteq L_{e\theta f} \cap (f]$. Let $q \in S_{\beta}^{(\alpha,\beta)}(x\beta e\theta f, f)$. Then $q\alpha(x\beta e\theta f) = q = f\beta q$ which together with

$$q\beta f = q\alpha x\beta e\theta f\beta f = q\alpha x\beta e\theta f = q,$$

it implies that $q \leqslant f$.

We will show that $e\theta q = e\theta f$. It follows that

$$e\theta q = e\theta f\beta q\beta f = (e\theta f\alpha x\beta e\theta f)\beta q\beta f = e\theta f\alpha x\beta e\theta f\beta f = e\theta f.$$

Thus $e\theta q = e\theta f$.

Next, we need to show that $e\theta f\mathcal{L}q$. Since $q = q\alpha x\beta(e\theta f)$ and $e\theta f = e\theta q = e\theta f\beta q$, we get that $e\theta f\mathcal{L}q$. Therefore $e\theta q = e\theta f\mathcal{L}q \leqslant f$. Hence $q \in L_{e\theta f} \cap (f]$.

Finally, we prove that $D \subseteq S_{\theta}^{(\alpha,\beta)}(e,f)\theta f$. Let $q \in D$. Then $e\theta q = e\theta f\mathcal{L}q \leqslant f$ and let $w \in S_{\theta}^{(\alpha,\beta)}(e,f)$. Consider

$$(q\beta w)\theta(q\beta w) = q\beta w\alpha e\theta q\beta w = q\beta w\alpha e\theta f\beta w = q\beta w\theta w = q\beta w,$$

$$(q\beta w)\alpha e = q\beta w = f\beta(q\beta w)$$

and

$$e\theta(q\beta w)\theta f = e\theta f\beta w\theta f = e\theta w\theta f = e\theta f.$$

Then $q\beta w \in S_{\theta}^{(\alpha,\beta)}(e,f)$. Since $e\theta f\mathcal{L}q$, there exist $u \in S, \delta \in \Gamma$ such that $q = u\delta(e\theta f)$. Thus

$$\begin{aligned} q &= u\delta(e\theta f) \\ &= u\delta(e\theta w\theta f) \\ &= u\delta(e\theta f\beta w\theta f) \\ &= q\beta w\theta f \in S_{\theta}^{(\alpha,\beta)}(e,f)\theta f. \end{aligned}$$

Hence $\{q \in S \mid e\theta q = e\theta f\mathcal{L}q \leqslant f\} \subseteq S_{\theta}^{(\alpha,\beta)}(e,f)\theta f$.

Therefore $S_{\theta}^{(\alpha,\beta)}(e,f)\theta f = S_{\beta}^{(\alpha,\beta)}(x\beta e\theta f, f) = \{q \in S \mid e\theta q = e\theta f\mathcal{L}q \leqslant f\} \subseteq L_{e\theta f} \cap (f]$. $\square$

Dually, we get the following statement.

Lemma 4.1.13. *Suppose that S is a regular Γ-semigroup and $\alpha, \beta, \theta \in \Gamma$. Let $e \in E_\alpha(S), f \in E_\beta(S)$ and $x \in V_\alpha^\beta(e\theta f)$. Then*

$$e\theta S_\theta^{(\alpha,\beta)}(e,f) = S_\alpha^{(\alpha,\beta)}(e, e\theta f\alpha x) = \{r \in S \mid r\theta f = e\theta f \mathcal{R} r \leqslant e\} \subseteq R_{e\theta f} \cap (e].$$

Proof. This Lemma can be proved dually Lemma 4.1.12. $\square$

Theorem 4.1.14. *Suppose that S is a regular Γ-semigroup and $\alpha, \beta, \theta \in \Gamma$. Let $e \in E_\alpha(S), f \in E_\beta(S)$. Then the mappings*

$$\varphi : x \mapsto (x\theta f, e\theta x), \qquad \text{and } \psi : (y,z) \mapsto y\alpha w\beta z$$

(where $w \in V_\alpha^\beta(e\theta f)$) are mutually inverse θ-isomorphisms between sub Γ-semigroup $S_\theta^{(\alpha,\beta)}(e,f)$ and $S_\theta^{(\alpha,\beta)}(e,f)\theta f \times e\theta S_\theta^{(\alpha,\beta)}(e,f)$.

Proof. Suppose that $y \in S_\theta^{(\alpha,\beta)}(e,f)\theta f$ and $w, w' \in V_\alpha^\beta(e\theta f)$. Let $z \in e\theta S_\theta^{(\alpha,\beta)}(e,f)$. Then $y = u\theta f, z = e\theta v$ for some $u, v \in S_\theta^{(\alpha,\beta)}(e,f)$ and thus

$$\begin{aligned} y\alpha w\beta z &= (u\theta f)\alpha w\beta(e\theta v) \\ &= u\alpha e\theta f\alpha w\beta e\theta f\beta v \\ &= u\alpha e\theta f\beta v \\ &= u\alpha e\theta f\alpha w'\beta e\theta f\beta v \\ &= u\theta f\alpha w'\beta e\theta v \\ &= y\alpha w'\beta z \end{aligned}$$

and ψ is a single valued. With the same notation and set $x := y\alpha w\beta z$, we get

$$\begin{aligned} w\beta z\theta y\alpha w &= w\beta z\theta u\theta f\alpha w \\ &= w\beta z\theta f\beta u\theta f\alpha w \\ &= w\beta z\theta f\beta y\alpha w \\ &= w\beta e\theta v\theta f\beta y\alpha w \\ &= w\beta e\theta f\beta y\alpha w \end{aligned}$$

$$\begin{aligned} w\beta z\theta y\alpha w &= w\beta e\theta f\beta u\theta f\alpha w \\ &= w\beta e\theta u\theta f\alpha w \\ &= w\beta e\theta f\alpha w \\ &= w \end{aligned}$$

which implies that $x = y\alpha w\beta z \in E_\theta(S)$. We will show that $x \in S_\theta^{(\alpha,\beta)}(e,f)$. It follows that

$$\begin{aligned} x\alpha e &= y\alpha w\beta z\alpha e \\ &= y\alpha w\beta e\theta f\beta v\alpha e \\ &= y\alpha w\beta e\theta f\beta v \\ &= y\alpha w\beta z \\ &= x \\ &= u\alpha e\theta f\alpha w\beta z \\ &= f\beta u\alpha e\theta f\alpha w\beta z \\ &= f\beta y\alpha w\beta z \\ &= f\beta x \end{aligned}$$

and

$$\begin{aligned} e\theta x\theta f &= e\theta y\alpha w\beta z\theta f \\ &= e\theta u\theta f\alpha w\beta e\theta v\theta f \\ &= e\theta f\alpha w\beta e\theta f \\ &= e\theta f. \end{aligned}$$

Thus $x \in S_\theta^{(\alpha,\beta)}(e,f)$.
Therefore $\psi : S_\theta^{(\alpha,\beta)}(e,f)\theta f \times e\theta S_\theta^{(\alpha,\beta)}(e,f) \to S_\theta^{(\alpha,\beta)}(e,f)$.

For $x \in S_\theta^{(\alpha,\beta)}(e,f)$ and $w \in V_\alpha^\beta(e\theta f)$, we have

$$\begin{aligned} x\varphi\psi &= (x\theta f, e\theta x)\psi \\ &= (x\theta f)\alpha w\beta(e\theta x) \\ &= (x\alpha e)\theta f\alpha w\beta e\theta(f\beta x) \\ &= x\alpha e\theta f\beta x \\ &= x \end{aligned}$$

and for $(y,z) \in S_\theta^{(\alpha,\beta)}(e,f)\theta f \times e\theta S_\theta^{(\alpha,\beta)}(e,f)$, we have

$$(y,z)\psi\varphi = (y\alpha w\beta z)\varphi = (y\alpha w\beta z\theta f, e\theta y\alpha w\beta z).$$

Next, we will show that $y\alpha w\beta z\theta f = y$ and $e\theta y\alpha w\beta z = z$. Since $y = u\alpha e\theta f$ and $z = e\theta f\beta v$ for some $u, v \in S_\theta^{(\alpha,\beta)}(e,f)$, we obtain that

$$\begin{aligned} y\alpha w\beta z\theta f &= u\alpha e\theta f\alpha w\beta e\theta f\beta v\theta f \\ &= u\alpha e\theta f\alpha w\beta e\theta v\theta f \\ &= u\alpha e\theta f\alpha w\beta e\theta f \\ &= u\alpha e\theta f \\ &= y \end{aligned}$$

and

$$\begin{aligned} e\theta y\alpha w\beta z &= e\theta u\alpha e\theta f\alpha w\beta e\theta f\beta v \\ &= e\theta u\theta f\alpha w\beta e\theta f\beta v \\ &= e\theta f\alpha w\beta e\theta f\beta v \\ &= e\theta f\beta v \\ &= z. \end{aligned}$$

Thus $(y,z)\psi\varphi = (y,z)$.

For all $y, y' \in S_\theta^{(\alpha,\beta)}(e,f)\theta f$ and by Lemma 4.1.12, we have $S_\theta^{(\alpha,\beta)}(e,f)\theta f = S_\beta^{(\alpha,\beta)}(w\beta e\theta f, f) = \{q \in S \mid e\theta q = e\theta f\mathcal{L}q \leqslant f\}$, it follows that

$$y\theta y' = y\alpha w\beta e\theta f\theta y' = y\alpha w\beta e\theta y'\theta y' = y\alpha w\beta e\theta y' = y\alpha w\beta e\theta f = y.$$

Similarly, we can show that $z\theta z' = z'$ for all $z, z' \in e\theta S_\theta^{(\alpha,\beta)}(e,f)$.

