

Full Paper

On statistical convergence of order α of difference sequence of functions

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Abstract: In this paper we introduce the concepts of uniform and pointwise Δ^m – statistical convergence of order α of sequence of functions. We also give notions of uniform and pointwise Δ^m – statistically Cauchy sequence of order α of sequence of functions and show that pointwise Δ^m – statistically Cauchy sequence (uniform Δ^m – statistical convergence) is equivalent to pointwise Δ^m – statistical convergence (uniform Δ^m – statistically Cauchy sequence) of order α . Moreover, some relations between pointwise Δ^m – statistical convergence of order α and strongly pointwise Δ_q^m – Cesàro summable of order α of sequence of functions are given.

Key words: statistical convergence, sequence of functions, Cesàro summability

INTRODUCTION

Fast [1] and Schoenberg [2] independently introduced the notion of statistical convergence. The idea hinges on the density of subsets of the set \mathbf{N} . The density of E , a subset of \mathbf{N} , is defined by $\delta(E) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \chi_E(k)$, provided that the limit exists, where χ_E is the characteristic function of E . A sequence $x = (x_k)$ is statistically convergent to a number L if, for every $\varepsilon > 0$, $\delta\{k \in \mathbf{N} : |x_k - L| \geq \varepsilon\} = 0$. The statistical convergence has been studied by Basar [3], Connor [4], Cinar et al. [5], Et et al. [6-7], Duman and Ankara [8], Fridy [9], Gokhan and Gungor [10], Gungor et al. [11], Gungor and Gokhan [12, 13], Mohiuddine et al. [14], Mursaleen [15], Salat [16] and many others.

Recently Gadjiev and Orhan [17] introduced the order of statistical convergence of a sequence of numbers and later Colak [18] studied statistical convergence of order α and strong q – Cesàro summability of order α and defined the α – density of a subset E of \mathbf{N} as

$$\delta_\alpha(E) = \lim_n \frac{1}{n^\alpha} |\{k \leq n : k \in E\}|, \text{ provided that the limit exists and } \alpha \in (0,1],$$

where $|\{k \leq n : k \in E\}|$ denotes the number of elements of E not exceeding n .

If $x = (x_k)$ is a sequence such that x_k satisfies the property $P(k)$ for all k except a set of zero α -density, then we say that x_k satisfies the property $P(k)$ for ‘almost all k according to α ’ and we abbreviate this by ‘*a.a.k* (α)’. It can be shown that any finite subset of \mathbb{N} has zero α density and $\delta_\alpha(E^c) = 1 - \delta_\alpha(E)$ does not hold for $\alpha \in (0,1]$. If $\alpha = 1$, then α -density reduces to the natural density.

Kizmaz [19] introduced the difference sequence spaces and Et and Colak [20] generalised the notion afterwards as follows:

$$\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\},$$

for $X = \ell_\infty, c$ or c_0 , where $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta^m x = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ and so

$\Delta^m x_k = \sum_{i=0}^m (-1)^i \binom{m}{i} x_{k+i}$. Recently, Basar and Altay [21], Altay and Basar [22], Altin et al. [23], Et

[24], Altinok and Mursaleen [25], Bektas et al. [26], Colak et al. [27], Djolovic and Malkowsky [28], Gungor and Et [29], Malkowsky et al. [30] and Mursaleen et al. [31] have studied difference sequences spaces.

MAIN RESULTS

In this section we give the main results of this article.

Definition 1. Let A be a subset of \mathbb{R} and $\{f_k\}$ be a sequence of real valued functions defined on A . The pointwise Δ^m -convergent sequence $\{f_k\}$ converges to f if there exists a natural number $N = N(x, \varepsilon)$ for $x \in A$ and $\varepsilon > 0$ such that $|\Delta^m f_k(x) - f(x)| < \varepsilon$ for every $k \geq N$. The notation $N = N(x, \varepsilon)$ means that the natural number N depends on x and ε . In this case we write $\lim_k \Delta^m f_k(x) = f(x)$ for every $x \in A$. The set of all pointwise Δ^m -convergent sequences of functions is denoted by $c(\Delta^m, F(p))$, where $\Delta^m f_k(x) = \sum_{v=0}^m (-1)^v \binom{m}{v} f_{k+v}(x)$.

