CHAPTER III

METHODS FOR FINDING SOLUTIONS OF GENERALIZED NON-STRONGLY MONOTONE VARIATIONAL INEQUALITY PROBLEMS

In this chapter, the main results of this thesis will be presented. Recall that we will consider the following problem: Find $u^* \in H, g(u^*) \in S$ such that

$$\langle A(u^*), g(v) - g(u^*)\rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{3.0.1}$$

where A is a kind of monotone mapping, $\{A_i\}_{i=1}^N$ is a finite family of λ_i-inverse strongly monotone mappings from K into H and $S = \bigcap_{i=1}^N S_i$, $S_i = \{x \in K : A_i(x) = 0\}$, denoted by $GVI_K(A, g, S)$ for a set of solutions of the problem (3.0.1).

3.1 Regularization for general variational inequality problem in Hilbert spaces.

For each $\alpha \in (0, 1)$ we now construct a regularization solution u_α for (3.0.1) by solving the following general variational inequality problem: find $u_\alpha \in H$, $g(u_\alpha) \in K$ such that

$$\langle A(u_\alpha) + \alpha^\mu \sum_{i=1}^{N} (A_i \circ g)(u_\alpha) + \alpha g(u_\alpha), g(v) - g(u_\alpha)\rangle \geq 0, \tag{3.1.1}$$

$\forall v \in H,\ g(v) \in K, 0 < \mu < 1.$

We start with useful lemmas.

Lemma 3.1.1. *Let K be a closed convex subset of a real Hilbert space H. If $A : H \to H$ is a hemicontinuous and g-monotone mapping and A_i is a λ_i-inverse strongly monotone mapping of K into H, then functions $F_0, F_i : g^{-1}(K) \times g^{-1}(K) \longrightarrow \mathbb{R}$ defined by*

$$F_0(u, v) = \langle A(u), g(v) - g(u)\rangle \text{ and } F_i(u, v) = \langle (A_i \circ g)(u), g(v) - g(u)\rangle, \tag{3.1.2}$$

for all $(u, v) \in g^{-1}(K) \times g^{-1}(K)$, *and* $i = 1, 2, ..., N$, *are equilibrium monotone bi-functions on* $g^{-1}(K)$, *for each* $i = 1, 2, ..., N$.

Proof. Let $u, v \in g^{-1}(K)$, Since A is a g-monotone mapping, we have

$$\begin{aligned} F_0(u,v) + F_0(v,u) &= \langle A(u), g(v) - g(u)\rangle + \langle A(v), g(u) - g(v)\rangle, \\ &= \langle A(u), g(v) - g(u)\rangle - \langle A(v), g(v) - g(u)\rangle, \\ &= \langle A(u) - A(v), g(v) - g(u)\rangle, \\ &= -\langle A(u) - A(v), g(u) - g(v)\rangle, \\ &\leq 0. \end{aligned}$$

Therefore F_0 is a monotone bi-function on $g^{-1}(K)$.

Next, we show that F_i is a monotone bi-function on $g^{-1}(K)$ for all $i = 1, 2, ..., N$. Let $u, v \in g^{-1}(K)$, Since A_i is a λ_i-inverse strongly monotone mapping, we have

$$\begin{aligned} F_i(u,v) + F_i(v,u) &= \langle (A_i \circ g)(u), g(v) - g(u)\rangle + \langle (A_i \circ g)(v), g(u) - g(v)\rangle, \\ &= \langle (A_i \circ g)(u), g(v) - g(u)\rangle + \langle (A_i \circ g)(v), g(v) - g(u)\rangle, \\ &= \langle A_i \circ g)(u) - A_i \circ g)(v), g(v) - g(u)\rangle, \\ &= \langle A_i(g(u)) - A_i(g(v)), g(v) - g(u)\rangle, \\ &= -\langle A_i(g(u)) - A_i(g(v)), g(u) - g(v)\rangle, \\ &\leq 0. \end{aligned}$$

Therefore F_i is a monotone bi-function on $g^{-1}(K)$, for all $i = 1, 2, ..., N$. □

Let K be a closed convex subset of a real Hilbert space H. For each $\alpha \in (0, 1)$, defined a function $F_\alpha : g^{-1}(K) \times g^{-1}(K) \longrightarrow \mathbb{R}$ by

$$F_\alpha(u,v) = F_0(u,v) + \alpha^\mu \sum_{i=1}^{N} F_i(u,v) + \alpha\langle g(u), g(v) - g(u)\rangle, \tag{3.1.3}$$

for all $(u, v) \in g^{-1}(K) \times g^{-1}(K)$, where F_0 and F_i defined by 3.1.2.

Lemma 3.1.2. *If $A : H \to H$ is a hemicontinuous and g-monotone mapping, A_i is a λ_i-inverse strongly monotone mapping of K into H and g is a continuous mapping, then a function F_α defined by (3.1.3) is a monotone hemicontinuous mapping in the variable u for each fixed $v \in g^{-1}(K)$, for each $\alpha \in (0,1)$.*

Proof. Let $u, v \in g^{-1}(K)$, by Lemma 3.1.1 we have F_0 and F_i are equilibrium monotone bi-functions on $g^{-1}(K)$, for each $i = 1, 2, ..., N$. This gives

$$\begin{aligned} F_\alpha(u,v) + F_\alpha(v,u) &= F_0(u,v) + \alpha^\mu \sum_{i=1}^{N} F_i(u,v) + \alpha\langle g(u), g(v) - g(u)\rangle \\ &\quad + F_0(v,u) + \alpha^\mu \sum_{i=1}^{N} F_i(v,u) + \alpha\langle g(v), g(u) - g(v)\rangle, \\ &= F_0(u,v) + F_0(v,u) + \alpha^\mu \sum_{i=1}^{N} [F_i(u,v) + F_i(v,u)] \\ &\quad + \alpha\langle g(u) - g(v), g(v) - g(u)\rangle \\ &= F_0(u,v) + F_0(v,u) + \alpha^\mu \sum_{i=1}^{N} [F_i(u,v) + F_i(v,u)] \\ &\quad - \alpha\|g(u) - g(v)\|^2 \\ &\leq 0. \end{aligned} \tag{3.1.4}$$

Therefore F_α is a monotone bi-function on $g^{-1}(K)$.