Next, we now prove that φ and ψ are θ-homomorphisms. Let $x, x' \in S_\theta^{(\alpha,\beta)}(e,f)$. Then

$$\begin{aligned}(x\theta x')\varphi &= (x\theta x'\theta f, e\theta x\theta x')\\ &= (x\alpha e\theta x'\theta f, e\theta x\theta f\beta x')\\ &= (x\alpha e\theta f, e\theta f\beta x')\\ &= (x\theta f, e\theta x')\end{aligned}$$

and

$$\begin{aligned}(x\varphi)\theta(x'\varphi) &= (x\theta f, e\theta x)\theta(x'\theta f, e\theta x')\\ &= ((x\theta f)\theta(x'\theta f), (e\theta x)\theta(e\theta x'))\\ &= (x\theta f, e\theta x').\end{aligned}$$

It follows that $(x\theta x')\varphi = (x\varphi)\theta(x'\varphi)$.
Let $y, y' \in S_\theta^{(\alpha,\beta)}(e,f)\theta f$ and $z, z' \in e\theta S_\theta^{(\alpha,\beta)}(e,f)$. Then $y' = v\theta f$ and $z = e\theta u$ for some $u, v \in S_\theta^{(\alpha,\beta)}(e,f)$. Thus we have

$$((y,z)\theta(y',z'))\psi = ((y\theta y'),(z\theta z'))\psi = (y\theta y')\alpha w\beta(z\theta z') = y\alpha w\beta z',$$

and

$$\begin{aligned}((y,z)\psi)\theta((y',z')\psi) &= (y\alpha w\beta z)\theta(y'\alpha w\beta z')\\ &= y\alpha w\beta e\theta u\theta v\theta f\alpha w\beta z'\\ &= y\alpha w\beta e\theta u\alpha e\theta v\theta f\alpha w\beta z'\\ &= y\alpha w\beta e\theta u\alpha e\theta f\alpha w\beta z'\\ &= y\alpha w\beta e\theta u\theta f\alpha w\beta z'\\ &= y\alpha w\beta e\theta f\alpha w\beta z'\\ &= y\alpha w\beta z'.\end{aligned}$$

Hence φ and ψ are θ-homomorphisms. □

Recall relations on $E(S)$ as follows : For $f \in E(S)$,

$$(f]_l := \{e \in E(S) \mid e \preccurlyeq^l f\},$$

$$(f]_r := \{e \in E(S) \mid e \preccurlyeq^r f\}$$

and

$$(f] := \{e \in E(S) \mid e \preccurlyeq f\}.$$

Proposition 4.1.15. *Let S be a regular Γ-semigroup and $\alpha, \beta \in \Gamma, e \in E_\alpha(S), f \in E_\beta(S)$. Then the following statements hold.*

(1) *If $e \preccurlyeq^r f$ then* $S_\alpha^{(\alpha,\beta)}(e,f) = (f]_r \cap E_\alpha(L_e)$ *and*

$$S_\beta^{(\beta,\alpha)}(f,e) = (f] \cap E_\beta(R_e).$$

(2) *If $e \preccurlyeq^l f$ then* $S_\alpha^{(\alpha,\beta)}(e,f) = (f] \cap E_\alpha(L_e)$ *and*

$$S_\beta^{(\beta,\alpha)}(f,e) = (f]_l \cap E_\beta(R_e).$$

Proof. (1) Suppose that $e \preccurlyeq^r f$. Then $e = f\beta e$. Let $p \in S_\alpha^{(\alpha,\beta)}(e,f)$. Clearly, $p = f\beta p$ then $p \in (f]_r$ and $p = p\alpha e$. Thus

$$e = e\alpha f\beta e = e\alpha p\alpha f\beta e = e\alpha p\alpha e = e\alpha p,$$

which implies that $p \in L_e$. Conversely, let $q \in (f]_r \cap E_\alpha(L_e)$. Then $q = f\beta q$ and $q\mathcal{L}e$. Thus $q = q\alpha e$ and $e = e\alpha q$. Hence $q \in S_\alpha^{(\alpha,\beta)}(e,f)$.

Let $x \in S_\beta^{(\beta,\alpha)}(f,e)$. Then $x = x\beta f = e\alpha x$ and $f\beta e = f\beta x\beta e$. Thus

$$f\beta x = f\beta e\alpha x = e\alpha x = x$$

which implies that $x \in (f]$. It follows that $e = f\beta e = f\beta x\beta e = x\beta e$, so $x \in R_e$. Hence $x \in (f] \cap E_\beta(R_e)$. Conversely, it is obvious.

(2) The proof is similar to the proof of (1). □

Recall $E(X) := E(S) \cap X$ where X is a subset of a Γ-semigroup and we denote the cardinality of a set X by $\mid X \mid$

Theorem 4.1.16. *Let S be a Γ-semigroup and $E(g\Gamma S\Gamma g)$ is a commutative sub Γ-semigroup of S for all $g \in E(S)$. Then the following statements hold.*

(1) $\mid S_{\theta}^{(\alpha,\beta)}(e,f) \mid \leqslant 1$ *for all* $\alpha,\beta,\theta \in \Gamma, e \in E_{\alpha}(S), f \in E_{\beta}(S)$.

(2) *If* $a,b,x,y \in Reg(S)$ *with* $a \leqslant x$, $b \leqslant y$ *where* $Reg(S)$ *is a sub* Γ-*semigroup then* $a\theta b \leqslant x\theta y$ *for some* $\theta \in \Gamma$.

Proof. (1) Let $p,q \in S_{\theta}^{(\alpha,\beta)}(e,f)$. It is easy to show that $e\theta p, e\theta q \in E_{\alpha}(S)$ and $p\theta f, q\theta f \in E_{\beta}(S)$. Consider

$$\begin{aligned} e\theta p &= e\theta p\alpha e \in e\Gamma S\Gamma e, \\ e\theta q &= e\theta q\alpha e \in e\Gamma S\Gamma e, \\ p\theta f &= f\beta p\theta f \in f\Gamma S\Gamma f \end{aligned}$$

and

$$q\theta f = f\beta q\theta f \in f\Gamma S\Gamma f.$$

Thus $e\theta p, e\theta q \in E_{\alpha}(e\Gamma S\Gamma e)$ and $p\theta f, q\theta f \in E_{\beta}(f\Gamma S\Gamma f)$. Since the idempotent in $E(e\Gamma S\Gamma e)$ and $E(f\Gamma S\Gamma f)$ commute, we have $(e\theta p)\alpha(e\theta q) = (e\theta q)\alpha(e\theta p)$ and $(p\theta f)\beta(q\theta f) = (q\theta f)\beta(p\theta f)$. This implies that $e\theta p\theta q = e\theta q\theta p$ and $p\theta q\theta f = q\theta p\theta f$. Consider

$$e\theta p\theta q = e\theta p\theta f\beta q = e\theta f\beta q = e\theta q$$

and

$$e\theta q\theta p = e\theta q\theta f\beta p = e\theta f\beta p = e\theta p,$$

we obtain that $e\theta q = e\theta p$. Similarly, we can show that $p\theta f = q\theta f$. Thus

$$p = p\theta p = p\theta f\beta p = q\theta f\beta p = q\theta p = q\alpha e\theta p = q\alpha e\theta q = q\theta q = q.$$

(2) Let $x' \in V_{\alpha}^{\beta}(x)$ and $y' \in V_{\gamma}^{\delta}(y)$ for some $\alpha,\beta,\gamma,\delta \in \Gamma$. Then $x'\beta x \in E_{\alpha}(L_x)$ and $y\gamma y' \in E_{\delta}(R_y)$. By Remark 1, there exists $e \in E_{\alpha}(L_a)$ such that

$e \preccurlyeq x'\beta x$ and $a = x\alpha e$. By Remark 1, there exists $f \in E_\delta(R_b)$ such that $f \preccurlyeq y\gamma y'$ and $b = f\delta y$. Since $e\mathcal{L}a$, we have $e\delta b\mathcal{L}a\delta b$. By assumption and Lemma 3.1.6(1), $e\delta b$ is regular. Since $f\mathcal{R}b$, we have $e\delta f\mathcal{R}e\delta b$. By Lemma 3.1.6(2), $e\delta f$ is regular. By Lemma 4.1.7(1), $S_\delta^{(\alpha,\delta)}(e,f) \neq \emptyset$. Let $p \in S_\delta^{(\alpha,\delta)}(e,f)$. Then

$$a\delta b = (x\alpha e)\delta(f\delta y) = x\alpha(e\delta p\delta f)\delta y = x\alpha(e\delta p)\delta(p\delta f)\delta y \tag{4.1}$$

and

$$p\alpha x'\beta x = p\alpha e\alpha x'\beta x = p\alpha e = p. \tag{4.2}$$

Thus $p\alpha x'\beta x\delta p = p\delta p = p$, so $x'\beta x\delta p\mathcal{L}p$. Since $p\alpha e = p$, we have $p\alpha e\delta p = p$ which implies that $e\delta p\mathcal{L}p$. Thus $e\delta p\mathcal{L}x'\beta x\delta p$. Consider

$$(x'\beta x\delta p)\alpha(x'\beta x\delta p) = x'\beta x\delta p\delta p = x'\beta x\delta p,$$

$$x'\beta x\delta p = x'\beta x\delta p\alpha x'\beta x,$$

and

$$x'\beta x\delta p = x'\beta x\alpha x'\beta x\delta p$$

which implies that $x'\beta x\delta p \preccurlyeq x'\beta x$ and

$$(e\delta p)\alpha(e\delta p) = e\delta p\delta p = e\delta p,$$

$$e\delta p = e\delta p\alpha e, \quad e\delta p = e\alpha e\delta p,$$

so $e\delta p \preccurlyeq e$. Thus $x'\beta x\delta p, e\delta p \preccurlyeq x'\beta x$. Now,

$$e\delta p = x'\beta x\alpha e\delta p = x'\beta x\alpha e\delta p\alpha x'\beta x \in (x'\beta x)\Gamma S\Gamma(x'\beta x)$$

and

$$x'\beta x\delta p = x'\beta x\delta p\alpha x'\beta x \in (x'\beta x)\Gamma S\Gamma(x'\beta x).$$

Since the idempotent in $E((x'\beta x)\Gamma S\Gamma(x'\beta x))$ commute, we obtain that

$$(e\delta p)\alpha(x'\beta x\delta p) = (x'\beta x\delta p)\alpha(e\delta p).$$

Thus $e\delta p = x'\beta x\delta p$. Similarly, we can show that $p\delta y\gamma y' = p\delta f$. By (4.2), we have $x\delta p = x\delta p\alpha x'\beta x$. Thus

$$x\delta p\alpha x' = x\delta(p\delta p)\alpha x' = (x\delta p\alpha x')\beta(x\delta p\alpha x'). \tag{4.3}$$

Similarly, we have $p\delta y = y\gamma y'\delta p\delta y$. Thus

$$y'\delta p\delta y = y'\delta(p\delta p)\delta y = (y'\delta p\delta y)\gamma(y'\delta p\delta y). \tag{4.4}$$

By 4.3 and 4.4, we obtain that $x\delta p\alpha x' \in E_\beta(S)$ and $y'\delta p\delta y \in E_\gamma(S)$. By 4.1, we get that

$$\begin{aligned} a\delta b &= x\alpha(e\delta p)\delta(p\delta f)\delta y \\ &= x\alpha x'\beta x\delta p\delta p\delta y\gamma y'\delta y \\ &= x\delta p\delta y \\ &= (x\delta y)\gamma(y'\delta p\delta y), \end{aligned}$$

and

$$a\delta b = x\delta p\delta y = x\delta p\alpha x'\beta x\delta y = (x\delta p\alpha x')\beta(x\delta y).$$

Hence $a\delta b \leqslant x\delta y$. □

4.2 The Sandwich Set of an Element on Regular Γ-semigroups

In this section, we introduce an (α,β,θ)-sandwich set of idempotent elements and study some properties of an (α,β,θ)-sandwich set of elements in regular Γ-semigroups. The next result shows that (α,β,θ)-sandwich set does not depend on the choice of (α,β)-inverse.