Definition 2. Let A be a subset of \mathbb{R} , $\{f_k\}$ be a sequence of real valued functions defined on A , and α be any real number such that $0 < \alpha \leq 1$. The sequence $\{f_k\}$ is said to be pointwise Δ^m -statistically convergent of order α to f if there exists a natural number $N = N(x, \varepsilon)$ for $x \in A$ and $\varepsilon > 0$ such that

$$|\Delta^m f_k(x) - f(x)| < \varepsilon \text{ a.a.k}(\alpha);$$

that is, for $x \in A$ and $\varepsilon > 0$,

$$\lim_n \frac{1}{n^\alpha} |\{k \leq n : |\Delta^m f_k(x) - f(x)| \geq \varepsilon\}| = 0.$$

In this case we write $S^\alpha - \lim \Delta^m f_k(x) = f(x)$ on A . The set of all pointwise Δ^m -statistically convergent sequences of functions of order α is denoted by $S^\alpha(\Delta^m, F(p))$.

It can be shown that the pointwise Δ^m -statistical convergence of order α for sequence of

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functions is well defined for $\alpha \in (0, 1]$, but not well defined for $\alpha > 1$. For this, let $\{f_k\}$ be defined as follows:

$$f_k(x) = \begin{cases} 1 & k = 2n \\ x^k & k \neq 2n \end{cases} \quad n = 1, 2, 3, \dots, x \in [0, \frac{1}{2}].$$

Then we calculate $\Delta f_k(x)$ as

$$\Delta f_k(x) = \begin{cases} 1 - x^{k+1} & k = 2n \\ x^k - 1 & k \neq 2n \end{cases} \quad n = 1, 2, 3, \dots, x \in [0, \frac{1}{2}].$$

In this case

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} \left| \left\{ k \leq n : \left| \Delta f_k(x) - (x^k - 1) \right| \geq \varepsilon, \text{ for every } x \in A \right\} \right| = \lim_{n \rightarrow \infty} \frac{n}{2n^\alpha} = 0$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} \left| \left\{ k \leq n : \left| \Delta f_k(x) - (1 - x^{k+1}) \right| \geq \varepsilon, \text{ for every } x \in A \right\} \right| = \lim_{n \rightarrow \infty} \frac{n}{2n^\alpha} = 0$$

for $\alpha > 1$. Therefore, $S^\alpha - \lim \Delta f_k(x) = 1$ and $S^\alpha - \lim \Delta f_k(x) = -1$, which is impossible.

Theorem 1. Let $\alpha \in (0, 1]$, and $\{f_k\}$ and $\{g_k\}$ be two sequences of real valued functions defined on A .

(i) If $S^\alpha - \lim \Delta^m f_k(x) = f(x)$ and $c \in \mathbb{R}$, then $S^\alpha - \lim c \Delta^m f_k(x) = cf(x)$.

(ii) If $S^\alpha - \lim \Delta^m f_k(x) = f(x)$ and $S^\alpha - \lim \Delta^m g_k(x) = g(x)$, then

$$S^\alpha - \lim \Delta^m (f_k(x) + g_k(x)) = f(x) + g(x).$$

Proof. Let $\{f_k\}$ and $\{g_k\} \in S^\alpha(\Delta^m, F(p))$ with $S^\alpha - \lim \Delta^m f_k(x) = f(x)$ and

$S^\alpha - \lim \Delta^m g_k(x) = g(x)$ on $A \subset \mathbb{R}$, and $c \in \mathbb{R}$. Then one can easily see by these assumptions that

$$S^\alpha - \lim \Delta^m (f_k(x) + g_k(x)) = f(x) + g(x),$$

$$S^\alpha - \lim c \Delta^m f_k(x) = cf(x),$$

as desired.

