CHAPTER II

PRELIMINARIES

In this chapter, we give some definitions, notations, and some useful results that will be used in the later chapter.

Throughout this thesis, we assume that $\mathbb{R}$ stands for the set of all real numbers and $\mathbb{N}$ is the set of all natural numbers.

2.1 Normed spaces and Hilbert spaces.

Definition 2.1.1. Let X be a linear space over the field $\mathbb{R}$. A function $\|\cdot\| : X \to \mathbb{R}$ is said to be *a norm on* X if it satisfies the following conditions:

1) $\|x\| \geq 0, \forall x \in X$;

2) $\|x\| = 0 \Leftrightarrow x = 0$;

3) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$;

4) $\|\alpha x\| = |\alpha|\|x\|, \forall x \in X$ and $\forall \alpha \in \mathbb{R}$.

Definition 2.1.2. Let $(X, \|\cdot\|)$ be a normed space.

1) A sequence $\{x_n\} \subset X$ is said to *converge strongly* in X if there exists $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. That is, if for any $\varepsilon > 0$ there exists a positive integer N such that $\|x_n - x\| < \varepsilon, \forall n \geq N$. We often write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ to mean that x is the limit of the sequence $\{x_n\}$.

2) A sequence $\{x_n\} \subset X$ is said to be *a Cauchy sequence* if for any $\varepsilon > 0$ there exists a positive integer N such that $\|x_m - x_n\| < \varepsilon, \forall\, m, n \geq N$.

3) A sequence $\{x_n\} \subset X$ is said to be *a bounded sequence* if there exists $M > 0$ such that $\|x_n\| \leqslant M, \forall n \in \mathbb{N}$.

Definition 2.1.3. A subset K of a normed linear space X is said to be *a convex set in* X if $\lambda x + (1 - \lambda)y \in K$ for each $x, y \in C$ and for each scalar $\lambda \in [0, 1]$.

Definition 2.1.4. A normed space X is called *complete* if every Cauchy sequence in X converges to an element in X.

Definition 2.1.5. Let X be a real vector space. Then a real valued function of two variables $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is called an *inner product* on a real vector space X if it satisfies the following conditions:

1) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y, z \in X$ and all real numbers α and β;

2) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in X$; and

3) $\langle x, x \rangle \geq 0$ for each $x \in X$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

A *real inner product space* is a real vector space equipped with an inner product.

Definition 2.1.6. A *Hilbert space* is an inner product space which is complete under the norm induced by its inner product.

Definition 2.1.7. Let H be an inner product space, let $\{x_n\}$ be a sequence of H and let x be an element of H. Then $\{x_n\}$ is said to *converge weakly* to x, denoted by $x_n \rightharpoonup x$, if for any $y \in H$ we have $\langle x_n, y \rangle \to \langle x, y \rangle$.

Lemma 2.1.8. [8]. Let H be a real Hilbert space, if $\{x_n\} \subset H$ is a bounded sequence, then there exists a weakly convergent subsequence of $\{x_n\}$.

Lemma 2.1.9. [8]. Let $\{x_n\}$ be a sequence of a real Hilbert space H and let $x_0 \in H$. Then $x_n \to x_0$ if and only if, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_{ij}}\}$ of $\{x_{n_i}\}$ converging to x_0.

Lemma 2.1.10. [8]. **[The Schwarz inequality]** If x and y are any two vectors in Hilbert space H, then

$$|\langle x, y \rangle| \leq ||x||||y||.$$

Lemma 2.1.11. [8]. **[The parallelogram law]** If x and y are any two vectors in a Hilbert space H, then

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2.$$

Lemma 2.1.12. *Let H be a real Hilbert space and $\{x_n\}$ be a sequence of H. If $x_n \rightharpoonup x_0$ and $\|x_n\| \to \|x_0\|$, then $x_n \to x_0$.*

Lemma 2.1.13. [8]. Let H be a real Hilbert space, let K be a nonempty closed convex subset of H and let $x \in H$. Then there exists a unique element $y_0 \in K$ such that $\|x - y_0\| = \inf\{\|x - y\| : y \in K\}$.

Definition 2.1.14. The *metric (nearest point) projection* P_K from a Hilbert space H to a closed convex subset K of H is defined as follows: Given $x \in H$, $P_K x$ is the only point in K with the property

$$\|x - P_K x\| = \inf\{\|x - y\| : y \in K\}.$$

Lemma 2.1.15. Let H be a real Hilbert space and K be a closed convex subset of H. Given $x \in H$ and $y \in K$. Then $y = P_K x$ if and only if there holds the inequality

$$\langle x - y, y - z\rangle \geq 0, \forall z \in K.$$

Lemma 2.1.16. Let H be a real Hilbert space and $C \subset H$ a strongly closed and convex set. Then C is weakly closed also.