Proposition 4.2.1. *Let S be a regular Γ-semigroup and for all $a \in S, \theta \in \Gamma$. Then there exist $\alpha,\beta \in \Gamma, a' \in V_\alpha^\beta(a)$ such that $S_\theta^{(\alpha,\beta)}(a'\beta a, a\alpha a') = a\alpha V_\alpha^\beta(a\theta a)\beta a$.*

Furthermore, If $a'' \in V_\alpha^\beta(a)$ then $S_\theta^{(\alpha,\beta)}(a''\beta a, a\alpha a'') = a\alpha V_\alpha^\beta(a\theta a)\beta a$.

Proof. Let $x \in S_\theta^{(\alpha,\beta)}(a'\beta a, a\alpha a')$. Then $x = x\alpha(a'\beta a) = (a\alpha a')\beta x$ and $(a'\beta a)\theta x\theta(a\alpha a') = a'\beta a\theta a\alpha a'$. Thus $x = a\alpha a'\beta x\alpha a'\beta a$.

Claim that $a'\beta x\alpha a' \in V_\alpha^\beta(a\theta a)$. Consider

$$(a'\beta x\alpha a')\beta(a\theta a)\alpha(a'\beta x\alpha a') = a'\beta x\theta x\alpha a' = a'\beta x\alpha a'$$

and

$$\begin{aligned}(a\theta a)\alpha(a'\beta x\alpha a')\beta(a\theta a) &= a\theta x\theta a\\ &= a\alpha a'\beta a\theta x\theta a\alpha a'\beta a\\ &= a\alpha a'\beta a\theta a\alpha a'\beta a\\ &= a\theta a,\end{aligned}$$

which together with $a'\beta x\alpha a' \in V_\alpha^\beta(a\theta a)$, so $x = a\alpha a'\beta x\alpha a'\beta a \in a\alpha V_\alpha^\beta(a\theta a)\beta a$. Hence $S_\theta^{(\alpha,\beta)}(a'\beta a, a\alpha a') \subseteq a\alpha V_\alpha^\beta(a\theta a)\beta a$.

We will show that $a\alpha V_\alpha^\beta(a\theta a)\beta a \subseteq S_\theta^{(\alpha,\beta)}(a'\beta a, a\alpha a')$ and let $y \in V_\alpha^\beta(a\theta a)$. Then $y = y\beta(a\theta a)\alpha y$ and $a\theta a = (a\theta a)\alpha y\beta(a\theta a)$. For $a\alpha y\beta a = a\alpha y\beta a\theta a\alpha y\beta a$ and then $a\alpha y\beta a \in E_\theta(S)$. Thus

$$(a\alpha y\beta a)\alpha(a'\beta a) = a\alpha y\beta a,$$

$$(a\alpha a')\beta(a\alpha y\beta a) = a\alpha y\beta a$$

and

$$(a'\beta a)\theta(a\alpha y\beta a)\theta(a\alpha a') = (a'\beta a)\theta(a\alpha a').$$

Therefore $a\alpha y\beta a \in S_\theta^{(\alpha,\beta)}(a'\beta a, a\alpha a')$.

Hence $a\alpha V_\alpha^\beta(a\theta a)\beta a \subseteq S_\theta^{(\alpha,\beta)}(a'\beta a, a\alpha a')$.

The proof that $S_\theta^{(\alpha,\beta)}(a''\beta a, a\alpha a'') = a\alpha V_\alpha^\beta(a\theta a)\beta a$ is similar for $a'' \in V_\alpha^\beta(a)$.

□

Remark 3. In Proposition 4.2.1, we see that $S_\theta^{(\alpha,\beta)}(a'\beta a, a\alpha a') = a\alpha V_\alpha^\beta(a\theta a)\beta a$ for all choice $a' \in V_\alpha^\beta(a)$.

Note that we write $S_\theta^{(\alpha,\beta)}(a) := S_\theta^{(\alpha,\beta)}(a'\beta a, a\alpha a')$ where $a' \in V_\alpha^\beta(a)$.

For $\theta \in \Gamma$, a Γ-semigroup S is called a **right (left) θ-zero semigroup** if $a\theta b = b$ $(a\theta b = a)$ for all $a, b \in S$.

We shall give some necessary and sufficient conditions for a right θ-zero semigroup $S_\theta^{(\alpha,\beta)}(a)$.

Theorem 4.2.2. *Let S be a regular Γ-semigroup and $\alpha, \beta, \theta \in \Gamma$. Then the following conditions are equivalent.*

(1) *For any $a \in S$, $S_\theta^{(\alpha,\beta)}(a)$ is a right θ-zero semigroup.*

(2) *If $e \in E_\alpha(S), f \in E_\beta(S)$ such that $e\mathcal{D}f$ then $S_\theta^{(\alpha,\beta)}(e, f)$ is a right θ-zero semigroup.*

(3) *If $a \in S$ and $x, y \in V_\alpha^\beta(a\theta a)$ then $(a\alpha x\beta a)\theta(a\alpha y\beta a) = a\alpha y\beta a$.*

(4) *If $a, x, y \in S$ with $a\theta a = a\theta a\alpha x\beta a\theta a = a\theta a\alpha y\beta a\theta a$ then*

$$(a\alpha x\beta a)\theta(a\alpha y\beta a) = (a\alpha y\beta a)\theta(a\alpha y\beta a).$$

(5) *If $x, y \in S$, $e \in E_\alpha(S), f \in E_\beta(S)$ such that $e\mathcal{D}f$,*

$$e\theta x = e\theta y = e\theta f\mathcal{L}x\mathcal{L}y, \text{ and } x, y \leqslant f \text{ then } x = y.$$

Proof. (1) $\Rightarrow$ (2) Let $e \in E_\alpha(S), f \in E_\beta(S)$ such that $e\mathcal{D}f$. By Lemma 3.1.5, there exist $a \in S$ and $a' \in V_\alpha^\beta(a)$ such that $e = a'\beta a$ and $f = a\alpha a'$. Thus

$$S_\theta^{(\alpha,\beta)}(e, f) = S_\theta^{(\alpha,\beta)}(a'\beta a, a\alpha a') = S_\theta^{(\alpha,\beta)}(a)$$

is a right θ-zero semigroup.

(2) $\Rightarrow$ (3) Let $a \in S, \theta \in \Gamma$ and $a' \in V_\alpha^\beta(a)$. Then $a'\beta a\mathcal{L}a$ and $a\mathcal{R}a\alpha a'$ which implies that $a'\beta a\mathcal{D}a\alpha a'$. By assumption, we have that $S_\theta^{(\alpha,\beta)}(a)$ is a right θ-zero semigroup. Let $x, y \in V_\alpha^\beta(a\theta a)$. By Proposition 4.2.1, we obtain $a\alpha x\beta a, a\alpha y\beta a \in S_\theta^{(\alpha,\beta)}(a)$. Thus $(a\alpha x\beta a)\theta(a\alpha y\beta a) = a\alpha y\beta a$.

(3) $\Rightarrow$ (4) Let $a, x, y \in S$ be such that $a\theta a = a\theta a\alpha x\beta a\theta a = a\theta a\alpha y\beta a\theta a$. Then $x\beta a\theta a\alpha x, y\beta a\theta a\alpha y \in V_\alpha^\beta(a\theta a)$. By hypothesis, we have

$$[a\alpha(x\beta a\theta a\alpha x)\beta a]\theta[a\alpha(y\beta a\theta a\alpha y)\beta a] = a\alpha(y\beta a\theta a\alpha y)\beta a.$$

Thus

$$\begin{aligned}(a\alpha y\beta a)\theta(a\alpha y\beta a) &= a\alpha x\beta a\theta a\alpha x\beta a\theta a\alpha y\beta a\theta a\alpha y\beta a \\ &= a\alpha x\beta a\theta a\alpha y\beta a\theta a\alpha y\beta a \\ &= a\alpha x\beta a\theta a\alpha y\beta a.\end{aligned}$$

(4) $\Rightarrow$ (5) Let $e \in E_\alpha(S)$ and $f \in E_\beta(S)$ be such that $e\mathcal{D}f$. By Lemma 3.1.5, there exist $a \in S$ and $a' \in V_\alpha^\beta(a)$ such that $e = a'\beta a$ and $f = a\alpha a'$. By Proposition 4.2.1, we have

$$a\alpha V_\alpha^\beta(a\theta a)\beta a = S_\theta^{(\alpha,\beta)}(a) = S_\theta^{(\alpha,\beta)}(e,f).$$

Let $x, y \in S$ be such that $e\theta x = e\theta y = e\theta f\mathcal{L}x\mathcal{L}y$ and $x, y \leqslant f$. By Lemma 4.1.12, we have $x, y \in S_\theta^{(\alpha,\beta)}(e,f)\theta f$. By Proposition 4.2.1, we get that $x, y \in a\alpha V_\alpha^\beta(a\theta a)\beta a\theta f$. Then there exist $s, t \in V_\alpha^\beta(a\theta a)$ such that $x = a\alpha s\beta a\theta f$ and $y = a\alpha t\beta a\theta f$. Since $s, t \in V_\alpha^\beta(a\theta a)$, we have $a\theta a = a\theta a\alpha s\beta a\theta a = a\theta a\alpha t\beta a\theta a$. By hypothesis, we have

$$(a\alpha s\beta a)\theta(a\alpha t\beta a) = (a\alpha t\beta a)\theta(a\alpha t\beta a) = a\alpha t\beta a.$$

It follows that

$$\begin{aligned}x &= a\alpha s\beta a\theta f \\ &= a\alpha s\beta a\alpha a'\beta a\theta f \\ &= a\alpha s\beta a\alpha a'\beta a\theta a\alpha a' \\ &= a\alpha s\beta a\alpha a'\beta a\theta a\alpha t\beta a\theta a\alpha a' \\ &= a\alpha t\beta a\theta a\alpha a' \\ &= a\alpha t\beta a\theta f \\ &= y.\end{aligned}$$

(5) $\Rightarrow$ (1) Let $a \in S$ and $a' \in V_\alpha^\beta(a)$. Set $e := a'\beta a$ and $f := a\alpha a'$. We will show that $S_\theta^{(\alpha,\beta)}(e,f)\theta f$ is trivial. Let $x, y \in S_\theta^{(\alpha,\beta)}(e,f)\theta f$. By Lemma 4.1.12, we

have $e\theta x = e\theta f\mathcal{L}x \leqslant f$ and $e\theta y = e\theta f\mathcal{L}y \leqslant f$ which implies that

$$e\theta x = e\theta y = e\theta f\mathcal{L}x\mathcal{L}y \text{ and } x, y \leqslant f.$$

Since $e = a'\beta a$ and $a = a\alpha a'\beta a = a\alpha e$, we get that $e\mathcal{L}a$. And since $f = a\alpha a'$ and $a = a\alpha a'\beta a = f\beta a$, we obtain $a\mathcal{R}f$. Thus $e\mathcal{D}f$. By hypothesis, we have $x = y$. Hence $S_\theta^{(\alpha,\beta)}(e, f)\theta f$ is trivial. Therefore $S_\theta^{(\alpha,\beta)}(a)$ is a right θ-zero semigroup. □

The main result shows that an (α, β, θ)-sandwich set $S_\theta^{(\alpha,\beta)}(a)$ has only one element.