It can be shown that every pointwise Δ^m -convergent sequence of functions is pointwise Δ^m -statistically convergent of order α ($0 < \alpha \leq 1$). However, the converse of this does not hold. Indeed if we take the sequence $\{f_k\}$ defined by

$$f_k(x) = \begin{cases} 1 & k = n^2 \\ \frac{kx}{1+k^2x^2} & \text{otherwise} \end{cases},$$

then we have

$$\Delta f_k(x) = \begin{cases} 1 - \frac{(k+1)x}{1+(k+1)^2x^2} & k = n^2 \\ \frac{kx}{1+k^2x^2} - 1 & k = n^2 - 1 \\ \frac{kx}{1+k^2x^2} - \frac{(k+1)x}{1+(k+1)^2x^2} & \text{otherwise} \end{cases}.$$

Therefore, $\{f_k\}$ is pointwise Δ -statistically convergent of order α with $S^\alpha - \lim \Delta f_k(x) = 0$ for $\alpha > \frac{1}{2}$, but it is not pointwise Δ -convergent.

Definition 3. Let A be a subset of \mathbb{R} , $\{f_k\}$ be a sequence of real valued functions defined on A ,

and α be any real number such that $0 < \alpha \leq 1$. The sequence $\{f_k\}$ is a pointwise Δ^m – statistically Cauchy sequence of order α provided that there exists a number $N = N(\varepsilon, x)$ for $x \in A$ and $\varepsilon > 0$ such that

$$\left| \Delta^m f_k(x) - \Delta^m f_N(x) \right| < \varepsilon \quad a.a.k(\alpha);$$

that is, for $x \in A$ and $\varepsilon > 0$,

$$\lim_n \frac{1}{n^\alpha} \left| \left\{ k \leq n : \left| \Delta^m f_k(x) - \Delta^m f_N(x) \right| \geq \varepsilon \right\} \right| = 0.$$

Using the techniques for proving Theorem 3.4 in Cinar et al. [5], we obtain the proof of the following theorem.

Theorem 2. Let $\{f_k\}$ be a sequence of functions defined on a set A . The following statements are equivalent:

- (i) $\{f_k\}$ is a pointwise Δ^m – statistically convergent sequence of order α on A ;
- (ii) $\{f_k\}$ is a pointwise Δ^m – statistically Cauchy sequence of order α on A ;
- (iii) $\{f_k\}$ is a sequence of functions for which there is a pointwise convergent sequence of function $\{\Delta^m g_k\}$ such that $\Delta^m f_k(x) = \Delta^m g_k(x) \quad a.a.k(\alpha)$ for every $x \in A$.

Theorem 3. If $0 < \alpha \leq \beta \leq 1$, then the inclusion $S^\alpha(\Delta^m, F(p)) \subseteq S^\beta(\Delta^m, F(p))$ strictly holds.

Proof. Since the proof of the inclusion $S^\alpha(\Delta^m, F(p)) \subset S^\beta(\Delta^m, F(p))$ is easy, we omit detail. To show the strictness of the inclusion, consider a sequence $\{f_k\}$ defined by

$$f_k(x) = \begin{cases} 1, & k = n^2 \\ \frac{k^2 x}{1+k^3 x^2}, & \text{otherwise} \end{cases} \quad n = 1, 2, 3, \dots, x \in [0, 1].$$

Then one can calculate $\Delta f_k(x)$ as follows:

$$\Delta f_k(x) = \begin{cases} 1 - \frac{(k+1)^2 x}{1+(k+1)^3 x^2}, & k = n^2 \\ \frac{k^2 x}{1+k^3 x^2} - 1, & k = n^2 - 1 \\ \frac{k^2 x}{1+k^3 x^2} - \frac{(k+1)^2 x}{1+(k+1)^3 x^2}, & \text{otherwise} \end{cases}.$$

Therefore, for $\frac{1}{2} < \alpha \leq 1$, we get

$$\begin{aligned} & \frac{1}{n^\alpha} \left| \left\{ k \leq n : \left| \Delta f_k(x) - 0 \right| \geq \varepsilon, \text{ for every } x \in [0, 1] \right\} \right| \\ &= \frac{1}{n^\alpha} \left| \left\{ k \leq n : \left| \Delta f_k(x) \right| \geq \varepsilon, \text{ for every } x \in [0, 1] \right\} \right| \\ &\leq \frac{2\sqrt{n}}{n^\alpha} \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

Then $S^\beta - \lim \Delta f_k(x) = 0$, i.e. $x \in S^\beta(\Delta, f)$ for $\frac{1}{2} < \beta \leq 1$, but $x \notin S^\alpha(\Delta, f)$ for $0 < \alpha \leq \frac{1}{2}$.

