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Some second-derivative-free sixth-order convergent iterative methods for non-linear equations

Farooq Ahmed Shah [1,*], Muhammad Aslam Noor [2] and Muhammad Waseem [3]

[1] Department of Mathematics, COMSATS Institute of Information Technology, Attock, Pakistan
[2] Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan
[3] Department of Computer Science, COMSATS Institute of Information Technology, Vehari, Pakistan

*Corresponding author, e-mail: farooqhamdani@gmail.com

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Abstract: We use the variational iteration technique to suggest some new iterative methods for solving non-linear equations $f(x)=0$. We discuss the convergence criteria of these new iterative methods. A comparison with other similar methods is also given. These new methods can be considered as an alternative to the existing methods. Several examples to illustrate the efficiency and performance of these methods are given. This technique can be used to suggest a wide class of new iterative methods for solving a system of non-linear equations.

Keywords: non-linear equations, variational iteration technique, Newton's method, Taylor series

INTRODUCTION

It is well known that a wide class of problems which arises in various branches of mathematical and engineering sciences can be studied by the non-linear equation of the form $f(x)=0$. Numerical methods for finding approximate solutions of the non-linear equation are being developed by using several different techniques including Taylor series, quadrature formulas, homotopy perturbation method, decomposition techniques and variational iteration technique [1–27 and the references therein]. The classical Newton's method for solving non-linear equations is written as:

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}, \qquad n=1,2,3\cdots.$$

This is an important and basic method [27] which converges quadratically. To improve the order of convergence, many modified methods have been suggested in the literature. Motivated and inspired

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by the research going on in this direction, we suggest and analyse new iterative methods for solving the non-linear equations.

In this paper we implement the variational iteration technique by considering two auxiliary functions, $\phi(x)$ and $\psi(x)$. The former plays the role as predictor function having convergence order $q \geq 1.$ The predictor function helps to obtain iterative methods of convergence order $q+r,$ where r is the order of convergence of the second auxiliary function $\psi(x)$. This is the modified form of variational iteration technique for finding simple roots of non-linear equations. Using the described technique, we present the iterative methods of higher-order convergence. The variational iteration technique was introduced by Inokuti et al. [10]. This technique was developed to solve a variety of diverse problems [7–9]. Essentially using the same idea, Noor and Shah [12, 14] suggested and analysed some iterative methods for finding simple roots and multiple roots of the non-linear equations. Now we apply the variational iteration technique to obtain higher-order methods. New methods are modified with fewer numbers of functional evaluations, which raises the efficiency index of these methods. We also discuss the convergence criteria of these new iterative methods.

A comparison with other similar methods is also given. Several examples are given to confirm the efficiency of the suggested methods, which can be considered as an alternative to the existing methods.

TECHNIQUE AND CONSTRUCTION OF ITERATIVE METHODS

We use the variational iteration technique to derive some new iterative methods. These are multi-step methods consisting of predictor and corrector steps. The convergence of the methods is better than the one-step methods. The variational iteration technique is used to obtain some new iterative methods of order $q+r,$ where $q, r \geq 1$ is the order of convergence of the auxiliary iteration functions $\phi(x)$ and $\psi(x)$.

Consider the non-linear equation of the type

$$f(x) = 0. \tag{1}$$

For simplicity, we assume that p is a simple root and γ is an initial guess sufficiently close to p. For the sake of completeness and to give the basic idea, we consider the approximate solution x_n of (1) such that

$$f(x_n) \neq 0.$$

We consider $\phi(x_n)$ and $\psi(x_n)$, two iteration functions of order q and r respectively. Then

$$x_{n+1} = \phi(x_n) + \lambda[f(\psi(x_n))g(\psi(x_n))]^t, \qquad \text{where } t = \frac{q}{r}, \tag{2}$$

is a recurrence relation which generates iterative methods of order $q + r$; $g(x)$ is any arbitrary function which later on is converted to $g(\psi(x_n))$, and λ is a parameter which is usually called the Lagrange's multiplier and can be identified by the optimality condition.

