CHAPTER III

THE NATURAL PARTIALLY ORDERED SET

3.1 Basic Properties of Regular Γ-semigroups

We start elementary properties of idempotent elements of Green's relations.

Proposition 3.1.1. *Let S be a regular Γ-semigroup, $a, b \in S, \alpha, \beta, \gamma \in \Gamma$ and $a' \in V_\alpha^\beta(a), b' \in V_\gamma^\delta(b)$. Then the following statements hold.*

(1) *$a\mathcal{R}b$ if and only if $a\alpha a'\beta b\gamma b' = b\gamma b'$ and $b\gamma b'\delta a\alpha a' = a\alpha a'$.*

(2) *$a\mathcal{L}b$ if and only if $a'\beta a\gamma b'\delta b = a'\beta a$ and $b'\delta b\alpha a'\beta a = b'\delta b$.*

(3) *If $a\mathcal{H}b$ and $b^* = a'\beta a\gamma b'\delta a\alpha a'$ then $b^* \in V_\alpha^\beta(b)$ and $a'\mathcal{H}b^*$.*

Proof. (1) Let $a\mathcal{R}b$. By Lemma 2.2.3, we have $a = b$ or $a = b\theta_1 s$ and $b = a\theta_2 t$ for some $s, t \in S, \theta_1, \theta_2 \in \Gamma$.

If $a = b$ then $b = a\alpha a'\beta a$ and $a = b\gamma b'\delta b$. Thus $b\gamma b' = a\alpha a'\beta a\gamma b' = a\alpha a'\beta b\gamma b'$ and $a\alpha a' = b\gamma b'\delta b\alpha a' = b\gamma b'\delta a\alpha a'$.

If $a = b\theta_1 s$ and $b = a\theta_2 t$ then

$$\begin{aligned} a\alpha a' &= a\alpha a'\beta b\theta_1 s\alpha a' \\ &= a\alpha a'\beta b\gamma b'\delta b\theta_1 s\alpha a' \\ &= a\alpha a'\beta a\theta_2 t\gamma b'\delta b\theta_1 s\alpha a' \\ &= b\gamma b'\delta b\theta_1 s\alpha a' \\ &= b\gamma b'\delta a\alpha a' \end{aligned}$$

and

$$b\gamma b' = a\theta_2 t\gamma b'\delta b\gamma b' = a\alpha a'\beta a\theta_2 t\gamma b'\delta b\gamma b' = a\alpha a'\beta b\gamma b'.$$

Conversely, if $a\alpha a'\beta b\gamma b' = b\gamma b'$ and $b\gamma b'\delta a\alpha a' = a\alpha a'$ then $a\mathcal{R}b$.

(2) The proof of this is similar to the proof of (1).

(3) Suppose that $a\mathcal{H}b$ and $b^* = a'\beta a\gamma b'\delta a\alpha a'$. By (1) and (2) we have that

$$\begin{aligned}
b^*\beta b\alpha b^* &= (a'\beta a\gamma b'\delta a\alpha a')\beta b\alpha(a'\beta a\gamma b'\delta a\alpha a') \\
&= a'\beta a\gamma b'\delta a\alpha a'\beta b\gamma b'\delta b\alpha a'\beta a\gamma b'\delta a\alpha a' \\
&= a'\beta a\gamma b'\delta b\gamma b'\delta b\alpha a'\beta a\gamma b'\delta a\alpha a' \\
&= a'\beta a\gamma b'\delta b\alpha a'\beta a\gamma b'\delta a\alpha a' \\
&= a'\beta a\gamma b'\delta b\gamma b'\delta a\alpha a' \\
&= a'\beta a\gamma b'\delta a\alpha a' \\
&= b^*.
\end{aligned}$$

Thus $b^* \in V_\alpha^\beta(b)$.

Next, we will show that $a'\mathcal{H}b^*$. Since $b^* = a'\beta a\gamma b'\delta a\alpha a'$, we get that

$$\begin{aligned}
a' &= a'\beta a\alpha a' \\
&= a'\beta b\gamma b'\delta a\alpha a' \\
&= a'\beta b\gamma b'\delta b\gamma b'\delta a\alpha a' \\
&= a'\beta b\gamma b'\delta b\alpha a'\beta a\gamma b'\delta a\alpha a' \\
&= a'\beta b\alpha b^*.
\end{aligned}$$

Therefore $a'\mathcal{L}b^*$.

$$\begin{aligned}
a' &= a'\beta a\alpha a' \\
&= a'\beta a\gamma b'\delta b\alpha a' \\
&= a'\beta a\gamma b'\delta b\gamma b'\delta b\alpha a' \\
&= a'\beta a\gamma b'\delta a\alpha a'\beta b\gamma b'\delta b\alpha a' \\
&= b^*\beta b\alpha a'.
\end{aligned}$$

Thus $a'\mathcal{R}b^*$. Hence $a'\mathcal{H}b^*$. □

Note that, for a Γ-semigroup S, $a \in S$ and $\alpha \in \Gamma$. we define the set $E_\alpha(R_a)$ and $E_\alpha(L_a)$ on S by

$$E_\alpha(R_a) := E_\alpha(S) \cap R_a$$

and

$$E_\alpha(L_a) := E_\alpha(S) \cap L_a.$$

Lemma 3.1.2. *Let S be a regular Γ-semigroup and $a \in S$. Then the following statements hold.*

(1) For all $\alpha \in \Gamma, e \in E_\alpha(R_a)$ if and only if there exist $\gamma \in \Gamma, a' \in V_\gamma^\alpha(a)$ such that $e = a\gamma a'$.

(2) For all $\alpha \in \Gamma, e \in E_\alpha(L_a)$ if and only if there exist $\gamma \in \Gamma, a' \in V_\alpha^\gamma(a)$ such that $e = a'\gamma a$.

(3) For all $\alpha, \beta \in \Gamma, e \in E_\alpha(L_a), f \in E_\beta(R_a)$ if and only if there exists $a' \in V_\alpha^\beta(a)$ such that $e = a'\beta a$ and $f = a\alpha a'$.