Corollary 1. If a sequence of functions $\{f_k\}$ is Δ^m – statistically convergent of order α to f , then it is Δ^m – statistically convergent to f .

Definition 4. Let A be a subset of \mathbb{R} , $\{f_k\}$ be a sequence of real valued functions defined on A , and α be any real number such that $0 < \alpha \leq 1$. A sequence of functions $\{f_k\}$ is said to be strongly pointwise Δ^m – Cesàro summable of order α for $x \in A$ and $\varepsilon > 0$ if there exists a function f such

that

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} \sum_{k=1}^n |\Delta^m f_k(x) - f(x)|^q = 0.$$

In this case we write $w_q^\alpha - \lim \Delta^m f_k(x) = f(x)$ on A . The set of all strongly pointwise Δ_q^m -Cesàro summable sequences of functions of order α is denoted by $w_q^\alpha(\Delta^m, F(p))$. We write $w_{\alpha, q}^\alpha(\Delta^m, F(p))$ in the case of $f(x) = 0$.

Theorem 4. Let $\alpha, \beta \in (0, 1]$ with $\alpha \leq \beta$ and $0 < q < \infty$. Then $w_q^\alpha(\Delta^m, F(p)) \subseteq w_q^\beta(\Delta^m, F(p))$ and the inclusion is strict for some α and β such that $\alpha \leq \beta$.

Proof. The inclusion part of the proof is easy, so we omit it. To show the strictness of the inclusion, consider a sequence $\{f_k\}$ defined by

$$f_k(x) = \begin{cases} \frac{kx}{1+kx} & k = n^2 \\ 0 & \text{otherwise} \end{cases} \quad x \in [1, 2].$$

Then we calculate $\Delta f_k(x)$ as

$$\Delta f_k(x) = \begin{cases} \frac{kx}{1+kx} & k = n^2 \\ -\frac{(k+1)x}{1+(k+1)x} & k = n^2 - 1 \\ 0 & \text{otherwise} \end{cases},$$

and so

$$\frac{1}{n^\beta} \sum_{k=1, x \in A}^n |\Delta f_k(x) - 0|^p \leq \frac{2\sqrt{n}}{n^\beta} = \frac{2}{n^{\beta-\frac{1}{2}}}.$$

Since $2/(n^{\beta-\frac{1}{2}}) \rightarrow 0$ as $n \rightarrow \infty$, then $w_q^\beta - \lim \Delta f_k(x) = 0$, which means that the sequence $\{f_k\}$ is strongly pointwise Δ_q -Cesàro summable of order α for $\frac{1}{2} < \beta \leq 1$. But since

$$\frac{2\sqrt{n}}{2n^\alpha} \leq \frac{1}{n^\alpha} \sum_{k=1, x \in A}^n |\Delta f_k(x) - 0|^p$$

and $2\sqrt{n}/2n^\alpha \rightarrow \infty$ as $n \rightarrow \infty$, the sequence $\{f_k\}$ is not strongly Δ_q -Cesàro summable of order α for $0 < \alpha < \frac{1}{2}$.

Corollary 2. Let q be a positive real number. Then the inclusion $w_q^\alpha(\Delta^m, F(p)) \subseteq w_q(\Delta^m, F(p))$ strictly holds for some $\alpha \in (0, 1]$.

Theorem 5. Let $\alpha, \beta \in (0, 1]$ with $\alpha \leq \beta$ and $0 < q < \infty$. If a sequence of functions $\{f_k\}$ is strongly pointwise Δ_q^m -Cesàro summable of order α to f , then it is pointwise Δ^m -statistically convergent of order β to f .