Next, we show that F_α is hemicontinuous in the variable u for each fixed $v \in g^{-1}(K)$. For each fixed $v \in g^{-1}(K)$, let $(u, z) \in g^{-1}(K) \times g^{-1}(K)$, we have

$$\begin{aligned} \lim_{t \to +0} F_\alpha(u + t(z-u), v) &= \lim_{t \to +0} [F_0(u + t(z-u), v) + \alpha^\mu \sum_{i=1}^{N} F_i(u + t(z-u), v) \\ &+ \alpha\langle g(u + t(z-u)), g(v) - g(u + t(z-u))\rangle]. \end{aligned}$$

Consider, since A is a hemicontinuous mapping, we have

$$\begin{aligned} \lim_{t \to +0} F_0(u + t(z-u), v) &= \lim_{t \to +0} \langle A(u + t(z-u)), g(v) - g(u + t(z-u))\rangle, \\ &= \langle \lim_{t \to +0} A(u + t(z-u)), g(v) - \lim_{t \to +0} g(u + t(z-u))\rangle, \\ &= \langle A(u), g(v) - g(u)\rangle, \end{aligned}$$

$$
\begin{aligned}
&= \langle A(u), g(v) - g(u)\rangle, \\
&= F_0(u,v),
\end{aligned}
$$

and, since A_i is a Lipschitz continuous mapping, we have

$$
\begin{aligned}
\lim_{t\to+0} F_i(u+t(z-u),v) &= \lim_{t\to+0}\langle (A_i\circ g)(u+t(z-u)), g(v) - g(u+t(z-u))\rangle, \\
&= \langle \lim_{t\to+0}(A_i\circ g)(u+t(z-u)), g(v) - \lim_{t\to+0} g(u+t(z-u))\rangle, \\
&= \langle (A_i\circ g)(u), g(v)-g(u)\rangle, \\
&= F_i(u,v).
\end{aligned}
$$

It implies that

$$
\begin{aligned}
\lim_{t\to+0} F_\alpha(u+t(z-u),v) &= F_0(u,v) + \alpha^\mu \sum_{i=1}^{N} F_i(u,v) + \alpha\langle g(u), g(v)-g(u)\rangle, \\
&= F_\alpha(u,v).
\end{aligned}
$$

Therefore F_α is a hemicontinuous mapping in the variable u for each fixed $v \in g^{-1}(K)$. □

Lemma 3.1.3. *Let $A : H \to H$ be a hemicontinuous and g-monotone mapping and A_i be a λ_i-inverse strongly monotone mapping of K into H. If $g : H \to H$ is an ξ-expanding mapping such that $K \subset g(H)$, then a function F_α defined by (3.1.3) is a strongly monotone mapping with constant $\alpha\xi^2 > 0$, for each $\alpha \in (0,1)$.*

Proof. By (3.1.4) and g is an ξ-expanding mapping, we have

$$
\begin{aligned}
F_\alpha(u,v) + F_\alpha(v,u) &= F_0(u,v) + F_0(v,u) + \alpha^\mu \textstyle\sum_{i=1}^{N} [F_i(u,v) + F_i(v,u)] \\
&\quad -\alpha\|g(u) - g(v)\|^2, \\
&\le -\alpha\|g(u)-g(v)\|^2, \\
&\le -\alpha\xi^2\|u-v\|^2.
\end{aligned}
$$

Therefore F_α is a strongly monotone mapping with constant $\alpha\xi > 0$. □

Theorem 3.1.4. *Let K be a closed convex subset of a real Hilbert space H and $g : H \to H$ be a mapping such that $K \subset g(H)$. Let $A : H \to H$ be a hemicontinuous and g-monotone mapping. Let A_i be a λ_i-inverse strongly monotone mapping of K into H, for each $i = 1, 2, ..., N$. If g is an expanding affine continuous mapping and $GVI_K(A, g, S) \neq \emptyset$, then the following conclusions are true:*

(a) For each $\alpha \in (0, 1)$, the problem (3.1.1) has the unique solution u_α.

(b) $\lim_{\alpha \to 0+} g(u_\alpha) = g(u^*)$, for some $u^* \in GVI_K(A, g, S)$.

(c) There exists a positive constant M such that

$$\|g(u_\alpha) - g(u_\beta)\|^2 \leq \frac{M|\alpha - \beta|}{\alpha^2}, \quad \text{for all } \alpha, \beta \in (0, 1). \tag{3.1.5}$$

Proof. Firstly, due to the definition of the λ_i-inverse strongly monotone mapping A_i, we may always assume that $\lambda_i \in \left(0, \frac{1}{2}\right]$ for each $i = 1, 2, ..., N$. Now, we define functions F_0, F_i by (3.1.2). Moreover, by Lemma 3.1.1, F_0 and F_i are equilibrium monotone bi-functions on $g^{-1}(K)$, for each $i = 1, 2, ..., N$.