Using the optimality criteria from (2), we have

$$\lambda = -\frac{\phi'(x_n)[f(\psi(x_n))g(\psi(x_n))]^{-t+1}}{t\psi'(x_n)[f'(\psi'(x_n))g(\psi(x_n)) + f(\psi(x_n))g'(\psi'(x_n))]}. \tag{3}$$

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From (2) and (3), we have

$$x_{n+1} = \phi(x_n) - \frac{\phi'(x_n)}{t\,\psi'(x_n)} \frac{f(\psi(x_n))g(\psi(x_n))}{[f'(\psi(x_n))g(\psi(x_n)) + f(\psi(x_n))g'(\psi(x_n))]}. \tag{4}$$

The relation (4) is the main recurrence relation for the iterative methods. We use (4) to construe the iterative methods for solving non-linear equations and generate higher-order convergent methods as provided in the form of $\phi(x_n)$ and $\psi(x_n)$.

Now we apply (4) to build up the iterative methods of a diversified form for solving non-linear equations. Let us suppose that

$$\psi(x_n) = y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{5}$$

the well-known Newton's method of 2^{nd} -order convergence. Then (4) becomes

$$x_{n+1} = \phi(x_n) - \frac{\phi'(x_n) f(y_n) g(y_n)}{t[f'(y_n)g(y_n) + f(y_n)g'(y_n)]y_n'}. \tag{6}$$

Let

$$\phi(x_n) = z_n = y_n - \frac{f(y_n)}{f'(y_n)}. \tag{7}$$

Then we have

$$\phi'(x_n) = \frac{f(y_n)f''(y_n)}{f'^2(y_n)} y_n', \tag{8}$$

and from Taylor series, we have

$$f(z_n) \approx f(y_n) + (z_n - y_n) f'(y_n) + \frac{(z_n - y_n)^2}{2} f''(y_n) = \frac{f^2(y_n) f''(y_n)}{2f'^2(y_n)}. \tag{9}$$

From (7) and (8), we have

$$\phi'(x_n) = \frac{2f(z_n)}{f(y_n)} y_n'. \tag{10}$$

Using (10) in (6), we obtain

$$x_{n+1} = z_n - \frac{2f(z_n)g(y_n)}{t[g'(y_n)f(y_n) + g(y_n)f'(y_n)]}. \tag{11}$$

Here, $t = \frac{4}{2} = 2,$ which is according to the above described technique. The relation (11) then becomes

$$x_{n+1} = z_n - \frac{f(z_n)g(y_n)}{[g'(y_n)f(y_n) + g(y_n)f'(y_n)]}. \tag{12}$$

Now equation (12) is the main recurrence relation for the iterative methods. We use (12) to deduce iterative methods for solving non-linear equations by considering some special cases of the auxiliary functions g :

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I. Let $g(x_n)=e^{-\alpha x_n}$. Then from (12), we obtain the following iterative method.

Algorithm 1. For a given x_0, find the approximate solution x_{n+1} by the iterative scheme:

$$x_{n+1}=z_n-\frac{f(z_n)}{f'(y_n)-\alpha f(y_n)},$$

$$\text{where} \quad y_n=x_n-\frac{f(x_n)}{f'(x_n)} \quad \text{and} \quad z_n=y_n-\frac{f(y_n)}{f'(y_n)}.$$

II. Let $g(x_n)=e^{\frac{\alpha}{f'(x_n)}}$. Then from (12), we obtain after combining with (9) the following iterative scheme for solving the non-linear equation (1).

Algorithm 2. For a given x_0, find the approximate solution x_{n+1} by the iterative scheme:

$$x_{n+1}=z_n-\frac{f(z_n)f(y_n)}{f'(y_n)f(y_n)-2\alpha f(z_n)},$$

$$\text{where} \quad y_n=x_n-\frac{f(x_n)}{f'(x_n)} \quad \text{and} \quad z_n=y_n-\frac{f(y_n)}{f'(y_n)}.$$

III. Let $g(x_n)=e^{\frac{\alpha f(x_n)}{f'(x_n)}}$. Then from (12), we have after combining with (9) the following iterative scheme for solving the non-linear equation (1).

Algorithm 3. For a given x_0, find the approximate solution x_{n+1} by the iterative scheme:

$$x_{n+1}=z_n-\frac{f(z_n)f(y_n)}{f'(y_n)f(y_n)+\alpha\{f(y_n)-2f(z_n)\}},$$

$$\text{where} \quad y_n=x_n-\frac{f(x_n)}{f'(x_n)} \quad \text{and} \quad z_n=y_n-\frac{f(y_n)}{f'(y_n)}.$$

Note: Never choose such a value of α which makes the denominator zero. Select the sign of α to keep the denominator largest in magnitude in the above algorithms for the best implementation.