Proof. (1) Let $\alpha \in \Gamma$ and $e \in E_\alpha(R_a)$. By Lemma 2.2.3(2), we get $e = a$ or $e = a\gamma x$ for some $x \in S, \gamma \in \Gamma$. This is obvious when $a = e$. Assume that $e = a\gamma x$. By Lemma 2.2.4(2), we have

$$a = e\alpha a = a\gamma(x\alpha e)\alpha a$$

and

$$x\alpha e = x\alpha e\alpha e = x\alpha a\gamma x\alpha e = (x\alpha e)\alpha a\gamma(x\alpha e)$$

which implies that $x\alpha e \in V_\gamma^\alpha(a)$. Set $a' := x\alpha e$. We obtain that $a\gamma a' = (a\gamma x)\alpha e = e$. Conversely, assume that $e = a\gamma a'$ for some $\gamma \in \Gamma, a' \in V_\gamma^\alpha(a)$. Then $e = a\gamma a'\alpha a\gamma a' = e\alpha e$ which implies that $e \in E_\alpha(S)$. Since $a = a\gamma a'\alpha a = e\alpha a$ and $e = a\gamma a'$ we have that $e \in R_a$. Hence $e \in E_\alpha(R_a)$.

(2) The proof is similar to the proof of (1).

(3) Let $\alpha, \beta \in \Gamma, e \in E_\alpha(L_a), f \in E_\beta(R_a)$ be such that $e\mathcal{L}a$ and $f\mathcal{R}a$. By Lemma 2.2.3, we have that $e = a$ or there exist $\gamma \in \Gamma, x \in S$ such that $e = x\gamma a$ and $f = a$ or there exist $\theta \in \Gamma, y \in S$ such that $f = a\theta y$.

Case 1. $e = a = f$. Then we set $a' := a$.

Case 2. $e = a$ and $f = a\theta y$. Then we can set $a' := f$. Thus $e = a'\beta a$ and

$$f = a\theta y = e\theta y = e\alpha a\theta y = a\alpha f = a\alpha a'.$$

Case 3. $f = a$ and $e = x\gamma a$. Then this proof is similar to the second case and set $a' := e$.

Case 4. $e = x\gamma a$ and $f = a\theta y$. Then we choose $a' := e\theta y\beta f$. By Lemma 2.2.4, we have that

$$\begin{aligned} a'\beta a\alpha a' &= e\theta y\beta f\beta a\alpha e\theta y\beta f \\ &= e\theta y\beta a\alpha e\theta y\beta f \\ &= e\theta y\beta a\theta y\beta f \\ &= e\theta y\beta f \\ &= a' \end{aligned}$$

and

$$a\alpha a'\beta a = a\alpha e\theta y\beta f\beta a = a\theta y\beta a = a.$$

Thus $a' \in V_\alpha^\beta(a)$ and we obtain that

$$a'\beta a = e\theta y\beta f\beta a = x\gamma a\theta y\beta f\beta a = x\gamma f\beta f\beta a = x\gamma f\beta a = x\gamma a = e$$

and

$$a\alpha a' = a\alpha e\theta y\beta f = a\theta y\beta f = f.$$

The converse part is obvious. $\square$

In the proof of Lemma 3.1.2, we see that any two elements in $\mathcal{L}$-class [$\mathcal{R}$-class, $\mathcal{H}$-class] may be alike and the proof of them is obvious.

Definition 3.1.3. Let a and b be elements of a regular Γ-semigroup S. Define

$$R_a \leqslant R_b \text{ if and only if } a\Gamma S \cup \{a\} \subseteq b\Gamma S \cup \{b\},$$

$$L_a \leqslant L_b \text{ if and only if } S\Gamma a \cup \{a\} \subseteq S\Gamma b \cup \{b\},$$

and

$$H_a \leqslant H_b \text{ if and only if } R_a \leqslant R_b \text{ and } L_a \leqslant L_b.$$

Proposition 3.1.4. *Let S be a regular Γ-semigroup and $a, b \in S$. Then $H_a \leqslant H_b$ if and only if $a \in b\Gamma S\Gamma b$.*

Proof. Assume that $H_a \leqslant H_b$. Then $L_a \leqslant L_b$ and $R_a \leqslant R_b$. Since $a \in S$, we have $a = a\alpha c\beta a$ for some $\alpha, \beta \in \Gamma, c \in S$. If $a = b$, it is obvious. If $a = x\gamma b$ and $a = b\delta y$ for some $\gamma, \delta \in \Gamma, x, y \in S$, we get that $a = a\alpha c\beta a = b\delta y\alpha c\beta x\gamma b \in b\Gamma S\Gamma b$. Conversely, assume that $a \in b\Gamma S\Gamma b$. Then $a \in b\Gamma S$ which implies that $a\Gamma S \subseteq b\Gamma S\Gamma S \subseteq b\Gamma S$. Thus $a\Gamma S \cup \{a\} \subseteq b\Gamma S \subseteq b\Gamma S \cup \{b\}$, so $R_a \leqslant R_b$. Similarly, we can show that $L_a \leqslant L_b$. Therefore $H_a \leqslant H_b$. □

Lemma 3.1.5. *Let $\alpha, \beta \in \Gamma$ be such that $e \in E_\alpha(S), f \in E_\beta(S)$. If e, f are $\mathcal{D}$-related then there exist $a \in S$ and $a' \in V_\alpha^\beta(a)$ such that $e = a'\beta a$ and $f = a\alpha a'$.*

Proof. Suppose that $e\mathcal{D}f$. Then there exists $a \in S$ such that $e\mathcal{L}a$ and $a\mathcal{R}f$. Since $e\mathcal{L}a$, we have $e = a$ or $e = x\gamma a$ and $a = a\alpha e$ for some $x \in S, \gamma \in \Gamma$. Again, since $a\mathcal{R}f$, we have $a = f$ or $a = f\beta a$ and $f = a\delta k$ for some $k \in S, \delta \in \Gamma$.

Case 1. $e = a = f$. Set $a' := a$. Clearly, $a' \in V_\alpha^\beta(a)$ and $e = a'\beta a, f = a\alpha a'$.

Case 2. $e = a$ and $a = f\beta a$, $f = a\delta k$. Set $a' := f$. Clearly, $a' \in V_\alpha^\beta(a)$ and $e = a'\beta a, f = a\alpha a'$.

Case 3. $e = x\gamma a$, $a = a\alpha e$ and $a = f$. Set $a' := e$. Clearly, $a' \in V_\alpha^\beta(a)$ and $e = a'\beta a, f = a\alpha a'$.