Now, let $\alpha \in (0, 1)$ be a given real number. We construct a function F_α defined by (3.1.3).

(a) Observe that, the problem (3.1.1) is equivalent to the problem of finding $u_\alpha \in g^{-1}(K)$ such that

$$F_\alpha(u_\alpha, v) \geq 0, \qquad \forall v \in g^{-1}(K). \tag{3.1.6}$$

Moreover, by Lemma 3.1.2 we have $F_\alpha(u, v)$ is monotone hemicontinuous in the variable u for each fixed $v \in g^{-1}(K)$. By Lemma 3.1.3, since g is an ξ-expanding, we have $F_\alpha(u, v)$ is a strongly monotone mapping with constant $\alpha\xi^2 > 0$. Further, since g is an affine continuous mapping, we have $g^{-1}(K)$ is a closed convex subset of H. Thus, by Lemma 2.2.10(b), the problem (3.1.6) has the unique solution $u_\alpha \in g^{-1}(K)$, for each $\alpha > 0$. This proves (a).

(b) Observe that for each $y \in GVI_K(A,g,S)$, we have $F_0(y,u_\alpha) \geq 0$ and $F_i(y,u_\alpha) = 0$, for $i = 1, ..., N$. These imply

$$F_0(y,u_\alpha) + \alpha^\mu \sum_{i=1}^{N} F_i(y,u_\alpha) \geq 0, \quad \forall y \in GVI_K(A,g,S). \tag{3.1.7}$$

Consequently, by using the monotonicity of F_0 and F_i, we see that (3.1.3), (3.1.6) and (3.1.7) imply

$$\begin{aligned}
0 &\geq -F_\alpha(u_\alpha, y) \\
&= -[F_0(u_\alpha,y) + \alpha^\mu \sum_{i=1}^{N} F_i(u_\alpha,y) + \alpha\langle g(u_\alpha), g(y) - g(u_\alpha)\rangle] \\
&\geq -F_0(u_\alpha,y) - \alpha^\mu \sum_{i=1}^{N} F_i(u_\alpha,y) - \alpha\langle g(u_\alpha), g(y) - g(u_\alpha)\rangle \\
&\qquad - [F_0(y,u_\alpha) + \alpha^\mu \sum_{i=1}^{N} F_i(y,u_\alpha)] \\
&= -[F_0(u_\alpha,y) + F_0(y,u_\alpha)] - \alpha^\mu \sum_{i=1}^{N} [F_i(u_\alpha,y) + F_i(y,u_\alpha)] \\
&\qquad - \alpha\langle g(u_\alpha), g(y) - g(u_\alpha)\rangle \\
&\geq \alpha\langle g(u_\alpha), g(u_\alpha) - g(y)\rangle,
\end{aligned} \tag{3.1.8}$$

for all $y \in GVI_K(A,g,S)$.

Hence,

$$\langle g(u_\alpha), g(y) - g(u_\alpha)\rangle \geq 0, \quad \text{for all } y \in GVI_K(A,g,S).$$

This implies,

$$\|g(u_\alpha)\| \leq \|g(y)\|, \quad \text{for all } y \in GVI_K(A,g,S). \tag{3.1.9}$$

Thus $\{g(u_\alpha)\}$ is a bounded net of K. Consequently, the set of weak limit points as $\alpha \to 0$ of the net $(g(u_\alpha))$, denotes by $\omega_w(g(u_\alpha))$, is nonempty. This allows us to pick $z \in \omega_w(g(u_\alpha))$ and a null sequence $\{\alpha_k\}$ in the interval $(0,1)$ such that $\{g(u_{\alpha_k})\}$ weakly converges to z as $k \to \infty$. Notice that, since K is closed and convex, this implies that $z \in K$. Consequently, since $K \subset g(H)$, we let $u^* \in H$ be such that $z = g(u^*)$. We claim that $u^* \in GVI_K(A, g, S)$.

To prove such a claim, we divide it into two steps:

Step 1: We will show that $g(u^*) \in S$.

Let us observe that, $F_0(y, u_{\alpha_k}) \geq 0$ for each $y \in GVI_K(A, g, S)$ and $k \in \mathbb{N}$. By using this observation together with (3.1.3) and the monotonicity of F_0, we see that

$$\alpha_k^\mu \sum_{i=1}^{N} F_i(u_{\alpha_k}, y) + \alpha_k \langle g(u_{\alpha_k}), g(y) - g(u_{\alpha_k}) \rangle \geq -F_0(u_{\alpha_k}, y) \geq F_0(y, u_{\alpha_k}) \geq 0,$$

that is,

$$\sum_{i=1}^{N} F_i(u_{\alpha_k}, y) + \alpha_k^{1-\mu} \langle g(u_{\alpha_k}), g(y) - g(u_{\alpha_k}) \rangle \geq 0, \qquad (3.1.10)$$

for each $y \in GVI_K(A, g, S)$ and $k \in \mathbb{N}$.