2.2 Nonlinear mappings

Definition 2.2.1. Let H be a real Hilbert space, let K be a nonempty subset of H and let f be a function of K in to $\mathbb{R} \cup \{\infty\}$. The function f is said to be

(i) *lower semicontinuous* if for any $a \in \mathbb{R}$, the set $\{x \in K : f(x) \leq a\}$ is closed;

(ii) *convex* if for any $x_1, x_2 \in K$ and $t \in (0,1)$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)t(x_2).$$

(ii) *affine* if for any $x_1, x_2 \in K$ and $t \in (0,1)$,

$$f(tx_1 + (1-t)x_2) = tf(x_1) + (1-t)t(x_2).$$

Definition 2.2.2. Let $A : K \to H$ be a mapping of K into H.

(i) A is called *monotone* if

$$\langle Au - Av, u - v \rangle \geq 0, \quad \forall u, v \in K.$$

(ii) A is called *λ-inverse strongly monotone* if there exists a positive real number λ such that

$$\langle Au - Av, u - v \rangle \geq \lambda \|Au - Av\|^2, \quad \forall u, v \in K.$$

Definition 2.2.3. Let $A, g : H \to H$ be mappings then A is said to be g-monotone, if

$$\langle A(x) - A(y), g(x) - g(y) \rangle \geq 0, \qquad \forall\, x, y \in H.$$

For $g = I$, the identity operator, Definition 2.2.3 reduces to Definition 2.2.2(i). However, the converse is not true.

Definition 2.2.4. Let X be a real normed space and K a nonempty subset of X. A mapping $A : K \longrightarrow X$ is called *hemicontinuous* at a point x in K if

$$\lim_{t \to 0} \langle A(x + th), y \rangle = \langle A(x), y \rangle, \quad \forall y \in X,$$

where $x + th \in K$.

Definition 2.2.5. Let X be a real normed space and K a nonempty subset of X. A mapping $T : K \longrightarrow X$ is called *ξ-expanding* if there exists a positive number ξ, such that

$$\|Tx - Ty\| \geq \xi \|x - y\|,$$

for all $x, y \in K$.

Definition 2.2.6. Let X be a real normed space and K a nonempty subset of X. A mapping $T : K \longrightarrow K$ is called *Lipschitz continuous* on K if there exists a positive number L, named *Lipschitz constant,* such that

$$\|Tx - Ty\| \leq L \|x - y\|$$

for all $x, y \in K$. If $0 < L < 1$, then the mapping T is called a *contraction*, and if $L = 1$, then the mapping T is called *nonexpansive.*

Definition 2.2.7. Let X be a real normed space and K a nonempty subset of X. A mapping $T : K \to X$ is said to be *k-strictly pseudocontractive*, if there exists a constant $k \in [0, 1)$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2 \quad \forall x, y \in K. \tag{2.2.1}$$

Clearly, when $k = 0$, T is nonexpansive. Therefore, the class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings.

Remark 2.2.8. In a real Hilbert space H, if I is the identity operator on K, where K is a nonempty subset of H. It is well known that if $T : K \to H$ is a k-strictly pseudocontrative mapping then the mapping $A := I - T$ is a $\left(\frac{1-k}{2}\right)$-inverse strongly monotone. Converscly, if $A : K \to H$ is a λ-inverse strongly monotone mapping with $\lambda \in \left(0, \frac{1}{2}\right]$, then $T := I - A$ is a $(1 - 2\lambda)$-strictly pseudocontrative mapping.

Proof. Let $x, y \in K$. Since T is a k-strictly pseudocontrative mapping, we have

$$\begin{aligned} ||x - y||^2 + k||Ax - Ay||^2 &= ||x - y||^2 + k||(I - T)x - (I - T)y||^2, \\ &\geq ||Tx - Ty||^2, \\ &= ||(I - A)x - (I - A)y||^2, \\ &= \langle (I - A)x - (I - A)y, (I - A)x - (I - A)y \rangle, \\ &= \langle x - y - Ax + Ay, x - y - Ax + Ay \rangle, \\ &= ||x - y||^2 - 2\langle Ax - Ay, x - y \rangle + ||Ax - Ay||^2. \end{aligned}$$

Hence

$$\langle Ax - Ay, x - y \rangle \geq \frac{1 - k}{2} ||Ax - Ay||^2.$$

This implies that A is a $\left(\frac{1-k}{2}\right)$-inverse strongly monotone mapping.