Case 4. $e = x\gamma a$, $a = a\alpha e$ and $a = f\beta a$, $f = a\delta k$. Set $a' := e\delta k\beta f$. Then

$$\begin{aligned} a'\beta a\alpha a' &= e\delta k\beta(f\beta a\alpha e)\delta k\beta f \\ &= e\delta k\beta a\delta k\beta f \\ &= e\delta k\beta f \\ &= a' \end{aligned}$$

and

$$a\alpha a'\beta a = (a\alpha e)\delta k\beta(f\beta a) = a\delta k\beta a = f\beta a = a.$$

Thus $a' \in V_\alpha^\beta(a)$. Also

$$a'\beta a = e\delta k\beta f\beta a = e\delta k\beta a = x\gamma a\delta k\beta a = x\gamma f\beta a = x\gamma a = e,$$

and

$$a\alpha a' = a\alpha e\delta k\beta f = a\delta k\beta f = f.$$

Hence $e = a'\beta a$ and $f = a\alpha a'$. □

Lemma 3.1.6. *Let S be a Γ-semigroup and $a, b \in S, \alpha \in \Gamma$. Suppose that $a\alpha b$ is regular and $e \in E_\alpha(S)$.*

(1) *If $e\mathcal{L}a$ and $(e\alpha b)\mathcal{L}(a\alpha b)$ then $e\alpha b$ is regular.*

(2) *If $e\mathcal{R}b$ and $(a\alpha e)\mathcal{R}(a\alpha b)$ then $a\alpha e$ is regular.*

Proof. Suppose that $a\alpha b$ is regular. Let $x \in V_\gamma^\delta(a\alpha b)$ for some $\gamma, \delta \in \Gamma$.

(1) By assumption, we have $a = a\alpha e$ and $(e\alpha b)\mathcal{L}(a\alpha b)$. Then $e\alpha b = c\theta(a\alpha b)$ for some $c \in S, \theta \in \Gamma$. Thus

$$e\alpha b = c\theta a\alpha b = c\theta(a\alpha b)\gamma x\delta(a\alpha b) = (e\alpha b)\gamma(x\delta a)\alpha(e\alpha b).$$

Hence $e\alpha b$ is regular.

(2) The proof of this is similar to the proof of (1) and we can show that $a\alpha e = (a\alpha e)\alpha(b\gamma x)\delta(a\alpha e)$. □

3.2 Natural Partial Ordered Sets on Regular Γ-semigroups

In this section, we construct a relation on a regular Γ-semigroup S by extending the partial order in [3].

Let S be a Γ-semigroup. We define relations on $E(S)$ as follows :

For $e, f \in E(S)$, define

(1) $e \preccurlyeq^l f \Leftrightarrow e = e\alpha f$ if $e \in E_\alpha(S)$ for some $\alpha \in \Gamma$,

(2) $e \preccurlyeq^r f \Leftrightarrow e = f\beta e$ if $f \in E_\beta(S)$ for some $\beta \in \Gamma$,

(3) $e \preccurlyeq f \Leftrightarrow e \preccurlyeq^l f$ and $e \preccurlyeq^r f$

$\Leftrightarrow e = e\alpha f = f\beta e$ if $e \in E_\alpha(S), f \in E_\beta(S)$ for some $\alpha, \beta \in \Gamma$.

We will show that $\preccurlyeq$ is a partial order on $E(S)$.

Let $e \in E(S)$. Then $e \in E_\alpha(S)$ for some $\alpha \in \Gamma$. It is easy to show that $\preccurlyeq$ is reflexive. Let $e \preccurlyeq f$ and $f \preccurlyeq e$. If $e, f \in E_\alpha(S)$ for some $\alpha \in \Gamma$ then $e = f$. If $e \in E_\alpha(S), f \in E_\beta(S)$ for some $\alpha, \beta \in \Gamma$ then $e = e\alpha f = f\beta e$ and $f = f\beta e = e\alpha f$. Thus $e = f$. Therefore $\preccurlyeq$ is anti-symmetric.

Next, we will show that $\preccurlyeq$ is transitive. Assume that $e \preccurlyeq f$ and $f \preccurlyeq g$.

Case 1. $e, f, g \in E_\alpha(S)$ for some $\alpha \in \Gamma$. It is easy to show that $\preccurlyeq$ is transitive.

Case 2. $e, f \in E_\alpha(S), g \in E_\beta(S)$ for some $\alpha, \beta \in \Gamma$. Then $e = e\alpha f = f\alpha e$ and $f = f\alpha g = g\beta f$. Thus $e = e\alpha f = e\alpha f\alpha g = e\alpha g$ and $e = f\alpha e = g\beta f\alpha e = g\beta e$ which implies that $e \preccurlyeq g$.

Case 3. $e \in E_\alpha(S), f, g \in E_\beta(S)$ for some $\alpha, \beta \in \Gamma$. Then $e = e\alpha f = f\beta e$ and $f = f\beta g = g\beta f$. Thus $e = e\alpha f = e\alpha f\beta g = e\beta g$, $e = f\beta e = g\beta f\beta e = g\beta e$ and $e\beta e = e\alpha f\beta f\beta e = e\alpha f\beta e = e\alpha e = e$ which implies that $e \preccurlyeq g$.

Case 4. $e \in E_\alpha(S), f \in E_\beta(S), g \in E_\gamma(S)$ for some $\alpha, \beta, \gamma \in \Gamma$. Then $e = e\alpha f = f\beta e$ and $f = f\beta g = g\gamma f$. Thus $e = e\alpha f = e\alpha f\beta g = e\alpha g$ and $e = f\beta e = g\gamma f\beta e = g\gamma e$ which implies that $e \preccurlyeq g$. Hence $\preccurlyeq$ is a partial order on $E(S)$.

Let a and b be elements of a regular Γ-semigroup S.

Define

$$a \leqslant_n b \text{ if } R_a \leqslant R_b \text{ and } a = f\beta b \text{ for some } f \in E_\beta(R_a), \beta \in \Gamma.$$

Next, we will show that $\leqslant_n$ is a partial order on a regular Γ-semigroup S.

Let $a \in S$. Then there exist $x \in S, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Set $f := a\alpha x$, we have $a := f\beta a$ and $f \in R_a$. Clearly, $R_a \leqslant R_a$. Thus $a \leqslant_n a$.

Let $a \leqslant_n b$ and $b \leqslant_n a$. Then $R_a \leqslant R_b$ and $R_b \leqslant R_a$. Thus $R_a = R_b$ and $a = f\beta b$ for some $f \in E_\beta(R_a)$. Since $R_a = R_b$ and $f \in R_a = R_b$, we have that $b \in R_f$. By Lemma 2.2.4, we have $f\beta b = b$, so $a = b$.