Now we consider for each $j \in \{1, 2, ..., N\}$. Let $y \in GVI_K(A, g, S)$ be picked. Since A_j is a λ_j-inverse strongly monotone mapping, in view of (3.1.9) we have

$$\begin{aligned}
\lambda_j \|A_j \circ g(u_{\alpha_k}) - A_j \circ g(y)\|^2 &\leq \langle A_j(g(u_{\alpha_k})) - A_j(g(y)), g(u_{\alpha_k}) - g(y) \rangle \\
&= \langle A_j(g(u_{\alpha_k})), g(u_{\alpha_k}) - g(y) \rangle \\
&\leq \sum_{i=1}^{N} \langle A_i(g(u_{\alpha_k})), g(u_{\alpha_k}) - g(y) \rangle \\
&= -\sum_{i=1}^{N} F_i(u_{\alpha_k}, y) \\
&\leq \alpha_k^{1-\mu} \langle g(u_{\alpha_k}), g(y) - g(u_{\alpha_k}) \rangle \\
&\leq \alpha_k^{1-\mu} \left[\|g(u_{\alpha_k})\| \|g(y)\| - \|g(u_{\alpha_k})\|^2 \right] \\
&\leq \alpha_k^{1-\mu} \|g(y)\|^2, \qquad (3.1.11)
\end{aligned}$$

for each $k \in \mathbb{N}$. Letting $k \to \infty$ in (3.1.11), we obtain

$$\lim_{k \to \infty} \|A_j \circ g(u_{\alpha_k})\| = \lim_{k \to \infty} \|A_j \circ g(u_{\alpha_k}) - A_j \circ g(y)\| = 0. \qquad (3.1.12)$$

On the other hand, we know that the mapping $T_j := I - A_j$ is strictly pseudocontractive, hence by Lemma 2.2.12, we have $A_j = I - T_j$ is demiclosed at zero. It follows that $A_j(g(u^*)) = 0$. This means $g(u^*) \in S_j$. Consequently, since $j \in \{1, 2, ..., N\}$ is chosen arbitrary, we conclude that $g(u^*) \in \bigcap_{i=1}^{N} S_i =: S$. This proves Step 1.

Step 2: We will show that $u^* \in GVI_K(A, g)$, where $GVI_K(A, g)$ is denoted for the solution set of the problem (1.0.2).

From the monotonic property of F_α and (3.1.6), we have

$$\begin{aligned} F_0(v, u_{\alpha_k}) + \alpha_k^\mu \textstyle\sum_{i=1}^{N} F_i(v, u_{\alpha_k}) + \alpha_k \langle g(v), g(u_{\alpha_k}) - g(v)\rangle &= F_\alpha(v, u_{\alpha_k}) \\ &\leq -F_\alpha(u_{\alpha_k}, v) \\ &\leq 0, \end{aligned}$$

for all $v \in g^{-1}(K)$. This gives,

$$F_0(v, u_{\alpha_k}) + \alpha_k^\mu \sum_{i=1}^{N} F_i(v, u_{\alpha_k}) \leq \alpha_k \langle g(v), g(v) - g(u_{\alpha_k})\rangle, \qquad \forall v \in g^{-1}(K). \quad (3.1.13)$$

Notice that, since the operator A_i is a Lipschitzian mapping, we have F_i is a bounded mapping for each $i = 1, ..., N$. Thus by letting $k \to \infty$, since $\alpha_k \to 0^+$ and $g(u_{\alpha_k}) \rightharpoonup g(u^*)$ as $n \to \infty$, from (3.1.13) we see that $F_0(v, u^*) \leq 0$, for any $v \in H, g(v) \in K$. Consequently, in view of Lemma 2.2.10(a) and Lemma 3.1.2, Step 2 is proved.

Hence, from Steps (1) and (2), we conclude that $u^* \in GVI_K(A, g, S)$ as required.

Next, we observe that the sequence $\{g(u_{\alpha_k})\}$ actually converges to $g(u^*)$ strongly. In fact, by using a lower semi-continuous of norm, we know that

$$\|g(u^*)\| \leq \liminf_{k\to\infty} \|g(u_{\alpha_k})\|. \quad (3.1.14)$$

Consequently, since $u^* \in GVI_K(A, g, S)$, we see that (3.1.9) and (3.1.14) imply $\|g(u_{\alpha_k})\| \to \|g(u^*)\|$ as $k \to \infty$. Then, it is straightforward from Lemma 2.1.12, that the weak convergence to $g(u^*)$ of $\{g(u_{\alpha_k})\}$ implies strong convergence to $g(u^*)$ of $\{g(u_{\alpha_k})\}$. Moreover, in view of (3.1.9), we see that

$$\|g(u^*)\| = \inf\{\|g(y)\| : y \in GVI_K(A, g, S)\}. \tag{3.1.15}$$

Now we show that

$$\lim_{\alpha \to 0^+} g(u_\alpha) = g(u^*).$$

Let $\{g(u_{\alpha_j})\} \subset (g(u_\alpha))$, where $\{\alpha_j\}$ be any null sequence in the interval $(0,1)$. By following the lines proof as above, and passing to a subsequence if necessary, we know that there is $\tilde{u} \in GVI_K(A, g, S)$ such that $g(u_{\alpha_j}) \to g(\tilde{u})$ as $j \to \infty$. Moreover, from (3.1.9) and (3.1.15), we have $\|g(\tilde{u})\| = \|g(u^*)\|$. Consequently, since the function $\|g(\cdot)\|$ is a lower semi-continuous function and $GVI_K(A, g, S)$ is a closed convex subset of H, we see that (3.1.15) gives $u^* = \tilde{u}$. This implies that $g(u^*)$ is the strong limit of the net $(g(u_\alpha))$ as $\alpha \to 0^+$.

(c) Let $\alpha, \beta \in (0,1)$ and u_α, u_β are solutions of the problem (3.1.1), associated with α and β, respectively. Without loss of generality, we will assume that $\alpha < \beta$.