Proof. For any sequence of functions $\{f_k\}$ defined on A , we can write

$$\sum_{k=1, x \in A}^n |\Delta^m f_k(x) - f(x)|^q \geq \left| \left\{ k \leq n : |\Delta^m f_k(x) - f(x)| \geq \varepsilon, \text{ for every } x \in A \right\} \right| \varepsilon^q,$$

which leads to the consequence that

$$\begin{aligned} \frac{1}{n^\alpha} \sum_{k=1, x \in A}^n \left| \Delta^m f_k(x) - f(x) \right|^q &\geq \frac{1}{n^\alpha} \left| \left\{ k \leq n : \left| \Delta^m f_k(x) - f(x) \right| \geq \varepsilon, \text{ for every } x \in A \right\} \right| \varepsilon^q \\ &\geq \frac{1}{n^\beta} \left| \left\{ k \leq n : \left| \Delta^m f_k(x) - f(x) \right| \geq \varepsilon, \text{ for every } x \in A \right\} \right| \varepsilon^q. \end{aligned}$$

Corollary 3. Let α be a fixed real number such that $0 < \alpha \leq 1$ and $0 < q < \infty$. If a sequence of functions $\{f_k\}$ is strongly pointwise Δ^m -Cesàro summable of order α to f , then it is pointwise Δ^m -statistically convergent of order α to f .

Definition 5. Let A be a subset of \mathbb{R} , $\{f_k\}$ be a sequence of real valued functions defined on A , and α be a fixed real number such that $0 < \alpha \leq 1$. Then $\{f_k\}$ is said to be uniformly and Δ^m -statistically convergent of order α to f on A if there exists a natural number $N = N(\varepsilon)$ for $x \in A$ and $\varepsilon > 0$ such that

$$\left| \Delta^m f_k(x) - f(x) \right| < \varepsilon \quad a.a.k(\alpha) \text{ and for every } x \in A,$$

meaning that for $\varepsilon > 0$,

$$\lim_n \frac{1}{n^\alpha} \left| \left\{ k \leq n : \left| \Delta^m f_k(x) - f(x) \right| \geq \varepsilon \right\} \right| = 0 \text{ for every } x \in A.$$

In this case we write $S_u^\alpha - \lim \Delta^m f_k(x) = f(x)$ on A . The set of all uniform, Δ^m -statistically convergent sequences of function order α is denoted by $S^\alpha(\Delta^m, F(u))$. In this definition the natural number N depends only on ε . Therefore, if a sequence is uniformly and Δ^m -statistically convergent to f , then it is pointwise Δ^m -statistically convergent to f . However, the converse does not hold in general. To show this, consider a sequence $\{f_k\}$ defined by

$$f_k(x) = \begin{cases} 2 & k = n^2 \\ \frac{kx}{1+kx^2} & \text{otherwise} \end{cases} \quad n = 1, 2, \dots \text{ and } x \in [0, 1].$$

Then

$$\Delta f_k(x) = \begin{cases} 2 - \frac{(k+1)x}{1+(k+1)x^2} & k = n^2 \\ \frac{kx}{1+kx^2} - 2 & k = n^2 - 1, n = 1, 2, \dots \text{ and } x \in [0, 1]. \\ \frac{kx}{1+kx^2} - \frac{(k+1)x}{1+(k+1)x^2} & \text{otherwise} \end{cases}$$

Therefore, $\{f_k\}$ is pointwise Δ -statistically convergent of order α to $f(x) = 0$ on $[0, 1]$ for $0 < \alpha < \frac{1}{2}$, but $\{f_k\}$ is not uniformly and Δ -statistically convergent of order α in the following theorem since $\lim_{k \rightarrow \infty} c_k$ does not exist, where

$$c_k = \max_{x \in [0, 1]} |\Delta f_k(x) - 0| = \begin{cases} 2 & k = n^2 \\ \frac{\sqrt{k}}{2} - 2 & k = n^2 - 1, n = 1, 2, \dots, \\ A & \text{otherwise} \end{cases}$$

in which $A = \frac{(\sqrt{6k^2+6k})(\sqrt{-(2k+1)+\sqrt{16k^2+16k+1}})}{(4k+5+\sqrt{16k^2+16k+1})} - \frac{(\sqrt{6k^2+6k})(\sqrt{-(2k+1)+\sqrt{16k^2+16k+1}})}{(4k+5+\sqrt{16k^2+16k+1})}$.

Theorem 6. Let f and $\Delta^m f_k$ (for all $k \in \mathbb{N}$) be continuous functions on $A = [a, b] \subset \mathbb{R}$ and $0 < \alpha \leq 1$. Then $S^\alpha - \lim \Delta^m f_k(x) = f(x)$ uniformly on A if and only if $S^\alpha - \lim c_k = 0$, where $c_k = \max_{x \in A} |\Delta^m f_k(x) - f(x)|$.