Suppose that $a \leqslant_n b$ and $b \leqslant_n c$. Then $R_a \leqslant R_b$, $a = f\beta b$ for some $f \in E_\beta(R_a)$ and $R_b \leqslant R_c$, $b = e\alpha c$ for some $e \in E_\alpha(R_b)$. Thus $R_a \leqslant R_c$ and $a = f\beta b = f\beta e\alpha c$. Claim that $R_f \leqslant R_e$ and $e\alpha f = f$. Since $R_a = R_f$ and $R_e = R_b$, we have $R_f = R_a \leqslant R_b = R_e$. Since $f \in R_e$ and by Lemma 2.2.4, we get that $e\alpha f = f$ which implies that

$$(f\beta e)\alpha(f\beta e) = f\beta f\beta e = f\beta e,$$

so $f\beta e$ is an α-idempotent element of S .

Next, we will show that $f\beta e \in R_a$. Since $f \in R_a$, we get that $f = a\delta x$ for some $\delta \in \Gamma, x \in S$. Then $f\beta e = a\delta x\beta e$ and $a = f\beta e\alpha c$. Thus $f\beta e \in R_a$. Hence $a \leqslant_n c$. Therefore $\leqslant_n$ is a partial order on S.

A partial order on a regular Γ-semigroup S is called *natural partial order* on S. For convenience, we write a symbol $\leqslant$ for the natural partial order $\leqslant_n$.

Next, we show that the natural partial order has an alternative characterization:

Theorem 3.2.1. *Let a and b be elements of a regular Γ-semigroup S. Then the following statements are equivalent.*

(1) $a \leqslant b$.

(2) *$a \in b\Gamma S$ and there exist $\alpha, \beta \in \Gamma, a' \in V_\alpha^\beta(a)$ such that $a = a\alpha a'\beta b$.*

(3) *There exist $\beta, \gamma \in \Gamma, f \in E_\beta(S), g \in E_\gamma(S)$ such that $a = f\beta b = b\gamma g$.*

(4) *$H_a \leqslant H_b$ and for all $\alpha, \delta \in \Gamma, b' \in V_\alpha^\delta(b), a = a\alpha b'\delta a$.*

(5) *$H_a \leqslant H_b$ and there exist $\alpha, \delta \in \Gamma, b' \in V_\alpha^\delta(b), a = a\alpha b'\delta a$.*

Proof. For the case $a = b$, we have the theorem. Now, we may assume that $a \neq b$.

(1) $\Rightarrow$ (2) Let $a \leqslant b$. Then $R_a \leqslant R_b$ and $a = f\beta b$ for some $f \in E_\beta(R_a), \beta \in \Gamma$. By Lemma 3.1.2(1), there exist $\alpha \in \Gamma, a' \in V_\alpha^\beta(a)$ such that $a\alpha a' = f$. Thus $a = a\alpha a'\beta b$. Since $R_a \leqslant R_b$, we have $a\Gamma S \cup \{a\} \subseteq b\Gamma S \cup \{b\}$ which implies that $a \in b\Gamma S$.

(2) $\Rightarrow$ (3) By assumption, $a = b\gamma u$ for some $\gamma \in \Gamma, u \in S$. Set $f := a\alpha a' \in E_\beta(S)$ and $g = u\alpha a'\beta b$, so we have $a = f\beta b$. Thus $b\gamma g = b\gamma u\alpha a'\beta b = a\alpha a'\beta b = a$ with $g \in E_\gamma(S)$.

(3) $\Rightarrow$ (4) By assumption, $a \in b\Gamma S$ and $a\Gamma S \subseteq b\Gamma S$ which implies that $a\Gamma S \cup \{a\} \subseteq b\Gamma S \cup \{b\}$, so $R_a \leqslant R_b$. Similarly, we can show that $S\Gamma a \cup \{a\} \subseteq S\Gamma b \cup \{b\}$, so that $L_a \leqslant L_b$. Thus $H_a \leqslant H_b$. Let $\alpha, \delta \in \Gamma, b' \in V_\alpha^\delta(b)$, we have immediately that $a\alpha b'\delta a = a$.

(4) $\Rightarrow$ (5) This part is obvious.

(5) $\Rightarrow$ (1) By assumption, $R_a \leqslant R_b$ and $L_a \leqslant L_b$. Let $a' \in V_\gamma^\beta(a)$ for some $\beta, \gamma \in \Gamma$. Set $f := a\gamma a'\beta a\alpha b'$. Then $a = a\alpha b'\delta a = a\gamma a'\beta a\alpha b'\delta a = f\delta a$ and $f \in E_\delta(S)$, which prove that $f \in E_\delta(R_a)$. Since $L_a \leqslant L_b$, we get that $a = u\theta b$ for some $u \in S, \theta \in \Gamma$. Thus $f\delta b = a\alpha b'\delta b = u\theta b = a$. Therefore $a \leqslant b$. □

The next result give a relationship between the natural partial order and the partial order on $E(S)$.

Proposition 3.2.2. *Let S be a regular Γ-semigroup and $a, b \in S$. Then the following statements are equivalent.*

(1) $a \leqslant b$.

(2) *For every $f \in E(R_b)$, there exist $\alpha \in \Gamma, e \in E_\alpha(R_a)$ such that $e \preccurlyeq f$ and $a = e\alpha b$.*

(3) *For every $f' \in E(L_b)$, there exist $\alpha \in \Gamma, e' \in E_\alpha(L_a)$ such that $e' \preccurlyeq f'$ and $a = b\alpha e'$.*

Proof. (1) $\Rightarrow$ (2) Let $f \in E(R_b)$. Then there exits $\beta \in \Gamma$ such that $f \in E_\beta(S) \cap R_b$.

By assumption, $a = h\gamma b$ for some $h \in E_\gamma(R_a), \gamma \in \Gamma$ and $R_a \leqslant R_b$ which implies that $R_h = R_a \leqslant R_b = R_f$. By Lemma 2.2.4(2), we have $h = f\beta h$ and $h\gamma f \in E_\beta(S)$. Choose $e := h\gamma f$. Then $h = e\beta h$ which implies $e\mathcal{R}h$, and so $e\mathcal{R}a$. Thus $a = h\gamma b = h\gamma f\beta b = e\beta b$, and $e = e\beta f$, $e = h\gamma f = f\beta e$. Therefore $e \preccurlyeq f$.