Thus, since F_0 and F_i are monotone mappings, by applying (3.1.6), we have

$$(\alpha^\mu - \beta^\mu)\sum_{i=1}^{N} F_i(u_\alpha, u_\beta) + \alpha\langle g(u_\alpha), g(u_\beta) - g(u_\alpha)\rangle + \beta\langle g(u_\beta), g(u_\alpha) - g(u_\beta)\rangle \geq 0,$$

that is,

$$\left\langle g(u_\alpha) - \frac{\beta}{\alpha} g(u_\beta), g(u_\alpha) - g(u_\beta) \right\rangle \leq \frac{\beta^\mu - \alpha^\mu}{\alpha} \sum_{i=1}^{N} |F_i(u_\alpha, u_\beta)|. \tag{3.1.16}$$

Notice that,

$$\begin{aligned} \langle g(u_\alpha) - \tfrac{\beta}{\alpha} g(u_\beta), g(u_\alpha) - g(u_\beta)\rangle &= \|g(u_\alpha) - g(u_\beta)\|^2 + \tfrac{\alpha-\beta}{\alpha}\langle g(u_\beta), g(u_\alpha)\rangle \\ &\quad - \tfrac{\alpha-\beta}{\alpha}\|g(u_\beta)\|^2 \\ &\geq \|g(u_\alpha) - g(u_\beta)\|^2 + \tfrac{\alpha-\beta}{\alpha}\langle g(u_\beta), g(u_\alpha)\rangle, \end{aligned}$$

since $0 < \alpha < \beta$. Using this one together with (3.1.16), we have

$$\|g(u_\alpha) - g(u_\beta)\|^2 \le \frac{\beta - \alpha}{\alpha}\theta^2 + \frac{\beta^\mu - \alpha^\mu}{\alpha}\sum_{i=1}^{N}|F_i(u_\alpha, u_\beta)|, \tag{3.1.17}$$

where $\theta = \sup\{\|g(u_\alpha)\| : \alpha \in (0,1)\}$. Consequently, by using the boundedness of F_i, we have

$$\|g(u_\alpha) - g(u_\beta)\|^2 \le \frac{\beta - \alpha}{\alpha}\theta^2 + \frac{\beta^\mu - \alpha^\mu}{\alpha}M_2, \quad \text{for some } M_2 > 0.$$

Now, we consider a function $h : [1, +\infty) \to \mathbb{R}$ which is defined by

$$h(t) = t^{-\mu}, \qquad \text{for all } t \in [1, +\infty).$$

Observe that, we have

$$h'(t) = \frac{-\mu}{t^{\mu+1}}, \quad \forall t \in [1, +\infty).$$

Now, for $t_1, t_2 \in [1, +\infty)$ such that $\alpha = \frac{1}{t_1}$ and $\beta = \frac{1}{t_2}$ by the Mean-Value Theorem, the exists $k \in (t_2, t_1)$ such that

$$\frac{-\mu}{k^{\mu+1}} = \frac{t_1^{-\mu} - t_2^{-\mu}}{t_1 - t_2} = \frac{\alpha^\mu - \beta^\mu}{\frac{1}{\alpha} - \frac{1}{\beta}} = \frac{\alpha^\mu - \beta^\mu}{\frac{\beta-\alpha}{\alpha\beta}}.$$

Consequently,

$$\frac{\alpha^\mu - \beta^\mu}{\alpha} = \frac{-\mu}{k^{\mu+1}} \cdot \frac{\beta - \alpha}{\alpha\beta} \cdot \frac{1}{\alpha}.$$

Therefore

$$\begin{aligned}
\|g(u_\alpha) - g(u_\beta)\|^2 &\le \frac{\beta - \alpha}{\alpha}M_1 + \frac{\beta - \alpha}{\alpha^2\beta} \cdot \frac{\mu}{k^{\mu+1}}M_2, \\
&\le \frac{\beta - \alpha}{\alpha^2}M_1 + \frac{\beta - \alpha}{\alpha^2} \cdot \mu M_2, \\
&\le \left[\frac{(\beta - \alpha)}{\alpha^2}\right] 2M, \qquad \text{where } M = \max\{M_1,\ \mu M_2\}.
\end{aligned}$$

This completes the proof. □

3.2 Regularization inertial proximal point algorithm for general variational inequality problem in Hilbert spaces.

In this section, we will provide a regularization inertial proximal point algorithm for finding a solution of the problem (3.0.1). In fact, starting with an element $z_1 \in H$ such that $g(z_1) \in K$, we will consider the sequence $\{z_n\}$ which is defined by the following processes:

$$\langle c_n[A(z_{n+1}) + \alpha_n^{\mu} \sum_{i=1}^{N} (A_i \circ g)(z_{n+1}) + \alpha_n g(z_{n+1})] + g(z_{n+1}) - g(z_n), \quad g(v) - g(z_{n+1})\rangle \geq 0, \quad (3.2.1)$$

for all $v \in H, g(v) \in K$, where $\{c_n\}$ and $\{\alpha_n\}$ are sequences of positive real numbers.

It is worth to know that, the well-definedness of sequence $\{z_n\}$, defined in (3.2.1) is guaranteed by the following result.

Lemma 3.2.1. *Assume that all hypotheses of the Theorem 3.1.4 are satisfied. Let $z \in g^{-1}(K)$ be a fixed element. Define a bi-function $F_z : g^{-1}(K) \times g^{-1}(K) \to \mathbb{R}$ by*

$$F_z(u,v) := \langle c[A(u) + \alpha^{\mu} \sum_{i=1}^{N} (A_i \circ g)(u) + \alpha g(u)] + g(u) - g(z), g(v) - g(u)\rangle,$$

where c, α are positive real numbers. Then there exists the unique element $u^* \in g^{-1}(K)$ such that $F_z(u^*, v) \geq 0$ for all $v \in g^{-1}(K)$.