Theorem 4.2.3. *Let S be a regular Γ-semigroup and $\alpha, \beta, \theta \in \Gamma$. Then the following conditions are equivalent.*

(1) *For any $a \in S$, $\mid S_\theta^{(\alpha,\beta)}(a) \mid = 1$.*

(2) *If $a \in S$ and $x, y \in V_\alpha^\beta(a\theta a)$ then $a\alpha x\beta a = a\alpha y\beta a$.*

(3) *If $a, x, y \in S$ with $a\theta a = a\theta a\alpha x\beta a\theta a = a\theta a\alpha y\beta a\theta a$ then*

$$(a\alpha x\beta a)\theta(a\alpha x\beta a) = (a\alpha y\beta a)\theta(a\alpha y\beta a).$$

(4) *If $a, x, y \in S$ with $a\theta a = a\theta a\alpha x\beta a\theta a = a\theta a\alpha y\beta a\theta a$ then*

$$(a\alpha x\beta a)\theta(a\alpha y\beta a) = (a\alpha y\beta a)\theta(a\alpha x\beta a).$$

(5) *If $e \in E_\alpha(S), f \in E_\beta(S)$ such that $e\mathcal{D}f$ then $\mid S_\theta^{(\alpha,\beta)}(e, f) \mid = 1$.*

Proof. (1) ⇒ (2) Let $a \in S$ and $x, y \in V_\alpha^\beta(a\theta a)$. By Proposition 4.2.1, we have $S_\theta^{(\alpha,\beta)}(a) = a\alpha V_\alpha^\beta(a\theta a)\beta a$. By hypothesis, we have $a\alpha x\beta a = a\alpha y\beta a$.

(2) ⇒ (3) Let $a, x, y \in S$ be such that $a\theta a = a\theta a\alpha x\beta a\theta a = a\theta a\alpha y\beta a\theta a$. Then $x\beta(a\theta a)\alpha x, y\beta(a\theta a)\alpha y \in V_\alpha^\beta(a\theta a)$. By hypothesis, we have

$$a\alpha(x\beta a\theta a\alpha x)\beta a = a\alpha(y\beta a\theta a\alpha y)\beta a.$$

(3) ⇒ (4) Let $a, x, y \in S$ be such that $a\theta a = a\theta a\alpha x\beta a\theta a = a\theta a\alpha y\beta a\theta a$. By

the hypothesis, we have

$$\begin{aligned}(a\alpha x\beta a)\theta(a\alpha y\beta a) &= (a\alpha x\beta a)\theta(a\alpha x\beta a)\theta(a\alpha y\beta a)\\ &= (a\alpha y\beta a)\theta(a\alpha y\beta a)\theta(a\alpha y\beta a)\\ &= (a\alpha y\beta a)\theta(a\alpha x\beta a)\theta(a\alpha x\beta a)\\ &= a\alpha y\beta a\theta a\alpha x\beta a.\end{aligned}$$

(4) $\Rightarrow$ (5) Let $e \in E_\alpha(S), f \in E_\beta(S)$ be such that $e\mathcal{D}f$. By Lemma 3.1.5, there exist $a \in S$ and $a' \in V_\alpha^\beta(a)$ such that $e = a'\beta a$ and $f = a\alpha a'$. Let $x, y \in S_\theta^{(\alpha,\beta)}(a'\beta a, a\alpha a')$. Then $a\alpha a'\beta x\alpha a'\beta a = x$ and $a\alpha a'\beta y\alpha a'\beta a = y$. Thus

$$a\theta x\theta a = a\alpha a'\beta a\theta x\theta a\alpha a'\beta a = a\alpha a'\beta a\theta a\alpha a'\beta a = a\theta a.$$

Similarly, we can show that $a\theta y\theta a = a\theta a$. Consider

$$(a\theta a)\alpha(a'\beta x\alpha a')\beta(a\theta a) = a\theta a = (a\theta a)\alpha(a'\beta y\alpha a')\beta(a\theta a).$$

By hypothesis, we have that

$$[a\alpha(a'\beta x\alpha a')\beta a]\theta[a\alpha(a'\beta y\alpha a')\beta a] = [a\alpha(a'\beta y\alpha a')\beta a]\theta[a\alpha(a'\beta x\alpha a')\beta a]$$

which implies that $x\theta y = y\theta x$. By Proposition 4.1.9, we get that

$$x = x\theta y\theta x = x\theta x\theta y = x\theta y = x\theta y\theta y = y\theta x\theta y = y.$$

(5) $\Rightarrow$ (1) Let $a \in S$. Since S is a regular Γ-semigroup, there exist $\alpha, \beta \in \Gamma$ such that $a' \in V_\alpha^\beta(a)$. Set $e := a'\beta a$ and $f := a\alpha a'$. Then it is easy to show that $e\mathcal{D}f$. By hypothesis, we obtain that $\mid S_\theta^{(\alpha,\beta)}(a) \mid = 1$. □

Corollary 4.2.4. *Let S be a regular Γ-semigroup and $\alpha, \beta, \theta \in \Gamma$. Then the following conditions are equivalent.*

(1) *If $e \in E_\alpha(S), f \in E_\beta(S)$ such that $e\mathcal{D}f$ then $\mid S_\theta^{(\alpha,\beta)}(e, f) \mid = 1$.*

(2) *For any $x, y \in S$,*

(2.1) *if $e\theta x = e\theta y = e\theta f\mathcal{L}x\mathcal{L}y$ and $x, y \leqslant f$ then $x = y$,*

(2.2) *if $x\theta f = y\theta f = e\theta f\mathcal{R}x\mathcal{R}y$ and $x, y \leqslant e$ then $x = y$.*

Proof. (1) $\Rightarrow$ (2) Let $e \in E_\alpha(S), f \in E_\beta(S)$ be such that $e\mathcal{D}f$ and $x, y \in S$, $e\theta x = e\theta y = e\theta f \mathcal{L} x \mathcal{L} y$ and $x, y \leqslant f$. By Lemma 4.1.12, we have $x, y \in S_\theta^{(\alpha,\beta)}(e,f)\theta f$. Then there exist $p, q \in S_\theta^{(\alpha,\beta)}(e,f)$ such that $x = p\theta f$ and $y = q\theta f$. By assumption, $p = q$ which implies that $x = p\theta f = q\theta f = y$. Similarly, we can show that if $x\theta f = y\theta f = e\theta f \mathcal{R} x \mathcal{R} y$ and $x, y \leqslant e$. Then $x = y$.

(2) $\Rightarrow$ (1) Let $a \in S$. Set $e := a'\beta a$ and $f := a\alpha a'$. Then it is easy to show that $e\mathcal{D}f$. Claim that $\mid S_\theta^{(\alpha,\beta)}(e,f)\theta f \mid = 1$ and $\mid e\theta S_\theta^{(\alpha,\beta)}(e,f) \mid = 1$. Let $x, y \in S_\theta^{(\alpha,\beta)}(e,f)\theta f$. Then $e\theta x = e\theta f \mathcal{L} x \leqslant f$ and $e\theta y = e\theta f \mathcal{L} y \leqslant f$. Thus $e\theta x = e\theta y = e\theta f \mathcal{L} x \mathcal{L} y$ and $x, y \leqslant f$. By assumption, we have $x = y$. Hence $S_\theta^{(\alpha,\beta)}(e,f)\theta f$ is trivial. Similarly, we can show that $e\theta S_\theta^{(\alpha,\beta)}(e,f)$ is trivial. By Theorem 4.1.14, $S_\theta^{(\alpha,\beta)}(e,f) \cong_\theta S_\theta^{(\alpha,\beta)}(e,f)\theta f \times e\theta S_\theta^{(\alpha,\beta)}(e,f)$. Thus $\mid S_\theta^{(\alpha,\beta)}(e,f) \mid = 1$. □

Corollary 4.2.5. *Let S be a regular Γ-semigroup, $\alpha, \beta, \theta \in \Gamma$, $e \in E_\alpha(S)$ and $f \in E_\beta(S)$. Then the following conditions are equivalent.*

(1) *$S_\theta^{(\alpha,\beta)}(e,f)$ is a right θ-zero semigroup.*

(2) $\mid S_\theta^{(\alpha,\beta)}(e,f)\theta f \mid = 1$.

(3) $\mid S_\beta^{(\alpha,\beta)}(x\beta e\theta f, f) \mid = 1$ *for any $x \in V_\alpha^\beta(e\theta f)$.*

Proof. (1) $\Rightarrow$ (2) Let $x, y \in S_\theta^{(\alpha,\beta)}(e,f)\theta f$. Then $x = s\theta f$ and $y = t\theta f$ for some $s, t \in S_\theta^{(\alpha,\beta)}(e,f)$. By assumption, we have that

$$x = s\theta f = s\alpha e\theta f = s\alpha e\theta t\theta f = s\theta t\theta f = t\theta f = y.$$

Thus $\mid S_\theta^{(\alpha,\beta)}(e,f)\theta f \mid = 1$.

(2) $\Rightarrow$ (1) Suppose that $S_\theta^{(\alpha,\beta)}(e,f)\theta f$ is trivial. Let $x, y \in S_\theta^{(\alpha,\beta)}(e,f)$. Then $x\theta f = y\theta f$ which implies that $x\theta y = x\theta f\beta y = y\theta f\beta y = y$. Therefore $S_\theta^{(\alpha,\beta)}(e,f)$ is a right θ-zero semigroup.

(2) $\Leftrightarrow$ (3) It follows from Lemma 4.1.12. □

4.3 The Finest Primitive Congruence on Regular Γ-semigroups

A regular Γ-semigroup S is called a **locally inverse Γ-semigroup** if $e\Gamma S\Gamma e$ is an inverse Γ-semigroup for every $e \in E(S)$.

Theorem 4.3.1. *Let S be a regular Γ-semigroup. Then the following statements are equivalent.*

(1) *S is a locally inverse Γ-semigroup.*

(2) *$\leqslant$ is compatible.*

(3) $| S_\theta^{(\alpha,\beta)}(e,f) |= 1$ *for all* $\alpha,\beta,\theta \in \Gamma, e \in E_\alpha(S), f \in E_\beta(S)$.