Conversely, if $A : K \to H$ is a λ-inverse strongly monotone mapping with $\lambda \in \left(0, \frac{1}{2}\right]$, we have

$$\begin{aligned}
\|Tx - Ty\|^2 &= \|(I - A)x - (I - A)y\|^2, \\
&= \langle (I - A)x - (I - A)y, (I - A)x - (I - A)y \rangle, \\
&= \langle x - y - Ax + Ay, x - y - Ax + Ay \rangle, \\
&= \|x - y\|^2 - 2\langle Ax - Ay, x - y \rangle + \|Ax - Ay\|^2, \\
&\leq \|x - y\|^2 - 2\lambda\|Ax - Ay\|^2 + \|Ax - Ay\|^2, \\
&= \|x - y\|^2 + (1 - 2\lambda)\|Ax - Ay\|^2, \\
&= \|x - y\|^2 + (1 - 2\lambda)\|(I - T)x - (I - T)y\|^2.
\end{aligned}$$

It implies that T is a $(1 - 2\lambda)$-strictly pseudocontrative mapping. □

Definition 2.2.9. The equilibrium bifunction $F : K \times K \to \mathbb{R}$ is said to be

(*i*) monotone, if for all $u, v \in K$, we have

$$F(u, v) + F(v, u) \leq 0, \tag{2.2.2}$$

(*ii*) strongly monotone with constant τ, if, for all $u, v \in K$, we have

$$F(u, v) + F(v, u) \leq -\tau\|u - v\|^2, \tag{2.2.3}$$

(*iii*) hemicontinuous in the first variable u, if for each fixed v, we have

$$\lim_{t \to +0} F(u + t(z - u), v) = F(u, v), \quad \forall (u, z) \in K \times K. \tag{2.2.4}$$

Recall that the equilibrium problem for $F : K \times K \to \mathbb{R}$ is to find $u^* \in K$ such that

$$F(u^*, v) \geq 0, \qquad \forall v \in K. \tag{2.2.5}$$

Concerning to the problem (2.2.5), the following facts are very useful.

Lemma 2.2.10. [9]. *Let $F : K \times K \to \mathbb{R}$ be such that $F(u, v)$ is convex and lower semicontinuous in the variable v for each fixed $u \in K$. Then,*

(a) *if* $F(u,v)$ *is hemicontinuous in the first variable and has the monotonic property, then* $U^* = V^*$, *where* U^* *is the solution set of* (2.2.5), $V^* = \{v^* \in K | F(u,v^*) \leq 0, \forall u \in K\}$. *Moreover, in this case, they are closed and convex;*

(b) *if* $F(u,v)$ *is hemicontinuous in the first variable for each* $v \in K$ *and* F *is strongly monotone, then* U^* *is a nonempty singleton. In addition, if* F *is a strongly monotone mapping then* $U^* = V^*$ *is a singleton set.*

Definition 2.2.11. A mapping $T : K \longrightarrow H$ is said to be *demiclosed* at a point v if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $x_n \rightharpoonup x \in D(T)$ and $Tx_n \to v$, then $Tx = v$.

Lemma 2.2.12. [10]. *Let* K *be a nonempty closed convex subset of a Hilbert space* H *and* $B : K \to H$ *a k-strictly pseudocontractive mapping. Then* $I - B$ *is demiclosed at zero, that is, whenever* $\{x_n\}$ *is a sequence in* K *such that* $\{x_n\}$ *converges weakly to* $x \in K$ *and* $\{(I - B)(x_n)\}$ *converges strongly to* 0, *we must have* $(I - B)(x) = 0$.

Definition 2.2.13. An element $x \in K \subset X$ is said to be a *fixed point* of a mapping $T : K \to X$ provided $Tx = x$.

The set of all fixed points of T is denoted by $F(T)$.

Remark 2.2.14. Let I be the identity operator on K. It is well known that any 1-inverse strongly monotone mapping is a nonexpansive mapping. Moreover, if T is a nonexpansive mapping then it is well known that $(I - T)$ is a $\frac{1}{2}$-inverse strongly monotone mapping. Notice that, the problem of finding an element of $F(T)$, is equivalent to that of finding an element of $x \in S_{(I-T)}$, where $S_{(I-T)} = \{x \in K : (I - T)(x) = 0\}$. Also, we note that if $x \in S_A$ then x is called a zero of A.

Thus, from Remark 2.2.14, we see that the problem of finding zero of the inverse strongly monotone mappings contains the problem of finding fixed points of nonexpansive mappings as special case.

2.3 Useful lemmas

Lemma 2.3.1. [11]. *Let* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ *be the sequences of positive numbers satisfying the conditions:*

(*i*) $a_{n+1} \leq (1 - b_n)a_n + c_n, b_n < 1$,

(*ii*) $\sum_{n=0}^{\infty} b_n = +\infty, \lim_{n\to+\infty}(\frac{c_n}{b_n}) = 0$.

Then, $\lim_{n\to+\infty} a_n = 0$.