Proof. Assume that g is an ξ- expanding mapping. Then for each $u, v \in g^{-1}(K)$, we see that

$$\begin{aligned} F_z(u,v) + F_z(v,u) &\leq (1 + c\alpha)\langle g(u) - g(v), g(v) - g(u)\rangle \\ &= -(1 + c\alpha)\|g(u) - g(v)\|^2 \\ &\leq -\xi(1 + c\alpha)\|u - v\|^2. \end{aligned}$$

This means F is $\xi(1 + c\alpha)$-strongly monotone. Consequently, by Lemma 2.2.10(b), the proof is completed. $\square$

Now we provide some sufficient conditions for the convergent of regularization inertial proximal point algorithm (3.2.1).

Theorem 3.2.2. *Assume that all hypotheses of the Theorem 3.1.4 are satisfied. If the parameters c_n and α_n are chosen positive real numbers such that*

$$
\begin{aligned}
&(i) \quad \lim_{n\to\infty} \alpha_n = 0,\\
&(ii) \quad \lim_{n\to\infty} \tfrac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}^2} = 0,\\
&(iii) \quad \liminf_{n\to\infty}, c_n\alpha_n > 0.
\end{aligned}
\tag{3.2.2}
$$

Then the sequence $\{g(z_n)\}$ defined by (3.2.1) converges strongly to the element $g(u^*)$ as $n \to +\infty$.

Proof. From (3.2.1), we have

$$\langle c_n[A(z_{n+1}) + \alpha_n^\mu \sum_{i=1}^N (A_i \circ g)(z_{n+1})] + (1 + c_n\alpha_n)g(z_{n+1}) - g(z_n), g(v) - g(z_{n+1})\rangle \geq 0.$$

That is

$$\langle c_n[A(z_{n+1}) + \alpha_n^\mu \sum_{i=1}^N (A_i \circ g)(z_{n+1})] + (1 + c_n\alpha_n)g(z_{n+1}), g(v) - g(z_{n+1})\rangle$$
$$\geq \langle g(z_n), g(v) - g(z_{n+1})\rangle.$$

Thus

$$(1 + c_n\alpha_n)\left\langle \frac{c_n}{(1 + c_n\alpha_n)}[A(z_{n+1}) + \alpha_n^\mu \sum_{i=1}^N (A_i \circ g)(z_{n+1})] + g(z_{n+1}), g(v) - g(z_{n+1})\right\rangle$$
$$\geq \langle g(z_n), g(v) - g(z_{n+1})\rangle,$$

so

$$\left\langle \frac{c_n}{(1 + c_n\alpha_n)}[A(z_{n+1}) + \alpha_n^\mu \sum_{i=1}^N (A_i \circ g)(z_{n+1})] + g(z_{n+1}), g(v) - g(z_{n+1})\right\rangle$$
$$\geq \frac{1}{(1 + c_n\alpha_n)}\langle g(z_n), g(v) - g(z_{n+1})\rangle.$$

Hence

$$\langle \mu_n[A(z_{n+1}) + \alpha_n^{\mu}\sum_{i=1}^{N}(A_i \circ g)(z_{n+1})] + g(z_{n+1}), g(v) - g(z_{n+1})\rangle$$

$$\geq \quad \beta_n \langle g(z_n), g(v) - g(z_{n+1})\rangle,$$

$$\text{where} \quad \beta_n = \frac{1}{(1+c_n\alpha_n)}, \qquad \mu_n = c_n\beta_n. \tag{3.2.3}$$

By the similar argument, from (3.1.1), we have

$$\langle c_nA(u_n) + \alpha_n^{\mu}\sum_{i=1}^{N}(A_i \circ g)(u_n) + \alpha_n g(u_n) + g(u_n) - g(u_n), g(v) - g(u_n)\rangle \geq 0,$$

and so

$$\langle \frac{c_n}{(1+c_n\alpha_n)}[A(u_n) + \alpha_n^{\mu}\sum_{i=1}^{N}(A_i \circ g)(u_n)] + g(u_n), g(v) - g(u_n)\rangle$$

$$\geq \frac{1}{(1+c_n\alpha_n)}\langle g(u_n), g(v) - g(u_n)\rangle,$$

where u_n is the solution of (3.1.1) when α is replaced by α_n. Thus

$$\langle \mu_n[A(u_n) + \alpha_n^{\mu}\sum_{i=1}^{N}(A_i \circ g)(u_n)] + g(u_n), g(v) - g(u_n)\rangle$$

$$\geq \beta_n\langle g(u_n), g(v) - g(u_n)\rangle. \tag{3.2.4}$$

By setting $v = u_n$ in (3.2.3), we have

$$\langle \mu_n[A(z_{n+1}) + \alpha_n^{\mu}\sum_{i=1}^{N}(A_i \circ g)(z_{n+1})],\ g(u_n) - g(z_{n+1})\rangle \geq \beta_n\langle g(z_n), g(u_n) - g(z_{n+1})\rangle.$$

$$\langle \mu_n[A(z_{n+1}) + \alpha_n^{\mu}\sum_{i=1}^{N}(A_i \circ g)(z_{n+1})],\ g(u_n) - g(z_{n+1})\rangle \geq \beta_n\langle g(z_n), g(u_n) - g(z_{n+1})\rangle.$$

and by setting $v = z_{n+1}$ in (3.2.4), we have

$$\langle \mu_n[A(u_n) + \alpha_n^{\mu}\sum_{i=1}^{N}(A_i \circ g)(u_n)] + g(z_{n+1}), g(z_{n+1}) - g(u)\rangle \geq \beta_n\langle g(u_n), g(z_{n+1}) - g(u_n)\rangle.$$