Proof. (1) $\Rightarrow$ (2) Let $a \leqslant b, c \in S, \theta \in \Gamma$. Then $R_a \leqslant R_b$ and there exist $\beta \in \Gamma, f \in E_\beta(S) \cap R_a$ such that $a = f\beta b$. By Lemma 3.1.2(1), there exist $\alpha \in \Gamma, a' \in V_\alpha^\beta(a)$ such that $a\alpha a' = f$. Choose $c' \in V_\gamma^\delta(c)$ for some $\gamma, \delta \in \Gamma$. By Proposition 4.1.1, we get that $g \in S_\theta^{(\alpha,\delta)}(a'\beta a, c\gamma c')$. By Theorem 4.1.8, $c'\delta g\alpha a' \in V_\gamma^\beta(a\theta c)$ which implies that $a\theta c = a\theta c\gamma c'\delta g\alpha a'\beta a\theta c$. Set $h := a\theta c\gamma c'\delta g\alpha a'$. Clearly, $h \in R_{a\theta c}$ and

$$h\beta h = (a\theta c\gamma c'\delta g\alpha a')\beta(a\theta c\gamma c'\delta g\alpha a') = a\theta c\gamma c'\delta g\alpha a' = h \in E_\beta(S).$$

Then $h \in E_\beta(S) \cap R_{a\theta c}$. Consider

$$\begin{aligned} h\beta(b\theta c) &= a\theta c\gamma c'\delta g\alpha a'\beta b\theta c \\ &= a\theta g\alpha a'\beta b\theta c \\ &= a\theta g\alpha a'\beta a\alpha a'\beta b\theta c \\ &= a\theta g\alpha a'\beta f\beta b\theta c \\ &= a\theta g\alpha a'\beta a\theta c \\ &= a\theta g\theta c \\ &= a\alpha a'\beta a\theta g\theta c\gamma c'\delta c \\ &= a\alpha a'\beta a\theta c\gamma c'\delta c \\ &= a\theta c. \end{aligned}$$

Next, we will show that $R_{a\theta c} \leqslant R_{b\theta c}$. If $a = b$ then $R_{a\theta c} \leqslant R_{b\theta c}$.

Suppose $a \in b\Gamma S$ then there exist $u \in S, \gamma_1 \in \Gamma$ such that $a = b\gamma_1 u$. Let $b' \in V_{\alpha_2}^{\beta_2}(b)$ for some $\alpha_2, \beta_2 \in \Gamma$. Then

$$\begin{aligned}(b'\beta_2 a)\alpha_2(b'\beta_2 a) &= b'\beta_2 f\beta b\alpha_2 b'\beta_2 b\gamma_1 u \\ &= b'\beta_2 f\beta b\gamma_1 u \\ &= b'\beta_2 f\beta a \\ &= b'\beta_2 f\beta f\beta b \\ &= b'\beta_2 f\beta b \\ &= b'\beta a \in E_{\alpha_2}(S).\end{aligned}$$

Thus $b'\beta a \in E_{\alpha_2}(S) \cap R_{b'\beta_2 a}$. It follows that

$$b'\beta_2 b\alpha_2 b'\beta_2 a = b'\beta_2 a$$

and

$$b'\beta_2 a\alpha_2 b'\beta_2 b = b'\beta_2 f\beta b\alpha_2 b'\beta_2 b = b'\beta_2 f\beta b = b'\beta_2 a$$

which implies that $b'\beta_2 a \leqslant b'\beta_2 b$. Consider

$$a = b\gamma_1 u = b\alpha_2 b'\beta_2 b\gamma_1 u = b\alpha_2 b'\beta_2 a,$$

so $a\mathcal{L}b'\beta_2 a$. Thus $b'\beta_2 a \in E_{\alpha_2}(S) \cap L_a$. By Lemma 3.1.2(2), there exist $a'' \in V_{\alpha_2}^{\beta_3}(a), \beta_3 \in \Gamma$ such that $a''\beta_3 a = b'\beta_2 a$. Now, $a''\beta_3 a = b'\beta_2 a \leqslant b'\beta_2 b$. Thus $a = b\alpha_2 b'\beta_2 a = b\alpha_2 a''\beta_3 a$. Let $p \in S_\theta^{(\alpha_2,\delta)}(a''\beta_3 a, c\gamma c')$. Note that

$$(a''\beta_3 a\theta p)\alpha_2(a''\beta_3 a\theta p) = a''\beta_3 a\theta p\theta p = a''\beta_3 a\theta p.$$

Thus

$$\begin{aligned}(b'\beta_2 b\theta p)\alpha_2(b'\beta_2 b\theta p) &= b'\beta_2 b\theta p\alpha_2 a''\beta_3 a\alpha_2 b'\beta_2 b\theta p \\ &= b'\beta_2 b\theta p\alpha_2 a''\beta_3 b\alpha_2 b'\beta_2 a\alpha_2 b'\beta_2 b\theta p \\ &= b'\beta_2 b\theta p\alpha_2 a''\beta_3 b\alpha_2 b'\beta_2 a\theta p \\ &= b'\beta_2 b\theta p\alpha_2 a''\beta_3 a\theta p \\ &= b'\beta_2 b\theta p.\end{aligned}$$

So $a''\beta_3 a\theta p, b'\beta_2 b\theta p \in E_{\alpha_2}(S)$. Now,

$$\begin{aligned} a''\beta_3 a\theta p &= a''\beta_3 a\theta p\alpha_2 a''\beta_3 a \\ &= b'\beta_2 a\theta p\alpha_2 a''\beta_3 f\beta b \\ &= b'\beta_2 b\alpha_2 a''\beta_3 a\theta p\alpha_2 a''\beta_3 f\beta b\alpha_2 b'\beta_2 b \\ &= b'\beta_2 b\alpha_2 a''\beta_3 a\theta p\alpha_2 a''\beta_3 a\alpha_2 b'\beta_2 b \in (b'\beta_2 b)\Gamma S\Gamma(b'\beta_2 b) \end{aligned}$$

and $a''\beta_3 a\theta p$ and $b'\beta_2 b\theta p$ are α_2-idempotents within the inverse Γ-semigroup $(b'\beta_2 b)\Gamma S\Gamma(b'\beta_2 b)$. Thus

$$\begin{aligned} a''\beta_3 a\theta p &= a''\beta_3 a\theta p\theta p \\ &= a''\beta_3 a\theta p\alpha_2 a''\beta_3 a\theta p \\ &= a''\beta_3 a\theta p\alpha_2 a''\beta_3 a\alpha_2 b'\beta_2 b\theta p \\ &= (a''\beta_3 a\theta p)\alpha_2(b'\beta_2 b\theta p) \\ &= (b'\beta_2 b\theta p)\alpha_2(a''\beta_3 a\theta p) \\ &= b'\beta_2 b\theta p. \end{aligned}$$

Set $q := c'\delta p\alpha_2 a''\beta_3 a\theta c$. Then

$$\begin{aligned} (b\theta c)\gamma q &= b\theta c\gamma c'\delta p\alpha_2 a''\beta_3 a\theta c \\ &= b\theta p\alpha_2 a''\beta_3 a\theta c \\ &= b\theta p\theta c \\ &= b\alpha_2 b'\beta_2 b\theta p\theta c \\ &= b\alpha_2 a''\beta_3 a\theta p\theta c \\ &= a\theta p\theta c \\ &= a\alpha_2 a''\beta_3 a\theta p\theta c\gamma c'\delta c \\ &= a\alpha_2 a''\beta_3 a\theta c\gamma c'\delta c \\ &= a\theta c. \end{aligned}$$

Thus $a\theta c \in (b\theta c)\Gamma S \subseteq (b\theta c)\Gamma S \cup \{b\theta c\}$. Hence $R_{a\theta c} \leqslant R_{b\theta c}$.

(2) $\Rightarrow$ (3) Let $g, h \in S_\theta^{(\alpha,\beta)}(e, f)$ where $\alpha, \beta \in \Gamma, e \in E_\alpha(S), f \in E_\beta(S)$.

Then $f\beta g = g$ and

$$(g\theta f)\beta(g\theta f) = g\theta(f\beta g)\theta f = g\theta g\theta f = g\theta f,$$

$$f\beta(g\theta f) = g\theta f$$

and

$$(g\theta f)\beta f = g\theta f.$$

which implies that $g\theta f \in E_\beta(S)$. By Theorem 3.2.1, $g\theta f \leqslant f$. Similarly, we can show that $e\theta g \leqslant e$. By assumption, we have

$$g\theta h = g\theta(f\beta h) = (g\theta f)\beta h \leqslant f\beta h = h$$

and

$$h\theta g = (h\alpha e)\theta g = h\alpha(e\theta g) \leqslant h\alpha e = h.$$

Since $S_\theta^{(\alpha,\beta)}(e,f)$ is a sub θ-semigroup, we have $g\theta h, h\theta g \in S_\theta^{(\alpha,\beta)}(e,f)$ which implies that $g\theta h, h\theta g \in E_\theta(S)$. Since $g\theta h \leqslant h$, we have $g\theta h = (g\theta h)\theta h = h\theta(g\theta h)$ and since $h\theta g \leqslant h$, we have $h\theta g = (h\theta g)\theta h = h\theta(h\theta g)$ which implies that $h\theta g = (h\theta g)\theta h = h\theta(g\theta h) = g\theta h$. Consider

$$\begin{aligned} g &= g\alpha e\theta f\beta g = g\theta h\theta g = g\theta(g\theta h) = g\theta h = h\theta g \\ &= h\theta(h\theta g) = h\theta(g\theta h) = h\alpha e\theta g\theta f\beta h = h\alpha e\theta f\beta h \\ &= h. \end{aligned}$$

Hence $| S_\theta^{(\alpha,\beta)}(e,f) |= 1$.

(3) $\Rightarrow$ (1) Let $e \in E_\alpha(S)$ for some $\alpha \in \Gamma$ and $a \in e\Gamma S\Gamma e, a', a'' \in V_\gamma^\delta(a) \cap e\Gamma S\Gamma e, \gamma, \delta \in \Gamma$. We will show that $a' = a''$. Now, $a = e\gamma_1 x\gamma_2 e$ for some $\gamma_1, \gamma_2 \in \Gamma, x \in S$ and $a'\delta a\gamma a' = a' = e\beta_1 y\beta_2 e$ for some $\beta_1, \beta_2 \in \Gamma, y \in S$, Then

$$a'\delta a = a'\delta a\gamma a'\delta a,$$

$$a'\delta a = e\beta_1 y\beta_2 e\delta a = e\alpha e\beta_1 y\beta_2 e\delta a = e\alpha a'\delta a$$

and

$$a'\delta a\gamma e = a'\delta a\gamma a'\delta a\gamma e,$$

so $a'\delta a \in S_{\gamma}^{(\gamma,\alpha)}(a'\delta a, e)$. By assumption, $a'\delta a$ is the only one element in $S_{\gamma}^{(\gamma,\alpha)}(a'\delta a, e)$. Similarly, we can show that $a''\delta a \in S_{\gamma}^{(\gamma,\alpha)}(a''\delta a, e)$. Consider $a''\delta a = a''\delta a\gamma a'\delta a$ and $a'\delta a = a'\delta a\gamma a''\delta a$ then $(a''\delta a)\mathcal{L}(a'\delta a)$. By Proposition 4.1.3, $S_{\gamma}^{(\gamma,\alpha)}(a'\delta a, e) = S_{\gamma}^{(\gamma,\alpha)}(a''\delta a, e)$ which implies that $a''\delta a = a'\delta a$.