(2) $\Rightarrow$ (3) Let $f' \in E(L_b)$. Then $f' \in E(S) \cap L_b$ which implies that $f' \in E_\beta(S)$ for some $\beta \in \Gamma$. By Lemma 3.1.2(2), there exist $\gamma \in \Gamma, b' \in V_\beta^\gamma(b)$ such that $f' = b'\gamma b$. Clearly, $b\beta b' \in E_\gamma(R_b)$. Set $f := b\beta b'$. By assumption, there exist $\alpha \in \Gamma, e \in E_\alpha(R_a)$ such that $e \preccurlyeq f$ and $a = e\alpha b$. Set $e' := b'\gamma e\alpha b$. Then

$$e' = b'\gamma a, \quad a = e\alpha b = f\gamma e\alpha b = b\beta b'\gamma e\alpha b = b\beta e',$$

and

$$e' = b'\gamma e\alpha b = b'\gamma e\alpha f\gamma e\alpha b = b'\gamma e\alpha b\beta b'\gamma e\alpha b = e'\beta e'$$

which implies that $e' \in L_a$ and $e' \in E_\beta(S)$. Thus $e' \in E_\beta(L_a)$. Consider $e' = b'\gamma b\beta b'\gamma e\alpha b = f'\beta e'$ and $e' = b'\gamma e\alpha b\beta b'\gamma b = e'\beta f'$. Therefore $e' \preccurlyeq f'$.

(3) $\Rightarrow$ (1) Let $b' \in V_\gamma^\delta(b)$ for some $\gamma, \delta \in \Gamma$. Then $b'\delta b \in E_\gamma(L_b)$. By assumption, there exist $\alpha \in \Gamma, e' \in E_\alpha(L_a)$ such that $e' \preccurlyeq b'\delta b$ and $a = b\alpha e'$. Set $f := b\alpha e'\alpha b'$. Clearly, $f \in E_\delta(S)$. Thus $f\delta b = b\alpha e' = a$. By Theorem 3.2.1, we have $a \leqslant b$. □

The following remark follows immediately from the above propositions.

Remark 1. Let S be a regular Γ-semigroup and $a, b \in S$. Then the following statements are equivalent.

(1) $a \leqslant b$.

(2) If $f \in E_\beta(R_b)$ for some $\beta \in \Gamma$ then there exists $e \in E_\beta(R_a)$ such that $e \preccurlyeq f$ and $a = e\beta b$.

(3) If $f' \in E_\beta(L_b)$ for some $\beta \in \Gamma$ then there exists $e' \in E_\beta(L_a)$ such that $e' \preccurlyeq f'$ and $a = b\beta e'$.

Next, we study a relationship of natural partial order on the set of all idempotent and regular Γ-semigroups.

Lemma 3.2.3. *Let S be a regular Γ-semigroup. Then the following conditions hold:*

(1) $\preccurlyeq \circ \mathcal{L} = \mathcal{L} \circ \preccurlyeq$.

(2) $\preccurlyeq \circ \mathcal{R} = \mathcal{R} \circ \preccurlyeq$.

Proof. (1) Let $(e, f) \in \preccurlyeq \circ \mathcal{L}$ where $e, f \in E(S)$. Then there exists $h \in E(S)$ such that $e \preccurlyeq h$ and $h\mathcal{L}f$.

Case 1. $e, f, h \in E_\alpha(S)$ for some $\alpha \in \Gamma$. Then $e = e\alpha h = h\alpha e$, $h = h\alpha f$ and $f = f\alpha h$ which implies that $e = e\alpha f$ Thus $f\alpha e = f\alpha(f\alpha e) = (f\alpha e)\alpha f$ and $f\alpha e \in E_\alpha(S)$, so $f\alpha e \preccurlyeq f$. Since $e = e\alpha(f\alpha e)$, we have that $e\mathcal{L}f\alpha e$. Thus $(e, f) \in \mathcal{L}\circ \preccurlyeq$.

Case 2. $e, h \in E_\alpha(S), f \in E_\beta(S)$ for some $\alpha, \beta \in \Gamma$. By Lemma 2.2.4(1), $h = h\beta f$ and $f = f\alpha h$. Then

$$e = e\alpha h = e\alpha h\beta f = e\beta f,$$

and

$$f\alpha e = f\alpha e\beta f = f\beta f\alpha e,$$

which implies that $f\alpha e \preccurlyeq f$. Since $e = e\beta f\alpha e$, we obtain that $e\mathcal{L}f\alpha e$. Therefore $(e, f) \in \mathcal{L}\circ \preccurlyeq$.

Case 3. $e, f \in E_\alpha(S), h \in E_\beta(S)$ for some $\alpha, \beta \in \Gamma$. The proof is similar to the proof of Case 2.

Case 4. $e \in E_\alpha(S), f \in E_\beta(S)$ and $h \in E_\gamma(S)$ for some $\alpha, \beta, \gamma \in \Gamma$. Then $e = e\alpha h = h\gamma e$. By Lemma 2.2.4(1), we get that $e = e\beta f$ and $f\alpha e = f\alpha e\beta f = f\beta f\alpha e$, which prove that $f\alpha e \preccurlyeq f$. Since $e = e\beta f\alpha e$, we have that $e\mathcal{L}f\alpha e$. By Cases (1)-(4), we get $(e, f) \in \mathcal{L}\circ \preccurlyeq$ which implies that $\preccurlyeq \circ\mathcal{L} \subseteq \mathcal{L}\circ \preccurlyeq$.

Similarly, we can show that $\mathcal{L}\circ \preccurlyeq \subseteq \preccurlyeq \circ\mathcal{L}$.

(2) The proof is similar to the proof of (1). □

Proposition 3.2.4. *Let S be a regular Γ-semigroup. Then the following statements hold.*

(1) *If $e \in E(S), a \in S$ and $a \leqslant e$ then $a \in E(S)$.*

(2) *For any $a, b \in S, a\mathcal{R}b$ and $a \leqslant b$ implies $a = b$.*

(3) *If $a \leqslant c, b \leqslant c$ and $H_a \leqslant H_b$ then $a \leqslant b$.*

Proof. (1) Let $e \in E(S)$. Then $e \in E_\alpha(S)$ for some $\alpha \in \Gamma$. By assumption, there exist $\beta, \gamma \in \Gamma, f \in E_\beta(S), g \in E_\gamma(S)$ such that $a = f\beta e = e\gamma g$. Thus

$$a\alpha a = f\beta e\alpha e\gamma g = f\beta e\gamma g = f\beta a = f\beta f\beta e = f\beta e = a$$

which implies that $a \in E_\alpha(S)$.