Adding one obtained result to the other, we have

$$\mu_n\langle A(z_{n+1}) - A(u_n) + \alpha_n^{\mu}\sum_{i=1}^{N}(A_i(g(z_{n+1})) - A_i(g(u_n))), g(u_n) - g(z_{n+1})\rangle$$
$$+ \langle g(z_{n+1}) - g(u_n), g(u_n) - g(z_{n+1})\rangle \geq \beta_n\langle g(z_n) - g(u_n), g(u_n) - g(z_{n+1})\rangle. \tag{3.2.5}$$

Notice that, since A is a g-monotone mapping and $A_i, i = 1, ..., N$ is a λ_i-inverse strongly monotone mapping, we have

$$\langle A(z_{n+1}) - A(u_n), g(u_n) - g(z_{n+1})\rangle \leq 0,$$

and

$$\langle A_i(g(z_{n+1})) - A_i(g(u_n)), g(u_n) - g(z_{n+1})\rangle \leq 0, \forall i = 1, ..., N.$$

Using these facts, together with (3.2.5), we obtain

$$\langle g(z_{n+1}) - g(u_n), g(u_n) - g(z_{n+1})\rangle \geq \beta_n\langle g(z_n) - g(u_n), g(u_n) - g(z_{n+1})\rangle.$$

This means

$$\langle g(z_{n+1}) - g(u_n), g(z_{n+1}) - g(u_n)\rangle \leq \beta_n\langle g(z_n) - g(u_n), g(z_{n+1}) - g(u_n)\rangle.$$

Then

$$\|g(z_{n+1}) - g(u_n)\|^2 \leq \beta_n\|g(z_n) - g(u_n)\|\|g(z_{n+1}) - g(u_n)\|,$$

which implies that

$$\|g(z_{n+1}) - g(u_n)\| \leq \beta_n\|g(z_n) - g(u_n)\|. \tag{3.2.6}$$

From (3.2.3), (3.1.1) with $y = u^*$, we have

$$\begin{aligned}\|g(z_{n+1}) - g(u_{n+1})\| &\leq \|g(z_{n+1}) - g(u_n)\| + \|g(u_n) - g(u_{n+1})\| \\ &\leq \beta_n\|g(z_n) - g(u_n)\| + \sqrt{\frac{M(\alpha_n - \alpha_{n+1})}{\alpha_{n+1}^2}} \\ &\leq (1 - b_n)\|g(z_n) - g(u_n)\| + d_n,\end{aligned}$$

where

$$b_n = \frac{c_n \alpha_n}{(1 + c_n \alpha_n)}, d_n = \sqrt{\frac{M(\alpha_n - \alpha_{n+1})}{\alpha_{n+1}^2}}.$$

Consequently, by the condition (ii), we have $\sum_{n=1}^{\infty} b_n = \infty$. Meanwhile, the conditions (ii) and (iii) imply $\lim_{n \to \infty} \frac{d_n}{b_n} = 0$. Thus all conditions of Lemma 2.3.1 are satisfied, then it follows that $\|g(z_{n+1}) - g(u_{n+1})\| \to 0$ as $n \to \infty$. Moreover by (i) and Theorem 3.1.4, we know that there exists $u^* \in GVI_K(A, g, S)$ such that $g(u_n)$ converges strongly to $g(u^*)$. Consequently, we obtain that

$$\begin{aligned} 0 \leq \|g(z_n) - g(u^*)\| &= \|g(z_n) - g(u_n) + g(u_n) - g(u^*)\| \\ &\leq \|g(z_n) - g(u_n)\| + \|g(u_n) - g(u^*)\| \end{aligned}$$

it implies that $\|g(z_n) - g(u^*)\| \to 0$ as $n \to \infty$. Hence $\{g(z_n)\}$ converges strongly to $g(u^*)$ as $n \to +\infty$. This completes the proof. □

Remark 3.2.3. The sequences $\{\alpha_n\}$ and $\{c_n\}$ which are defined by

$$\alpha_n = \left(\frac{1}{n}\right)^p, \quad 0 < p < 1, \text{ and } c_n = \frac{1}{\alpha_n}$$

satisfy all conditions in Theorem 3.2.2.

Remark 3.2.4. It is worth to notice that, because of condition (ii) of Theorem 3.2.2, the important natural choice $\{1/n\}$ does not include in the class of parameters $\{\alpha_n\}$. This leads to a question: Can we find another regularization inertial proximal point algorithm for the problem (3.0.1) that include a natural parameter choice $\{1/n\}$?

Remark 3.2.5. It is worth to point out that although many authors have proved results for finding the solution of the variational inequality problem and the solution set of a finite family of inverse strongly monotone mappings, it is clear that it cannot be directly applied to the problem $GVI_K(A, g, S)$ due to the presence of g.

As a special case of Theorem 3.1.4, if g is the identity operator on H, we have the following results:

Theorem 3.2.6. *Let K be a closed convex subset of a real Hilbert space H. Let $A : K \to H$ be a hemicontinuous and monotone mapping. Let A_i be a λ_i-inverse strongly monotone mapping of K into H, for each $i = 1, 2, ..., N$. If $VI_K(A) \bigcap S \neq \emptyset$, where $VI_K(A)$ is denoted for the solution set of the problem (*), then the following conclusions are true:*

(a) For each $\alpha \in (0,1)$, the problem

$$\langle A(u_\alpha) + \alpha^\mu \sum_{i=1}^{N} A_i(u_\alpha) + \alpha u_\alpha, v - u_\alpha \rangle \geq 0, \forall v \in H, 0 < \mu < 1$$

has the unique solution u_α.