Similarly, we can show that $S_{\delta}^{(\alpha,\delta)}(e, a\gamma a') = S_{\delta}^{(\alpha,\delta)}(e, a\gamma a'')$ and we have $a\gamma a' = a\gamma a''$. Now,

$$a'' = a''\delta a\gamma a'' = a'\delta a\gamma a'' = a'\delta a\gamma a' = a'.$$

Hence $e\Gamma S\Gamma e$ is an inverse Γ-semigroup. □

Let $\phi : S \to S'$. Define the relation $\leqslant_{S'}$ on S' by for all $x', y' \in S'$

$$x' \leqslant_{S'} y' \Leftrightarrow x' = e'\alpha y' = y'\beta f' \text{ for some } e' \in E_{\alpha}(S\phi), f' \in E_{\beta}(S\phi), \alpha, \beta \in \Gamma.$$

Then $\leqslant_{S'}$ is a natural partial order on S'.

Lemma 4.3.2. *Let $\phi : S \to S'$ be a homomorphism of regular Γ-semigroups. Then ϕ preserves the relation $\preccurlyeq$ of $E(S)$ and $E(S')$.*

Proof. Let $e \in E_{\alpha}(S)$ and $f \in E_{\beta}(S)$ be such that $e \preccurlyeq f$. Then $e = e\alpha f = f\beta e$. Consider

$$e\phi = (e\alpha f)\phi = (e\phi)\alpha(f\phi)$$

and

$$e\phi = (f\beta e)\phi = (f\phi)\beta(e\phi).$$

Clearly, $e\phi \in E_{\alpha}(S')$ and $f\phi \in E_{\beta}(S')$, which implies that $e\phi \preccurlyeq f\phi$. Thus $\preccurlyeq$ preserves of $E(S)$ and $E(S')$. □

Lemma 4.3.3. *Let $\phi : S \to S'$ be a homomorphism of regular Γ-semigroups and $a, b \in S'$. Then the following statements are equivalent.*

(1) $a \leqslant_{S'} b$.

(2) *If $f' \in E(S\phi) \cap R_b$ then $e' \preccurlyeq f'$ and $a = e'\alpha b$ for some $\alpha \in \Gamma, e' \in E_\alpha(S\phi) \cap R_a$.*

(3) *If $f' \in E(S\phi) \cap L_b$ then $e' \preccurlyeq f'$ and $a = b\alpha e'$ for some $\alpha, \in \Gamma, e' \in E_\alpha(S\phi) \cap L_a$.*

Proof. (1) $\Rightarrow$ (2) Let $f' \in E(S\phi) \cap R_b$. Then there exists $\beta \in \Gamma$ such that $f' \in E_\beta(S')$. By assumption, $a = h'\gamma b$ for some $h' \in E_\gamma(S\phi) \cap R_a, \gamma \in \Gamma$ and $R_a \leqslant R_b$. Then $R_{h'} = R_a \leqslant R_b = R_{f'}$. Since $h' \in R_{f'}$, we have $h' = f'\beta h'$ and $h'\gamma f' \in E_\beta(S')$. Set $e' := h'\gamma f'$. Then $h' = h'\gamma f'\beta h' = e'\beta h'$, so $e'\mathcal{R}h'$ which implies that $e'\mathcal{R}a$. Then

$$a = h'\gamma b = h'\gamma f'\beta b = e'\beta b$$

and

$$e' = h'\gamma f' = e'\beta f', \; e' = f'\beta h'\gamma f' = f'\beta e'.$$

Thus $e' \preccurlyeq f'$.

(2) $\Rightarrow$ (3) Let $f' \in E(S\phi) \cap L_b$. Then $f' \in E_\beta(S\phi)$ for some $\beta \in \Gamma$. By Lemma 3.1.2(2), there exist $\gamma \in \Gamma, b' \in V_\beta^\gamma(b)$ such that $f' = b'\gamma b$. Clearly, $b\beta b' \in E_\gamma(R_b)$. Set $k' := b\beta b'$. By assumption, there exist $\delta \in \Gamma, e \in E_\delta(R_a)$ such that $e \preccurlyeq k'$ and $a = e\delta b$. Set $e' := b'\gamma e\delta b$. Then $e' \in E_\beta(S)$, $e' = b'\gamma e\delta b = b'\gamma a$ and

$$a = e\delta b = k'\gamma e\delta b = b\beta b'\gamma e\delta b = b\beta e'.$$

Thus $e' \in E_\beta(L_a)$. Consider

$$e' = b'\gamma e\delta b = b'\gamma b\beta b'\gamma e\delta b = b'\gamma b\beta e' = f'\beta e'$$

and

$$e' = b'\gamma e\delta b = b'\gamma e\delta b\alpha b'\gamma b = e'\beta f'.$$

Therefore $e' \preccurlyeq f'$.

(3) $\Rightarrow$ (1) Let $b' \in V_\gamma^\delta(b)$ for some $\gamma, \delta \in \Gamma$. Clearly, $b'\delta b \in E_\gamma(L_b)$. By assumption, there exist $\alpha \in \Gamma, e' \in E_\alpha(L_a)$ such that $e' \preccurlyeq b'\delta b$ and $a = b\alpha e'$. Then $b\alpha e'\alpha b' \in E_\delta(S')$. Set $f := b\alpha e'\alpha b'$. Thus

$$f\delta b = b\alpha e'\alpha b'\delta b = b\alpha e' = a.$$

By Theorem 3.2.1, we have $a \leqslant_{S'} b$. □

The next theory, we use Lemma 4.3.3 to prove Theorem.

Theorem 4.3.4. *Let $\phi : S \to S'$ be a homomorphism of regular Γ-semigroups. Then ϕ reflects natural partial orders of S and S'.*

Proof. Let $u, v \in S\phi$ with $u \leqslant_{S'} v$ and let $y \in S$ with $y\phi = v$. We want to find $x \in S$ such that $x \leqslant y$ and $x\phi = u$.

Since $y \in S$, we can choose $f \in E_\beta(S) \cap R_y$ for some $\beta \in \Gamma$. Since $f\mathcal{R}y$, there exist $b \in S, \theta \in \Gamma$ such that $f = y\theta b$ and $y = f\beta y$. Thus

$$v = y\phi = (f\beta y)\phi = (f\phi)\beta(y\phi) = (f\phi)\beta v$$

and

$$f\phi = (y\theta b)\phi = (y\phi)\theta(b\phi) = v\theta(b\phi).$$

Therefore $(f\phi)\mathcal{R}v$ which implies that $f\phi \in E_\beta(R_v)$. Set $f' := f\phi$. By Lemma 4.3.3 (2), there exist $e' \in E_\alpha(S\phi) \cap R_u, \alpha \in \Gamma$ such $e' \preccurlyeq f'$ and $u = e'\alpha v$. Then there exists $e \in E_\alpha(S)$ such that $e\phi = e'$. Choose $h \in S_\beta^{(\alpha,\beta)}(e, f)$ and $g \in S_\beta^{(\beta,\alpha)}(f, e)$. Thus $h\phi \in S_\beta^{(\alpha,\beta)}(e\phi, f\phi) = S_\beta^{(\alpha,\beta)}(e', f')$ and $g\phi \in S_\beta^{(\beta,\alpha)}(f', e')$. Since $e' \preccurlyeq f'$, we get that $e' \in E_\beta(S\phi)$. Then

$$e' = e'\beta f' = e'\beta(h\phi)\beta f'$$

and

$$(h\phi)\beta f' = (h\phi)\alpha e'\beta f' = (h\phi)\alpha e'$$

which implies that $e'\mathcal{L}((h\phi)\beta f')$. Now,

$$(h\phi)\beta f' = (h\beta f)\phi$$

and

$$(h\phi)\beta f' = (h\phi)\alpha e' = h\phi.$$

Dually, we have that

$$e' = f'\beta e' = f'\beta(g\phi)\beta e'$$

and

$$f'\beta(g\phi) = f'\beta e'\alpha(g\phi) = e'\alpha(g\phi)$$

which implies that $e'\mathcal{R}(f'\beta(g\phi))$. Consider

$$f'\beta(g\phi) = (f\phi)\beta(g\phi) = (f\beta g)\phi$$

and

$$f'\beta(g\phi) = e'\alpha(g\phi) = g\phi.$$

By Proposition 4.1.3(3) and 4.1.4, we have that

$$S_{\beta}^{(\beta,\beta)}((h\beta f)\phi, (f\beta g)\phi) = S_{\beta}^{(\beta,\beta)}(e', e') = \{e'\}.$$

Thus, if $k \in S_{\beta}^{(\beta,\beta)}(h\beta f, f\beta g)$ then $f\beta k = f\beta g\beta k = k$ and $k\beta f = k\beta h\beta f = k$. It implies that $k \preccurlyeq f$. Now, we get that $k\phi = e'$. Then $k = k\beta f = k\beta y\theta b$, so $k \in E_{\beta}(R_{k\beta y})$. By Lemma 4.3.3, $k\beta y \leqslant_S y$. Set $x := k\beta y$.

Therefore $x\phi = (k\beta y)\phi = (k\phi)\beta(y\phi) = u.$ □

Proposition 4.3.5. *Let S be a regular Γ-semigroup, $\alpha, \beta, \theta \in \Gamma, e \in E_{\alpha}(S)$ and $f \in E_{\beta}(S)$. If ρ is a congruence on a regular Γ-semigroup S and $h \in S_{\theta}^{(\alpha,\beta)}(e, f)$ then $h\rho \in S_{\theta}^{(\alpha,\beta)}(e\rho, f\rho)$.*

Proof. It is obvious. □

Proposition 4.3.6. *Let ρ be a congruence on a regular Γ-semigroup S and $a \in S$. If $a\rho \in E(S/\rho)$ then $e\rho = a\rho$ for some $e \in E(S)$. Moreover, $H_e \leqslant H_a$.*

Proof. Let $a\rho \in E(S/\rho)$. Then $a\rho \in E_\theta(S/\rho)$ for some $\theta \in \Gamma$. Thus $a\rho = (a\rho)\theta(a\rho) = (a\theta a)\rho$. Let $x \in V_\alpha^\beta(a\theta a)$ for some $\alpha, \beta \in \Gamma$. Set $e := a\alpha x\beta a$. Then

$$e\theta e = (a\alpha x\beta a)\theta(a\alpha x\beta a) = a\alpha(x\beta(a\theta a)\alpha x)\beta a = a\alpha x\beta a = e$$

which implies that $e \in E_\theta(S)$. Thus

$$\begin{aligned} a\rho &= (a\theta a)\rho \\ &= ((a\theta a)\alpha x\beta(a\theta a))\rho \\ &= ((a\theta a)\rho)\alpha(x\rho)\beta((a\theta a)\rho) \\ &= (a\rho)\alpha(x\rho)\beta(a\rho) \\ &= (a\alpha x\beta a)\rho \\ &= e\rho. \end{aligned}$$

Clearly, $e = a\alpha x\beta a \in S\Gamma a$ and $e = a\alpha x\beta a \in a\Gamma S$ implies that $R_e \leqslant R_a$ and $L_e \leqslant L_a$. □

