CHAPTER I

INTRODUCTION

The various relations are used in the study of the structure and properties of semigroups. Green's relations and the natural partial orders are important relations which are most notable and useful tool in semigroup theory. We know that Green's relations are the equivalence relations. The well-known natural partial order for a regular semigroup play an important role in the structure of regular semigroup. Many authors studied partial orders on semigroups and special class of semigroups.

In 1952, Vagner [1] introduced a natural partial order on inverse semigroups as follows:

$$a \leqslant b \text{ if and only if } a = eb \text{ for some } e \in E(S), \tag{1.1}$$

where $E(S)$ denotes the set of all idempotents of S. Later, Mitsch [2] defined the natural order on an inverse semigroup S by

$$a \leqslant b \text{ if and only if } ab' = aa' \tag{1.2}$$

where a', b' denote the unique inverse of a and b respectively and showed that the partial orders (1.1) and (1.2) are equivalent. Moreover, an inverse semigroup S is totally ordered with respect to its natural order if and only if $ab = ba = a$ or $ab = ba = b$ for all $a, b \in S$. Furthermore, Nambooripad [3] defined a partial order $\leqslant$ on a regular semigroup S by

$$a \leqslant b \text{ if and only if } R_a \leqslant R_b \text{ and } a = fb \text{ for some } f \in E(R_a) \tag{1.3}$$

where $R_a \leqslant R_b$ if and only if $S^1a \subseteq S^1b$ and $E(R_a)$ denotes the set of all idempotents in R_a, that coincides with the relation defined above on an inverse semigroup. Such the relation $\leqslant$ is called the natural partial order on inverse semigroups S. We see that the relation (1.3) is a generalization of the relation (1.1). Moreover Nambooripad [3] proved that the natural partial order on a regular subsemigroup

T of a regular semigroup S is the restriction of the natural partial order on S to T. Mitsch [4] used properties of the natural partial order on a regular semigroup to define the natural partial order on a semigroup. The relation [4] $\leqslant$ on a semigroup S is defined by

$$a \leqslant b \text{ if and only if } a = xb = by \text{ and } xa = a \text{ for some } x, y \in S^1. \quad (1.4)$$

Then the relation $\leqslant$ is a partial order on a semigroup.

In 1994, Mitsch [5] studied certain properties of the natural partial order with respect to the structure of a semigroup. For example, if S^2 is regular on a semigroup S then the natural partial order on S is the right compatible with multiplication if and only if S satisfies $\mathcal{L}$-majorization. If the natural partial order on a semigroup S is compatible with multiplication then

$$\omega := \{(a, b) \in S \times S \mid c \leqslant a \text{ and } c \leqslant b \text{ for some } c \in S\}$$

is a congruence on S.

Petrich [6] used a definition of the partial order on a regular semigroup S as follows:

$$a \leqslant b \text{ if and only if } a = eb = bf \text{ for some } e, f \in E(S) \qquad (1.5)$$

and showed that the following statements are equivalent.

(1) The natural partial order $\leqslant$ is compatible on regular semigroup S.

(2) S is locally inverse semigroup.

(3) S satisfies $\mathcal{L}$- and $\mathcal{R}$-majorization.

Next, Srinivas [7] proved that $E(S)$ is a normal band if and only if the natural partial order (1.5) and μ coincide on a subsemigroup $Reg(S)$ where

$$\mu := \{(a, b) \in S \times S \mid sa = sb = a = at = bt \text{ for some } s, t \in S^1\}.$$

The concept of sandwich sets in regular semigroups was first introduced by Nambooripad [8]. The sandwich set had a benefit in a structure theory and used to simplify the proof of regular semigroups. Nambooripad [8] studied the sandwich set $S(e, f)$ of idempotents e and f in semigroups and described successfully the structure of regular semigroups. He showed that the sandwich set $S(e, f)$ is a subsemigroup of a regular semigroup. In 1981, Nambooripad and Pastijn [9] used the concept of a sandwich set $S(e, f)$ and given a characterization of a V-regular semigroup in terms of its partial band of idempotents. Koch [10] studied properties of the sandwich set $S(x_1, x_2)$ in a regular semigroup S and presented functional relationships between elements of $S(x_1, x_2)$ pairs of idempotents in an $\mathcal{L}$-class containing x_1x_2 and an $\mathcal{R}$-class containing x_1x_2 and pairs of elements in the intersection of $\downarrow x_2$ and $\mathcal{L}$-class containing x_1x_2 and in the intersection of $\downarrow x_1$ and $\mathcal{R}$-class containing x_1x_2 where $\downarrow x_i = \{x \in x_iS \mid x = ex_i$ and $e\mathcal{R}x$ for some $e \in E(S)\}$. Zhu and He [11] used the concept of a sandwich set $S(e, f)$ and showed that the sandwich set on some special classes of semigroups has only one element. Petrich [12] introduced one-sided sandwich sets $S(e, f)f$ and $eS(e, f)$ where e, f are idempotents on a regular semigroup and characterized them. He showed that the mappings between $S(e, f)$ and $S(e, f)f \times eS(e, f)$ are mutually inverse isomorphisms. Gao and Wang [13] showed that an inverse subsemigroup $S^0 := S \cup \{0\}$ of a regular semigroup S is a strongly E-inverse transversal if and only if $fe \in S(e, f)$ for all $e, f \in E(S)$. Properties of partial orders and the sandwich set of semigroups are appearance in [14], [15].