(b) $\lim_{\alpha \to 0+} u_\alpha = u^*$, for some $u^* \in VI_K(A) \bigcap S$.

(c) There exists a positive constant M such that

$$\|u_\alpha - u_\beta\|^2 \leq \frac{M|\alpha - \beta|}{\alpha^2}, \quad \text{for all } \alpha, \beta \in (0,1). \tag{3.2.7}$$

Theorem 3.2.7. *Assume that all hypotheses of the Theorem 3.2.6 are satisfied. If the parameters c_n and α_n are chosen positive real numbers such that*

(i) $\lim_{n \to \infty} \alpha_n = 0$,

(ii) $\lim_{n \to \infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}^2} = 0$,

(iii) $\liminf_{n \to \infty} c_n \alpha_n > 0$. (3.2.8)

Then the sequence $\{z_n\}$ defined by, starting with an element $z_1 \in K$, we will consider the following processes:

$$\langle c_n[A(z_{n+1}) + \alpha_n^\mu \sum_{i=1}^{N} A_i(z_{n+1}) + \alpha_n z_{n+1}] + z_{n+1} - z_n, v - z_{n+1} \rangle \geq 0,$$

for all $v \in K$, where $\{c_n\}$ and $\{\alpha_n\}$ are sequences of positive real numbers, converges strongly to the element $u^* \in H$ as $n \to +\infty$.

On the other hand, also as a special case of Theorem 3.1.4, if A_1 is the zero operator on H, we have the following results:

Theorem 3.2.8. *Let K be a closed convex subset of a real Hilbert space H and $g : H \to H$ be a mapping such that $K \subset g(H)$. Let $A : H \to H$ be a hemicontinuous and g-monotone mapping. If g is an expanding affine continuous mapping and $GVI_K(A, g) \neq \emptyset$, where $GVI_K(A, g)$ is denoted for the solution set of the problem 1.0.2, then the following conclusions are true:*

(a) For each $\alpha \in (0, 1)$, the problem

$$\langle A(u_\alpha) + \alpha g(u_\alpha), g(v) - g(u_\alpha)\rangle \geq 0,$$

$\forall v \in H,\ g(v) \in K, 0 < \mu < 1$, has the unique solution u_α.

(b) $\lim\limits_{\alpha \to 0^+} g(u_\alpha) = g(u^*)$, for some $u^* \in GVI_K(A, g)$.

(c) There exists a positive constant M such that

$$\|g(u_\alpha) - g(u_\beta)\|^2 \leq \frac{M|\alpha - \beta|}{\alpha^2}, \quad \text{for all } \alpha, \beta \in (0, 1). \tag{3.2.9}$$

Theorem 3.2.9. *Assume that all hypotheses of the Theorem 3.2.8 are satisfied. If the parameters c_n and α_n are chosen positive real numbers such that*

$$(i)\quad \lim_{n \to \infty} \alpha_n = 0,$$

$$(ii)\quad \lim_{n \to \infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}^2} = 0,$$

$$(iii)\quad \liminf_{n \to \infty} c_n \alpha_n > 0. \tag{3.2.10}$$

Then the sequence $\{g(z_n)\}$ defined by, starting with an element $z_1 \in H$ such that $g(z_1) \in K$, we will consider the following processes:

$$\langle c_n[A(z_{n+1}) + \alpha_n g(z_{n+1})] + g(z_{n+1}) - g(z_n), g(v) - g(z_{n+1})\rangle \geq 0,$$

for all $v \in H, g(v) \in K$, where $\{c_n\}$ and $\{\alpha_n\}$ are sequences of positive real numbers, converges strongly to the element $g(u^*)$ as $n \to +\infty$.

Finally, we have the following results:

Theorem 3.2.10. *Let K be a closed convex subset of a real Hilbert space H. Let $A : K \to H$ be a hemicontinuous and monotone mapping. If $VI_K(A) \neq \emptyset$, then the following conclusions are true:*

(a) For each $\alpha \in (0, 1)$, the problem

$$\langle A(u_\alpha) + \alpha u_\alpha, v - u_\alpha \rangle \geq 0, \forall v \in H, 0 < \mu < 1$$

has the unique solution u_α.

(b) $\lim_{\alpha \to 0^+} u_\alpha = u^*$, for some $u^* \in VI_K(A)$.

(c) There exists a positive constant M such that

$$\|u_\alpha - u_\beta\|^2 \leq \frac{M|\alpha - \beta|}{\alpha^2}, \quad \textit{for all } \alpha, \beta \in (0, 1). \tag{3.2.11}$$

Theorem 3.2.11. *Assume that all hypotheses of the Theorem 3.2.10 are satisfied. If the parameters c_n and α_n are chosen positive real numbers such that*

(i) $\lim_{n \to \infty} \alpha_n = 0,$

(ii) $\lim_{n \to \infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}^2} = 0,$

(iii) $\liminf_{n \to \infty} c_n \alpha_n > 0.$ (3.2.12)

Then the sequence $\{z_n\}$ defined by, starting with an element $z_1 \in K$, we will consider the following processes:

$$\langle c_n[A(z_{n+1}) + \alpha_n z_{n+1}] + z_{n+1} - z_n, v - z_{n+1} \rangle \geq 0,$$

for all $v \in K$, where $\{c_n\}$ and $\{\alpha_n\}$ are sequences of positive real numbers, converges strongly to the element $u^* \in H$ as $n \to +\infty$.