Proposition 4.3.7. *Assume that ρ be a congruence on a regular Γ-semigroup S. Let $a, b \in S$, $\alpha, \beta, \gamma, \delta, \theta \in \Gamma$ and $a' \in V_\alpha^\beta(a), b' \in V_\gamma^\delta(b)$. If $a\rho \in E_\theta(S/\rho)$ then $S_\theta^{(\alpha,\beta)}(a) \subseteq a\rho$.*

Proof. Note that $S_\theta^{(\alpha,\beta)}(a) = S_\theta^{(\alpha,\beta)}(a'\beta a, a\alpha a')$. By Proposition 4.2.1, $S_\theta^{(\alpha,\beta)}(a) = a\alpha V_\alpha^\beta(a\theta a)\beta a$. By assumption, $a\rho = (a\rho)\theta(a\rho) = (a\theta a)\rho$.
Consider $x \in a\alpha V_\alpha^\beta(a\theta a)\beta a$, we have $x = a\alpha y\beta a$ for some $y \in V_\alpha^\beta(a\theta a)$. Thus

$x = a\alpha y\beta(a\theta a)\alpha y\beta a = x\theta x$. Consider

$$\begin{aligned}
a\rho &= (a\theta a)\rho \\
&= ((a\alpha a'\beta a)\theta(a\alpha a'\beta a))\rho \\
&= (a\rho)\alpha((a'\beta a\theta a\alpha a')\rho)\beta(a\rho) \\
&= (a\rho)\alpha((a'\beta a\theta x\theta a\alpha a')\rho)\beta(a\rho) \\
&= ((a\alpha a'\beta a)\rho)\theta(x\rho)\theta((a\alpha a'\beta a)\rho) \\
&= (a\rho)\theta(x\rho)\theta(a\rho) \\
&= (a\rho)\theta((a\alpha y\beta a)\rho)\theta(a\rho) \\
&= ((a\theta a)\rho)\alpha((y\beta a)\rho)\theta(a\rho) \\
&= (a\rho)\alpha((y\beta a)\rho)\theta(a\rho) \\
&= ((a\alpha y\beta a)\rho)\theta(a\rho) \\
&= (x\rho)\theta(a\rho) \\
&= ((x\theta x)\rho)\theta(a\rho) \\
&= (x\rho)\theta(x\rho)\theta(a\rho) \\
&= (x\rho)\theta((a\alpha y\beta a)\rho)\theta(a\rho) \\
&= (x\rho)\theta((a\alpha y)\rho)\beta((a\theta a)\rho) \\
&= (x\rho)\theta((a\alpha y)\rho)\beta(a\rho) \\
&= (x\rho)\theta((a\alpha y\beta a)\rho) \\
&= (x\rho)\theta(x\rho) \\
&= (x\theta x)\rho \\
&= x\rho.
\end{aligned}$$

Thus $x \in a\rho$. Hence $S_\theta^{(\alpha,\beta)}(a) \subseteq a\rho$. □

Let a and b be elements of a regular Γ-semigroup S. Define

$$I(a) := S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S \cup \{a\}$$

and

$$J_a \leqslant J_b \text{ if and only if } I(a) \subseteq I(b).$$

Theorem 4.3.8. *Let S be a Γ-semigroup without zero. Then the following conditions are equivalent.*

(1) *S is completely simple.*

(2) *S is regular and every idempotent is primitive.*

Proof. (1) $\Rightarrow$ (2) By Theorem 2.2.7, Corollary 2.2.9 and Theorem 2.2.12, a completely simple Γ-semigroup S is regular.

(2) $\Rightarrow$ (1) We will show that S is simple. Since S is regular, every $\mathcal{D}$-class (and so certainly every $\mathcal{J}$-class) contains an idempotent. Let $e \in E(S)$. Then there exists $\alpha \in \Gamma$ such that $e \in E_\alpha(S)$. We will show that J_e is a minimal J-class.

Suppose that $J_f \leqslant J_e$ where f is another idempotent. Then $f \in E_\beta(S)$ for some $\beta \in \Gamma$ and $f \in I(e)$.

Case 1. $f \in S\Gamma e$. Then $f = x\delta e$ for some $x \in S, \delta \in \Gamma$. Set $g := e\beta f\beta x\delta e$. Thus

$$\begin{aligned} g\alpha g &= (e\beta f\beta x\delta e)\alpha(e\beta f\beta x\delta e) \\ &= e\beta f\beta x\delta e\beta f\beta x\delta e \\ &= e\beta f\beta f\beta f\beta x\delta e \\ &= e\beta f\beta x\delta e \\ &= g \end{aligned}$$

and $g\alpha e = e\alpha g = g$, so $g \preccurlyeq e$. Since e is primitive, we have $g = e$. We now have $e = e\beta f\beta x\delta e \in S\Gamma f\Gamma S \subseteq I(f)$. It follows that $I(e) \subseteq I(f)$.

Case 2. $f \in e\Gamma S$. It is similar to the proof of Case 1.

Case 3. $f \in S\Gamma e\Gamma S$. Then $f = x\gamma e\delta y$ for some $x, y \in S, \gamma, \delta \in \Gamma$.

Set $g := e\delta y\beta f\beta x\gamma e$. Thus

$$\begin{aligned} g\alpha g &= (e\delta y\beta f\beta x\gamma e)\alpha(e\delta y\beta f\beta x\gamma e) \\ &= e\delta y\beta f\beta x\gamma e\delta y\beta f\beta x\gamma e \\ &= d\delta y\beta f\beta f\beta f\beta x\gamma e \\ &= e\delta y\beta f\beta x\gamma e \\ &= g \end{aligned}$$

and $g\alpha e = e\alpha g = g$, which implies that $g \preccurlyeq e$. Since e is primitive, we have $g = e$. Now, we have that $e = e\delta y\beta f\beta x\gamma e \in S\Gamma f\Gamma S \subseteq I(f)$. Thus $I(e) \subseteq I(f)$.

By case (1)-(3), we have $J_e \leqslant J_f$. Thus J_e is a minimal $\mathcal{J}$-class.

We will show that $J_e = S$. Let $a \in S$. Since S is regular, there exist $\alpha, \beta \in \Gamma, x \in S$ such that $a = a\alpha x\beta a$. Then $h := a\alpha x \in E_\beta(S)$. It implies that $a \in h\Gamma S \subseteq I(h)$, so $a \in J_h$. Note that $J_e \leqslant J_h$. Set $e := x\gamma h\delta y$ for some $x, y \in S, \gamma, \delta \in \Gamma$.

Let $g := h\delta y\alpha e\alpha x\gamma h$. Thus $g\beta g = g$ and $h\beta g = g = g\beta h$ which implies that $g \preccurlyeq h$. Since every idempotent is primitive, we get that $g = h$. Therefore $h = h\delta y\alpha e\alpha x\gamma h \in S\Gamma e\Gamma S$, so $J_e = J_h$. Hence $a \in J_e$. Therefore $S = J_e$.

Next, we will show that $S = S\Gamma a\Gamma S$ for all $a \in S$. Let $a \in S$. Then $a \in J_e$ which implies that $a\mathcal{J}e$. For $x \in S$ and S is regular, we have $x \in J_e$. Then $x\mathcal{J}a$. Hence $S \subseteq S\Gamma a\Gamma S$. Therefore $S = S\Gamma a\Gamma S$. □

Theorem 4.3.9. *A regular Γ-semigroup S without zero is completely simple if and only if the natural partial order on S is the identity relation.*

Proof. Assume that $x \leqslant y$. Since S is regular, there exist $\alpha, \beta \in \Gamma$ such that $y' \in V_\alpha^\beta(y)$. Set $f := y\alpha y' \in E_\beta(S)$. Then $f\mathcal{R}y$. By Remark 1, there exists $g \in E_\beta(R_x)$ such that $g \preccurlyeq f$ and $x = g\beta y$. Since S is completely simple, we have that $f = g$. Therefore $x = g\beta y = f\beta y = y\alpha y'\beta y = y$.

Conversely, assume that the natural partial order on S is the identity relation. Since S is regular, it contains an idempotent e. Suppose that $f \in E(S)$ such that $f \preccurlyeq e$. By assumption, we have $f = e$. By Theorem 4.3.8, S is completely simple. □

Theorem 4.3.10. *Let ρ be a congruence on a regular Γ-semigroup S. Then ρ is strictly compatible if and only if $e\rho$ is a completely simple sub Γ-semigroup of S for all $e \in E(S)$.*

Proof. Assume that ρ is strictly compatible and $e \in E(S)$. Then there exists $\alpha \in \Gamma$ such that $e \in E_\alpha(S)$. If $\rho^* : S \to S/\rho$ is the canonical homomorphism, $x \in e\rho, f \in E_\beta(R_x)$ and $g \in E_\gamma(L_x), \beta, \gamma \in \Gamma$ then $f = x\delta b$ for some $b \in S, \delta \in \Gamma$. Thus

$$f\rho^* = (x\delta b)\rho^* = (x\rho^*)\delta(b\rho^*) = (x\rho)\delta(b\rho) = (e\rho)\delta(b\rho) = (e\rho^*)\delta(b\rho^*)$$

and

$$e\rho^* = e\rho = x\rho = x\rho^* = (f\beta x)\rho^* = (f\rho^*)\beta(x\rho^*).$$

So $f\rho^*\mathcal{R}e\rho^*$. Similarly, we can show that $e\rho^*\mathcal{L}g\rho^*$.
By Proposition 4.1.3(3), $S_\alpha^{(\gamma,\beta)}(g\rho^*, f\rho^*) = S_\alpha^{(\alpha,\alpha)}(e\rho^*, e\rho^*) = \{e\rho^*\}$.
Next, we will show that $S_\alpha^{(\gamma,\beta)}(g,f)\rho^* \subseteq S_\alpha^{(\gamma,\beta)}(g\rho^*, f\rho^*)$.