(2) Let $a\mathcal{R}b$. Then there exist $x \in S, \theta \in \Gamma$ such that $b = a\theta x$. Since $a \leqslant b$, we get that $a = f\beta b$ for some $\beta \in \Gamma, f \in E_\beta(S)$. Thus

$$a = f\beta b = f\beta a\theta x = f\beta f\beta b\theta x = f\beta b\theta x = a\theta x = b.$$

(3) Assume that $a \leqslant c, b \leqslant c$ and $H_a \leqslant H_b$. Let $c' \in V_\alpha^\beta(c)$ for some $\alpha, \beta \in \Gamma$. Then $c\alpha c' \in E_\beta(R_c)$. By Remark 1, there exist $e \in E_\beta(R_a), f \in E_\beta(R_b)$ such that $e \preccurlyeq c\alpha c', f \preccurlyeq c\alpha c'$ and $a = e\beta c, b = f\beta c$. By assumption and Proposition 3.1.4, we have $a \in b\Gamma S\Gamma b$. Then $a = b\delta x\theta b$ for some $\delta, \theta \in \Gamma, x \in S$. Thus

$$\begin{aligned} (c'\beta f)\beta b\alpha(c'\beta f) &= c'\beta c\alpha c'\beta f\beta b\alpha c'\beta f \\ &= c'\beta f\beta f\beta c\alpha c'\beta f \\ &= c'\beta f\beta c\alpha c'\beta f \\ &= c'\beta f \end{aligned}$$

and

$$\begin{aligned} b\alpha(c'\beta f)\beta b &= f\beta c\alpha c'\beta f\beta b \\ &= f\beta f\beta b \\ &= f\beta f\beta c \\ &= b \end{aligned}$$

from which get that $c'\beta f \in V_\alpha^\beta(b)$. Set $b' := c'\beta f$.

Since $e \preccurlyeq c\alpha c'$ and by Theorem 3.2.1, we obtain that

$$\begin{aligned} e &= e\beta c\alpha c' \\ &= a\alpha c' \\ &= b\delta x\theta b\alpha c' \\ &= f\beta c\delta x\theta b\alpha c' \\ &= f\beta b\delta x\theta b\alpha c' \\ &= f\beta a\alpha c' \\ &= f\beta e\beta c\alpha c' \\ &= f\beta e. \end{aligned}$$

Therefore $a\alpha b'\beta a = e\beta f\beta e\beta c = e\beta c = a$.

Again, by Theorem 3.2.1, we have that $a \leqslant b$. □

Note that by Proposition 3.2.4(1), if $e \in E_\alpha(S)$ and $a \leqslant e$ then $a \in E_\alpha(S)$.

Proposition 3.2.5. *Let e be an α-idempotent and f be a β-idempotent of a regular Γ-semigroup S. Then the following statements hold.*

(1) *If $e \preccurlyeq f$ then $e \in E_\beta(S)$.*

(2) $V_\alpha^\beta(f\beta e) \neq \emptyset$.

Proof. (1) This follows directly from the definition of the relation $\preccurlyeq$.

(2) Since $f\beta e$ is a regular element, we can choose $x \in S, \gamma, \delta \in \Gamma$ such that $f\beta e = (f\beta e)\gamma x\delta(f\beta e)$. It follows that

$$\begin{aligned} (e\gamma x\delta f\beta e\gamma x\delta f)\beta(f\beta e)\alpha(e\gamma x\delta f\beta e\gamma x\delta f) &= e\gamma x\delta f\beta e\gamma x\delta f\beta e\gamma x\delta f\beta e\gamma x\delta f \\ &= e\gamma x\delta f\beta e\gamma x\delta f \end{aligned}$$

and

$$(f\beta e)\alpha(e\gamma x\delta f\beta e\gamma x\delta f)\beta(f\beta e) = f\beta e$$

which proves that $e\gamma x\delta f\beta e\gamma x\delta f \in V_\alpha^\beta(f\beta e)$. Therefore $V_\alpha^\beta(f\beta e) \neq \emptyset$. □

A regular Γ-semigroup S is called an **$\mathcal{L}$-unipotent** [**$\mathcal{R}$-unipotent**] if every $\mathcal{L}$-class [$\mathcal{R}$-class] of S contains only one α-idempotent for all $\alpha \in \Gamma$.

Proposition 3.2.6. *Let S be a regular Γ-semigroup. If S is an $\mathcal{L}$-unipotent then $e\alpha f\beta e = f\beta e$ for all $e \in E_\alpha(S), f \in E_\beta(S)$ for some $\alpha, \beta \in \Gamma$.*

Proof. Let $e \in E_\alpha(S)$ and $f \in E_\beta(S)$ for some $\alpha, \beta \in \Gamma$. By Proposition 3.2.5(2), we can choose $x \in V_\alpha^\beta(f\beta e)$. Then

$$(x\beta f\beta e)\alpha(x\beta f\beta e) = x\beta f\beta e$$

and

$$(e\alpha x\beta f\beta e)\alpha(e\alpha x\beta f\beta e) = e\alpha x\beta f\beta e,$$

so $x\beta f\beta e, e\alpha x\beta f\beta e \in E_\alpha(S)$ and it follows immediately that $(x\beta f\beta e)\mathcal{L}(e\alpha x\beta f\beta e)$. The hypothesis implies that

$$x\beta f\beta e = e\alpha x\beta f\beta e. \tag{3.1}$$

Now, $x = e\alpha x\beta f\beta e\alpha x = e\alpha x$. It follows that $x = x\beta f\beta x$, that is $x\beta f \in E_\beta(S)$. Thus

$$(f\beta e\alpha x\beta f)\beta(f\beta e\alpha x\beta f) = f\beta e\alpha x\beta f,$$

which implies that $f\beta e\alpha x\beta f \in E_\beta(S)$ and $(x\beta f)\mathcal{L}(f\beta e\alpha x\beta f)$.
Again, the hypothesis implies that $x\beta f = f\beta e\alpha x\beta f$. Then

$$x\beta f\beta e = f\beta e. \tag{3.2}$$

By (3.1) and (3.2), we get that $f\beta e = e\alpha x\beta f\beta e$.

Therefore $e\alpha f\beta e = e\alpha x\beta f\beta e = f\beta e$. □

Proposition 3.2.7. *Let S be a regular Γ-semigroup. If e and f are α-idempotent with $e\alpha f\alpha e = f\alpha e$ then S is an $\mathcal{L}$-unipotent.*

Proof. Let $\alpha \in \Gamma$ and $e, f \in E_\alpha(S)$ be such that $e\mathcal{L}f$. By Lemma 2.2.4(1), we get that $e = e\alpha f$ and $f = f\alpha e$. By the hypothesis we get that

$$e = e\alpha f = e\alpha f\alpha e = f\alpha e = f.$$

□

Remark 2. By Proposition 3.2.6 and 3.2.7, we get that S is an $\mathcal{L}$-unipotent if and only if $e\alpha f\alpha e = f\alpha e$ for some $e, f \in E_\alpha(S), \alpha \in \Gamma$.