Proof. Omitted.

It is trivial that if $\lim \Delta^m f_k(x) = f(x)$ is uniform on A , then $S^\alpha - \lim \Delta^m f_k(x) = f(x)$ is uniform on A . However, the converse is not true in general. To show this, consider the sequence defined by

$$f_k(x) = \begin{cases} 1 & k = n^2 \\ x^k & \text{otherwise} \end{cases} \quad k = 1, 2, 3, \dots, x \in [0, 1].$$

So we have

$$\Delta f_k(x) = \begin{cases} 1 - x^{k+1} & k = n^2 \\ x^k - 1 & k = n^2 - 1 \\ x^k - x^{k+1} & \text{otherwise} \end{cases}.$$

If $x \in [0, 1]$ and $\alpha \in [\frac{1}{2}, 1]$, then $\{f_k\}$ is uniformly and Δ -statistically convergent of order α to $f(x) = 0$ on $[0, 1]$ since $S^\alpha - \lim c_k = 0$, where

$$c_k = \max_{x \in [0, 1]} |\Delta f_k(x) - 0| = \begin{cases} 1 & k = n^2 \\ 0 & k = n^2 - 1 \\ \left(\frac{k}{k+1}\right)^k \frac{1}{k+1} & \text{otherwise} \end{cases}.$$

However, $(\Delta f_k(x))$ is not uniformly convergent on $[0, 1]$ since $\lim_{k \rightarrow \infty} c_k$ does not exist.

Corollary 3. (i) $\lim \Delta^m f_k(x) = f(x)$ uniformly on $A \Rightarrow \lim \Delta^m f_k(x) = f(x)$ on $A \Rightarrow S^\alpha - \lim \Delta^m f_k(x) = f(x)$ pointwise on A .

(ii) $S_u^\alpha - \lim \Delta^m f_k(x) = f(x)$ uniformly on $A \Rightarrow S^\alpha - \lim \Delta^m f_k(x) = f(x)$ pointwise on A .

(iii) If $0 < \alpha \leq \beta \leq 1$, then $S^\alpha(\Delta^m, F(u)) \subseteq S^\beta(\Delta^m, F(u))$.

Definition 6. Let A be a subset of \mathbb{R} , $\{f_k\}$ be a sequence of real valued functions defined on A , and $\alpha \in (0, 1]$. Then $\{f_k\}$ is said to be uniform Δ^m -statistical Cauchy sequence of order α if there exists a natural number $N = N(\varepsilon)$ for any $\varepsilon > 0$ such that

$$|\Delta^m f_k(x) - \Delta^m f_N(x)| < \varepsilon \quad a.a.k(\alpha) \text{ and for every } x \text{ in } A,$$

meaning that, for $\varepsilon > 0$,

$$\lim_n \frac{1}{n^\alpha} \left| \left\{ k \leq n : |\Delta^m f_k(x) - f(x)| \geq \varepsilon \right\} \right| = 0, \text{ for every } x \text{ in } A.$$

In this definition the natural number N only depends on ε . By $S_u^\alpha(\Delta^m, F(p))$, we denote the set of uniform Δ^m -statistical Cauchy sequences of functions of order α .

The proofs of the following two theorems are similar to those of Theorems 1 and 2. Thus, we give them without proofs.

Theorem 7. The set $S_u^\alpha(\Delta^m, F(p))$ is a vector space under the usual operations, which are addition and scalar multiplication of the sequences of functions.

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Theorem 8. Let $\alpha \in (0,1]$ and $\{f_k\}$ be a sequence of functions defined on a set A . The following statements are equivalent:

- (i) $\{f_k\}$ is a uniform, Δ^m – statistically convergent sequence of order α on A ;
- (ii) $\{f_k\}$ is a uniform, Δ^m – statistically Cauchy sequence of order α on A ;
- (iii) $\{f_k\}$ is a sequence of functions for which there is a uniformly convergent sequence of functions $\{\Delta^m g_k\}$ such that $\Delta^m f_k(x) = \Delta^m g_k(x)$ a.a.k.(α) for all $x \in A$.

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