The concept of Γ-semigroups has been introduced by Sen [16] in 1981. In 1986, Sen and Saha [17] changed the definition, which is more general definition. Later, Sen and Saha [18] gave the definition of the Γ-semigroup via a mapping as follows: A nonempty set S is called a *Γ-semigroup* if there exists a mapping from $S \times \Gamma \times S$ to S written as $(a, \alpha, b) \mapsto a\alpha b$ satisfying the identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Many authors tried to transfer results of semigroups to Γ-semigroup and some of them used the definition of a Γ-semigroup introduced by Sen in 1981 and 1986. For examples, Sen and Saha [18] have been extended results of orthodox semigroups to orthodox Γ-semigroups and showed that every inverse Γ-semigroup is an orthodox Γ-semigroup. Moreover, they have also shown that the relation

$$\rho := \{(a,b) \in S \times S \mid V_\alpha^\beta(a) = V_\alpha^\beta(b) \text{ for all } \alpha, \beta \in \Gamma\}$$

is the minimum inverse Γ-semigroup congruence in an orthodox Γ-semigroup S.

Saha [19] has been established a maximum idempotent separating congruence on an inverse Γ-semigroup S and proved that the relation

$$\begin{aligned}\mu := \{(a,b) \in S \times S \mid & \text{ there exist } \gamma, \delta \in \Gamma, a' \in V_\gamma^\delta(a), b' \in V_\gamma^\beta(b) \\ & \text{such that } a\alpha e\gamma a' = b\alpha e\gamma b' \text{ for all } e \in E_\alpha(S)\}\end{aligned}$$

is the maximum idempotent separating congruence on S. Siripitukdet and Iampan [20] studied bands of weakly r-archimedean Γ-semigroups and showed that a Γ-semigroup S is a band of weakly r-archimedean left ideals of S if and only if S is a band of r-archimedean left ideals of S.

The notions of regular and orthodox Γ-semigroups have been studied and developed in [21] and [22]. Krishnamoorthy and Arul Doss [23] have first introduced an (α, α) sandwich set of α-idempotents e, f of a commuting regular Γ-semiring S by

$$S_\alpha^\alpha(e,f) := \{g \in V_\alpha^\alpha(e\alpha f) \cap E(S) \mid g\alpha e = f\alpha g = g\}$$

and showed that the (α, α) sandwich set is a regular sub Γ-semiring of S.

Motivated and inspired by the above works, the purposes of this research will be to introduce the (α, β, θ)-sandwich set and the natural partial order on regular Γ-semigroups and to find the least primitive congruence on a regular Γ-semigroup. Moreover, we find necessary and sufficient conditions for an (α, β, θ)-sandwich set of elements in a regular Γ-semigroup so that it is a right θ-zero semigroup and find the mutually inverse θ-isomorphism between (α, β, θ)-sandwich set and cartesian product of one-sided (α, β, θ)-sandwich set .

This dissertation is organized into five chapters as follows. Chapter I is an introduction to the research problem. Chapter II, we collect definitions, examples and results, mainly without proof, to be used throughout the entire dissertation. In Chapter III, we divide into three sections. The first section of Chapter III, we studied some properties of idempotent elements of Green's relations on regular Γ-semigroups. The second section of Chapter III, we defined the natural partial order on regular Γ-semigroups and partial order on the set of all idempotents of regular Γ-semigroups and studied relationship between the natural partial order on regular Γ-semigroups and partial order on the set of all idempotents of regular Γ-semigroups. The third section of Chapter III, we studied the least primitive congruence on regular Γ-semigroups. In Chapter IV, we divide into three sections. The first section of Chapter IV, we defined the (α, β, θ)-sandwich set of idempotent elements and studied the mutually inverse θ-isomorphism between (α, β, θ)-sandwich set of idempotent elements and cartesian product of one-sided (α, β, θ)-sandwich set of idempotent elements on a regular Γ-semigroup. The second section of Chapter IV, we define the (α, β, θ)-sandwich set of an element on regular Γ-semigroups and give necessary and sufficient conditions for an (α, β, θ)-sandwich set of elements so that it is a right θ-zero semigroup. The third section of Chapter IV, we studied the finest primitive congruence on regular Γ-semigroups and studied a Γ-semigroup which is completely simple.