Proposition 3.2.8. *Let S be a regular Γ-semigroup and $e \in E(S)$. If $a \in e\Gamma S\Gamma e$ then the following statements hold.*

(1) *There exist $\gamma, \delta \in \Gamma, a' \in V_\gamma^\delta(a) \cap e\Gamma S\Gamma e$ such that $a'\delta a \preccurlyeq e$.*

(2) *There exist $\gamma, \delta \in \Gamma, a'' \in V_\gamma^\delta(a) \cap e\Gamma S\Gamma e$ such that $a\gamma a'' \preccurlyeq e$.*

(3) *If $a', a'' \in V_\gamma^\delta(a) \cap e\Gamma S\Gamma e$ then $a'\delta a\mathcal{L}a''\delta a$ and $a\gamma a'\mathcal{R}a\gamma a''$.*

Proof. Let $e \in E_\alpha(S)$ for some $\alpha \in \Gamma$ and let $a \in e\Gamma S\Gamma e$. Then there exist $\beta, \gamma \in \Gamma, x \in S$ such that $a = e\beta x\gamma e$. Since a is a regular element of S, we get that $a = a\delta y\theta a$ for some $y \in S, \delta, \theta \in \Gamma$.

(1) Set $a' := e\delta y\theta a\delta y\theta e$. Then

$$\begin{aligned} a'\alpha a\alpha a' &= e\delta y\theta a\delta y\theta e\alpha e\beta x\gamma e\alpha e\delta y\theta a\delta y\theta e \\ &= e\delta y\theta a\delta y\theta a\delta y\theta a\delta y\theta e \\ &= e\delta y\theta a\delta y\theta e \\ &= a' \end{aligned}$$

and

$$a\alpha a'\alpha a = e\beta x\gamma e\alpha e\delta y\theta a\delta y\theta e\alpha e\beta x\gamma e = a\delta y\theta a\delta y\theta a = a$$

which then implies that $a' \in V_\alpha^\alpha(a) \cap e\Gamma S\Gamma e$. Then $a'\alpha a \preccurlyeq e$.

(2) The proof is similar to the proof of (1).

(3) Let $a', a'' \in V_\gamma^\delta(a) \cap e\Gamma S\Gamma e$. Then

$$a'\delta a = a'\delta a\gamma a''\delta a\ ,\ \ a''\delta a = a''\delta a\gamma a'\delta a$$

and

$$a\gamma a' = a\gamma a''\delta a\gamma a' \ , \ a\gamma a'' = a\gamma a'\delta a\gamma a''.$$

Therefore $a'\delta a\mathcal{L}a''\delta a$ and $a\gamma a'\mathcal{R}a\gamma a''$. □

Theorem 3.2.9. *The partial order on $E(S)$ of a regular semigroup S is the restriction of the natural partial order on S to $E(S)$.*

Proof. Let $e, f \in E(S)$ be such that $e \leqslant f$. Then there exists $\beta \in \Gamma$ such that $f \in E_\beta(S)$. By the proof of Proposition 3.2.4(1), we get that $e \in E_\beta(S)$. By Theorem 3.2.1, there exist $g \in E_\gamma(S), h \in E_\delta(S), \gamma, \delta \in \Gamma$ such that $e = f\gamma g = h\delta f$. Thus $e = f\beta f\gamma g = f\beta e$ and $e = h\delta f\beta f = e\beta f$. Therefore $e \preccurlyeq f$. The converse is obvious. □

3.3 Primitive Congruences on Regular Γ-semigroups

In this section, we find a relation on a regular Γ-semigroup S and show that this relation is a primitive congruence on S.

A regular Γ-semigroup S satisfies **$\mathcal{L}$-majorization** [**$\mathcal{R}$-majorization**] if for any $a, b, c \in S, a \leqslant c, b \leqslant c$ and $a\mathcal{L}b$ [$a\mathcal{R}b$] imply that $a = b$.

Theorem 3.3.1. *Let S be a regular Γ-semigroup. Then the following statements are equivalent.*

(1) *$\leqslant$ is right compatible.*

(2) *S satisfies $\mathcal{L}$-majorization for idempotents.*

(3) *S satisfies $\mathcal{L}$-majorization.*

Proof. (1) ⇒ (2) Let $e, f, g \in E(S)$ be such that $f \preccurlyeq e, g \preccurlyeq e$ and $f\mathcal{L}g$. Then there exists $\alpha \in \Gamma$ such that $e \in E_\alpha(S)$. By Proposition 3.2.5, we have that $f, g \in E_\alpha(S)$. Thus $f = f\alpha g$ and $g = g\alpha f$. By hypothesis, we get that

$$f = f\alpha g \preccurlyeq e\alpha g = g \text{ and } g = g\alpha f \preccurlyeq e\alpha f = f.$$

Therefore $f = g$.

(2) $\Rightarrow$ (3) Let $a, b, c \in S$ be such that $a \leqslant c$, $b \leqslant c$ and $a\mathcal{L}b$. Then $a = e\alpha c = c\beta f$ for some $e \in E_\alpha(S), f \in E_\beta(S), \alpha, \beta \in \Gamma$. Let $c' \in V_\gamma^\delta(c)$ for some $\gamma, \delta \in \Gamma$. It follows that

$$c'\delta a = (c'\delta c)\gamma(c'\delta a),$$

$$c'\delta a = c'\delta e\alpha c\gamma c'\delta c = (c'\delta a)\gamma(c'\delta c)$$

and

$$c'\delta a = c'\delta e\alpha c\beta f = (c'\delta a)\gamma(c'\delta a)$$

which proves that $c'\delta a \leqslant c'\delta c$. Thus $a = c\gamma c'\delta c\beta f = c\gamma c'\delta a$, we have that $a\mathcal{L}c'\delta a$. Similarly, since $b \leqslant c$ we have $b = e_1\alpha_1 c = c\beta_1 f_1$ for some $e_1 \in E_{\alpha_1}(S), f_1 \in E_{\beta_1}(S), \alpha_1, \beta_1 \in \Gamma$. Then

$$c'\delta b = (c'\delta c)\gamma(c'\delta b),$$

$$c'\delta b = c'\delta e_1\alpha_1 c\gamma c'\delta c = (c'\delta b)\gamma(c'\delta c)$$

and

$$c'\delta b = c'\delta e_1\alpha_1 c\beta_1 f_1 = (c'\delta b)\gamma(c'\delta b)$$

which proves that $c'\delta b \leqslant c'\delta c, b = c\gamma c'\delta b$ and $b\mathcal{L}c'\delta b$ with $c'\delta b \in E_\gamma(S)$. This implies that $c'\delta a\mathcal{L}c'\delta b$. By the hypothesis, we obtain $c'\delta a = c'\delta b$. Therefore $a = c\gamma c'\delta a = c\gamma c'\delta b = b$.