Let $a \in S_\alpha^{(\gamma,\beta)}(g,f)\rho^*$. Then $a = p\rho^*$ for some $p \in S_\alpha^{(\gamma,\beta)}(g,f)$. Thus

$$a\gamma(g\rho^*) = (p\rho^*)\gamma(g\rho^*) = (p\gamma g)\rho^* = p\rho^* = a$$

and

$$g\rho^*\alpha f\rho^* = (g\alpha f)\rho^* = (g\alpha p\alpha f)\rho^* = g\rho^*\alpha p\rho^*\alpha f\rho^* = g\rho^*\alpha a\alpha f\rho^*$$

which implies that $a \in S_\alpha^{(\gamma,\beta)}(g\rho^*, f\rho^*)$. Hence $S_\alpha^{(\gamma,\beta)}(g,f)\rho^* \subseteq S_\alpha^{(\gamma,\beta)}(g\rho^*, f\rho^*)$.
Clearly, $S_\alpha^{(\gamma,\beta)}(g,f) \subseteq e\rho$. Therefore, if $h \in S_\alpha^{(\gamma,\beta)}(g,f)$ then

$$(x\alpha h)\rho = (x\alpha h)\rho^* = (x\rho^*)\alpha(h\rho^*) = x\rho\alpha e\rho^* = e\rho\alpha e\rho^* = e\rho^*\alpha e\rho^* = e\rho^* = e\rho.$$

It follows that $x\alpha h \in e\rho$. Similarly, we can show that $h\alpha x \in e\rho$.
Next, we will show that $x\alpha h \leqslant x$ and $h\alpha x \leqslant x$. Since $g\mathcal{L}x$, there exist $c \in S, \theta \in \Gamma$

such that $g = c\theta x$. Now,

$$x\alpha h = x\alpha h\gamma g = x\alpha h\gamma c\theta x,$$

$$(x\alpha h\gamma c)\theta(x\alpha h\gamma c) = x\alpha h\gamma g\alpha h\gamma c = x\alpha h\gamma c,$$

$$h\alpha x = f\beta h\alpha x = x\delta(b\beta h\alpha x)$$

and

$$(b\beta h\alpha x)\delta(b\beta h\alpha x) = b\beta h\alpha f\beta h\alpha x = b\beta h\alpha x.$$

Then $x\alpha h \leqslant x$ and $h\alpha x \leqslant x$. Since ρ is strictly compatible, we get that $x = x\alpha h$ and $h\alpha x = x$. Thus

$$h = h\gamma g = h\gamma c\theta x = h\gamma c\theta(x\alpha h)$$

and

$$h = f\beta h = x\delta b\delta h = (h\alpha x)\delta(b\beta h)$$

which implies that $x\alpha h\mathcal{L}h$ and $h\mathcal{R}h\alpha x$, so $x \in H_h$. By Theorem 4.3.9, the natural partial order on it is the identity relation. By Theorem 4.3.8, $e\rho$ is completely simple.

Conversely, assume that $e\rho$ is a completely simple sub Γ-semigroup for all $e \in E(S)$ and $(x, y) \in \rho, x \leqslant y$. Since S is regular, there exist $\alpha, \beta \in \Gamma$ such that $y' \in V_\alpha^\beta(y)$. Set $f := y\alpha y' \in E_\beta(S)$. Then $f\mathcal{R}y$. By Remark 1, there exists $g \in E_\beta(R_x)$ such that $g \preccurlyeq f$ and $x = g\beta y$. Thus $g = g\beta f = g\beta y\alpha y' = x\alpha y'$, which implies that

$$g\rho = (x\alpha y')\rho = x\rho\alpha y'\rho = y\rho\alpha y'\rho = (y\alpha y')\rho = f\rho.$$

It implies that $(g, f) \in \rho$. By assumption, $f\rho$ is completely simple. By Theorem 4.3.8, we have that $f = g$. Therefore

$$x = g\beta y = f\beta y = y\alpha y'\beta y = y.$$

□

Let S be a regular Γ-semigroup. A non-empty subset X is called a *directed subset* of S if for all $x, y \in X$ there exists $z \in X$ such that $z \leqslant y$ and $z \leqslant x$.

Define the relation ρ on a regular Γ-semigroup S as follows:

$$\rho := \{(x, y) \in S \times S \mid z \leqslant x \text{ and } z \leqslant y \text{ for some } z \in S\}.$$

Theorem 4.3.11. *Let S be a regular Γ-semigroup. Then the following statements are equivalent.*

(1) *For all $e \in E_\alpha(S), \alpha \in \Gamma$, $(e]$ is directed.*

(2) *ρ is an equivalence relation.*

(3) *ρ is congruence.*

Proof. (1) $\Rightarrow$ (2) Clearly, ρ is reflexive and anti-symmetric. Assume that (x, y), $(y, z) \in \rho$. Then there exist $u_1, u_2 \in S$ such that $u_1 \leqslant x, u_1 \leqslant y$ and $u_2 \leqslant y, u_2 \leqslant z$. Since S is regular, there exist $\alpha, \beta \in \Gamma$ such that $y' \in V_\alpha^\beta(y)$. Set $f := y'\beta y \in E_\alpha(R_y)$. By Remark 1, there exists $e_1 \in E_\alpha(R_{u_1})$ such that $e_1 \preccurlyeq f$ and $u_1 = e_1\alpha y$. Again, by Remark 1, there exists $e_2 \in E_\alpha(R_{u_2})$ such that $e_2 \preccurlyeq f$ and $u_2 = e_2\alpha y$. By assumption and $e_1 \preccurlyeq f, e_2 \preccurlyeq f$, there is $g \in (f]$ such that $g \preccurlyeq e_1$ and $g \preccurlyeq e_2$. Since $e_1\mathcal{R}u_1$, there are $a \in S, \delta \in \Gamma$ such that $e_1 = u_1\delta a$. Thus

$$g\gamma u_1 = e_1\alpha g\gamma u_1 = u_1\delta a\alpha g\gamma u_1$$

where $a\alpha g\gamma u_1 \in E_\delta(S)$ and $u_1\delta a\alpha g \in E_\gamma(S)$ which implies that $g\gamma u_1 \leqslant u_1$. Since $\leqslant$ is transitive and $g\alpha y = g\gamma e_1\alpha y = g\gamma u_1$, we obtain that $g\alpha y \leqslant x$. In a similar way, it can be shown that $g\alpha y \leqslant z$. Thus $(x, z) \in \rho$. Therefore ρ is an equivalence relation.

(2) $\Rightarrow$ (3) We must be shown that ρ is compatible. Let $x \leqslant y$ and $c \in S, \theta \in \Gamma$. Then there exist $\alpha, \beta \in \Gamma$ such that $y' \in V_\alpha^\beta(y)$. Set $f := y\alpha y' \in E_\beta(S) \cap R_y$ and $f' := y'\beta y \in E_\alpha(S) \cap L_y$. By Remark 1, there exist $e \in E_\beta(R_x), e' \in E_\alpha(L_x)$ such that $e \preccurlyeq f$, $x = e\beta y$ and $e' \preccurlyeq f'$, $x = y\alpha e'$. Then

$$e' = f'\alpha e'\alpha f' = y'\beta y\alpha e'\alpha y'\beta y = y'\beta x\alpha y'\beta y = y'\beta e\beta y\alpha y'\beta y = y'\beta e\beta y.$$

Since $c \in S$, there exists $c' \in V_\gamma^\delta(c)$ for some $\gamma, \delta \in \Gamma$. Set $g := c'\delta c \in E_\gamma(S) \cap L_c$. Let $h \in S_\theta^{(\gamma,\beta)}(g, f)$ and $k \in S_\theta^{(\gamma,\beta)}(g, e)$. Then

$$y'\beta h\theta y = y'\beta h\gamma g\theta y = y'\beta h\gamma c'\delta c\theta y$$

and

$$c\theta y = c\gamma g\theta y = c\gamma g\theta f\beta y = c\gamma g\theta h\theta f\beta y = c\gamma g\theta y\alpha y'\beta h\theta y.$$

Thus $y'\beta h\theta y\mathcal{L}c\theta y$. It is clearly that $y'\beta h\theta y \preccurlyeq f'$ and $y'\beta h\theta y \in E_\alpha(S)$. Set $h' := y'\beta h\theta y \in E_\alpha(L_{c\theta y}) \cap (f']$.

Consider

$$\begin{aligned} y'\beta k\theta e\beta y\alpha y'\beta k\theta e\beta y &= y'\beta k\theta e\beta f\beta k\theta e\beta y \\ &= y'\beta k\theta e\beta k\theta e\beta y \\ &= y'\beta k\theta e\beta y, \end{aligned}$$

$$y'\beta k\theta e\beta y = y'\beta k\gamma g\theta x = y'\beta k\gamma c'\delta c\theta x$$

and

$$c\theta x = c\gamma g\theta k\theta e\beta y = c\theta e\beta f\beta k\theta e\beta y = c\theta x\alpha y'\beta k\theta e\beta y.$$

So $y'\beta k\theta e\beta y\mathcal{L}c\theta x$. It is easy to show that $y'\beta k\theta e\beta y \preccurlyeq f'$. Set $k' := y'\beta k\theta e\beta y \in E_\alpha(L_{c\theta x}) \cap (f']$. Thus $e', h', k' \in (f']$.

Now, every element of $(f']$ is ρ-related to f' and by assumption, we have $(e', h'), (h', k') \in \rho$. Hence $(e'] \cap (h'] \cap (k'] \neq \emptyset$. If $l \in (e'] \cap (h'] \cap (k']$ then

$$z := c\theta y\alpha l = c\theta y\alpha e'\alpha l = c\theta x\alpha l,$$

$$c\theta x\alpha l = c\theta x\alpha l\alpha k' = c\theta x\alpha l\alpha y'\beta k\gamma c'\delta c\theta x$$

and

$$c\theta y\alpha l = c\theta y\alpha l\alpha h' = c\theta y\alpha y'\beta h\gamma c'\delta c\theta y$$

where $c\theta x\alpha l\alpha y'\beta k\gamma c'$, $c\theta y\alpha l\alpha y'\beta h\gamma c' \in E_\delta(S)$ and $l\alpha y'\beta k\gamma c'\delta c\theta x$, $l\alpha y'\beta h\gamma c'\delta c\theta y \in E_\alpha(S)$. Thus $z \leqslant c\theta x$ and $z \leqslant c\theta y$. It implies that $(c\theta x, c\theta y) \in \rho$. In a similar way, it can be shown that $(c\theta x, c\theta y) \in \rho$. Therefore ρ is a congruence.

(3) $\Rightarrow$ (1) Assume that ρ is a congruence and $\alpha \in \Gamma$. Let $e \in E_\alpha(S)$ and $f, g \in (e]$. Then $f \preccurlyeq e$ and $g \preccurlyeq e$. Since $\preccurlyeq$ is reflexive, we obtain that $(f, e), (e, g) \in \rho$. By assumption, $(f, g) \in \rho$. It implies that $z \leqslant f$ and $z \leqslant g$ for some $z \in S$. By Proposition 3.2.4 and 3.2.5, $z \in E_\alpha(S)$. Thus $z \preccurlyeq f$ and $z \preccurlyeq g$. Since $\preccurlyeq$ is transitive, we have that $z \preccurlyeq e$. □

Theorem 4.3.12. *Let S be a regular Γ-semigroup and*

$$\rho = \{(x, y) \in S \times S \mid z \leqslant x \text{ and } z \leqslant y \text{ for some } z \in S\}.$$

Then the congruence ρ is the finest primitive congruence on S.

Proof. Let σ be a primitive congruence on S and $(x, y) \in \rho$. Then there exists $z \in S$ such that $z \leqslant x$ and $z \leqslant y$. By Theorem 4.3.4, $z\sigma^* \leqslant x\sigma^*$ and $z\sigma^* \leqslant y\sigma^*$ where σ^* is the canonical homomorphism of S onto S/σ. By Theorem 4.3.9, we conclude that $x\sigma^* = z\sigma^* = y\sigma^*$, that is $(x, y) \in \sigma$. Therefore $\rho \subseteq \sigma$. □