(3) $\Rightarrow$ (1) Let $a \leqslant b$. By Theorem 3.2.1, $a = e\alpha b = b\beta f$ for some $e \in E_\alpha(S), f \in E_\beta(S), \alpha, \beta \in \Gamma$. Also, let $c \in S, \theta \in \Gamma$ and $x \in V_\gamma^\delta(a\theta c)$ for some $\gamma, \delta \in \Gamma$. Then

$$b\theta(c\gamma x\delta a) = (b\theta c\gamma x\delta e)\alpha b,$$

$$(c\gamma x\delta a)\theta(c\gamma x\delta a) = c\gamma x\delta a,$$

and

$$(b\theta c\gamma x\delta e)\alpha(b\theta c\gamma x\delta e) = b\theta c\gamma x\delta a\theta c\gamma x\delta e = b\theta c\gamma x\delta e$$

which proves that $b\theta c\gamma x\delta a \leqslant b$. Again, we have that

$$a\theta c\gamma x\delta a = b\beta(f\theta c\gamma x\delta a) = (a\theta c\gamma x\delta e)\alpha b,$$

$$(f\theta c\gamma x\delta a)\beta(f\theta c\gamma x\delta a) = f\theta c\gamma x\delta a$$

and

$$(a\theta c\gamma x\delta e)\alpha(a\theta c\gamma x\delta e) = a\theta c\gamma x\delta e$$

which give $a\theta c\gamma x\delta a \leqslant b$. It is easy to show that $(b\theta c\gamma x\delta a)\mathcal{L}(a\theta c\gamma x\delta a)$. By the hypothesis, we get that $b\theta c\gamma x\delta a = a\theta c\gamma x\delta a$. Since $x \in V_\gamma^\delta(a\theta c)$, we get that

$$a\theta c = a\theta c\gamma x\delta a\theta c = b\theta c\gamma x\delta a\theta c \text{ and } a\theta c = e\alpha b\theta c$$

with $x\delta a\theta c \in E_\gamma(S)$. We conclude that $a\theta c \leqslant b\theta c$. □

Dually, we get the following statements.

Corollary 3.3.2. *Let S be a regular Γ-semigroup. Then the following statements are equivalent.*

(1) *$\leqslant$ is left compatible.*

(2) *S satisfies $\mathcal{R}$-majorization for idempotents.*

(3) *S satisfies $\mathcal{R}$-majorization.*

Proof. The proof is similar to that of Theorem 3.3.1. □

Theorem 3.3.3. *Let S be a regular Γ-semigroup. Then the following statements are equivalent.*

(1) *$\leqslant$ is compatible.*

(2) *S satisfies $\mathcal{L}$- and $\mathcal{R}$-majorization for idempotents.*

(3) *S satisfies $\mathcal{L}$- and $\mathcal{R}$-majorization.*

Proof. It follows from Theorem 3.3.1 and Corollary 3.3.2. □

Theorem 3.3.4. *Let S be a regular Γ-semigroup and the natural partial order on S be compatible with multiplication. Then*

$$\omega := \{(a,b) \in S \times S \mid c \leqslant a \text{ and } c \leqslant b \text{ for some } c \in S\}$$

is a congruence on S.

Proof. Note that ω is reflexive and symmetric. Next, we will show that ω is transitive. Let $(a,b),(b,c) \in \omega$. Then there exist $x,y \in S$ such that $x \leqslant a, x \leqslant b$ and $y \leqslant b, y \leqslant c$. It implies that $x = f\beta b$ and $y = b\alpha e$ for some $f \in E_\beta(S), e \in E_\alpha(S), \beta, \alpha \in \Gamma$. Then

$$x\alpha e = f\beta b\alpha e = f\beta y.$$

Set $z := x\alpha e = f\beta y$. By hypothesis and $x \leqslant b$ we get that $z = x\alpha e \leqslant b\alpha e = y$ and $y \leqslant b$ implies that $z = f\beta y \leqslant f\beta b = x$, so $z \leqslant x \leqslant a$ and $z \leqslant y \leqslant c$. It implies that $(a,c) \in \omega$. By hypothesis, ω is compatible.

Therefore ω is a congruence on S. □

A non-zero element of a regular Γ-semigroup S is **primitive** if it is minimal among the non-zero elements of S. A regular Γ-semigroup S is said to be **primitive** if each of its non-zero idempotents is primitive. A congruence ρ on a regular Γ-semigroup S is called **primitive** if S/ρ is primitive. Clearly, if S is trivially ordered then S is primitive.

A mapping $\phi : X \to Y$ of a quasi-ordered set $(X, \leqslant_X)$ into a quasi-ordered set $(Y, \leqslant_Y)$ **reflecting** [3] if for all $y, y' \in X\phi$ such that $y' \leqslant_Y y$ and $x \in X$ with $x\phi = y$ there is some $x' \in X$ such that $x' \leqslant_X x$ and $x'\phi = y'$.

Theorem 3.3.5. *Let S be a regular Γ-semigroup such that ω is a congruence and the natural homomorphism for ω is reflecting the natural partial order. Then ω is the least primitive congruence on S.*

Proof. Define the natural homomorphism $\varphi : S \to S/\omega$ by $s\varphi = s\omega$ for all $s \in S$. We will show that S/ω is trivially ordered. Let $y, z \in S/\omega$ be such that $y \leqslant z$. Since φ is reflecting the natural partial order, there exist $s, t \in S$ such that $s \leqslant t$ and $s\varphi = y, t\varphi = z$. Since $s \leqslant t$, we now get that $s\omega t$. Thus

$$y = s\varphi = s\omega = t\omega = t\varphi = z.$$

Therefore S/ω is trivially ordered.

Let ρ be any congruence on S/ω such that $(S/\rho, \leqslant)$ is trivially ordered and let ψ denotes the natural homomorphism corresponding ρ. Suppose that $s\omega t$. There exists $w \in S$ such that $w \leqslant s$ and $w \leqslant t$, giving $w\psi \leqslant s\psi$ and $w\psi \leqslant t\psi$ in S/ρ. Since S/ρ is trivially ordered, we obtain that $s\psi = w\psi = t\psi$, so $s\rho = t\rho$. Thus $s\rho t$ immediately implies that $\omega \subseteq \rho$.

Therefore ω is the least primitive congruence on S. $\square$