CONVERGENCE ANALYSIS

In this section we consider the convergence criteria of the main and general scheme described in equation (12), which is the main and general iterative scheme.

Theorem. Assume that the function $f: D\subset R\rightarrow R$ for an open interval D has a simple root $p\in D$. Let $f(x)$ be sufficiently smooth in some neighborhood of the root. Then the relation defined by (12) as

$$x_{n+1}=z_n-\frac{f(z_n)g(y_n)}{[g'(y_n)f(y_n)+g(y_n)f'(y_n)]},$$

$$\text{where} \quad y_n=x_n-\frac{f(x_n)}{f'(x_n)} \quad \text{and} \quad z_n=y_n-\frac{f(y_n)}{f'(y_n)},$$

has sixth-order convergence.

Proof. Here we study the iteration function of the type

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$$G(x) = z - \frac{f(z)g(y)}{[f'(y)g(y) + f(y)g'(y)]}, \tag{13}$$

where $y(x) = x - \dfrac{f(x)}{f'(x)}$ and $z(y(x)) = y - \dfrac{f(y)}{f'(y)}$.

If p is the zero of $f(x)$, then it can easily be computed that

$$y(p) = p, \tag{14}$$

$$y'(p) = 0, \tag{15}$$

$$y''(p) = \frac{f''(p)}{f'(p)}. \tag{16}$$

And also, it can be calculated that

$$z(p) = p, \tag{17}$$

$$z'(p) = 0, \tag{18}$$

$$z''(p) = 0, \tag{19}$$

$$z'''(p) = 0, \tag{20}$$

and

$$z^{(iv)}(p) = 3\frac{[f''(p)]^3}{[f(p)]^3}. \tag{21}$$

The above results lead us to the analysis of the convergence of the main iterative scheme (13) at $x = p.$ Using (14) to (21), we obtain

$$G(p) = p. \tag{22}$$

Differentiating (13) with respect to x, we get

$$G'(x) = z' - f'(z)\frac{g(y)}{[g'(y)f(y) + g(y)f'(y)]} - f(z)\left(\frac{g(y)}{[g'(y)f(y) + g(y)f'(y)]}\right)'. \tag{23}$$

Using (14) to (21), we obtain

$$G'(p) = 0 \tag{24}$$

and

$$G''(x) = z'' - 2f'(z)\left(\frac{g(y)}{[g'(y)f(y) + g(y)f'(y)]}\right)' - f(z)\left(\frac{g(y)}{[g'(y)f(y) + g(y)f'(y)]}\right)''$$
$$- f''(z)\left(\frac{g(x)}{[g'(y)f(y) + g(y)f'(y)]}\right). \tag{25}$$

Applying (14) to (21), we obtain

$$G''(p) = 0. \tag{26}$$

Now,

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$$G'''(x) = z''' - 3f''(z)\left(\frac{g(y)}{[g'(y)f(y)+g(y)f'(y)]}\right)' - 3f'(z)\left(\frac{g(y)}{[g'(y)f(y)+g(y)f'(y)]}\right)''$$
$$- f(z)\left(\frac{g(y)}{[g'(y)f(y)+g(y)f'(y)]}\right)''' - f'''(z)\left(\frac{g(y)}{[g'(y)f(y)+g(y)f'(y)]}\right) \quad (27)$$

Again, for $x = p$, applying (14) to (21), we obtain

$$G'''(p) = 0. \quad (28)$$

Now we calculate

$$G^{(iv)}(x) = (z)^{(iv)} - 4f'''(z)\left(\frac{g(y)}{[g'(y)f(y)+g(y)f'(y)]}\right)' - 6f''(z)\left(\frac{g(y)}{[g'(y)f(y)+g(y)f'(y)]}\right)''$$
$$-4f'(z)\left(\frac{g(y)}{[g'(y)f(y)+g(y)f'(y)]}\right)''' - f(z)^{(iv)}\left(\frac{g(y)}{[g'(y)f(y)+g(y)f'(y)]}\right)$$
$$- f(z)\left(\frac{g(y)}{[g'(y)f(y)+g(y)f'(y)]}\right)^{(iv)}. \quad (29)$$

Using (14) to (21), we consequently obtain

$$G^{(iv)}(p) = 0. \quad (30)$$

Using (14) to (21), and

$$\left((z(p))^{(iv)} - [f(z(p))]^{(iv)}\left(\frac{g(y(p))}{[g'(y(p))f(y(p)) + g(y(p))f'(y(p))]}\right)\right) = 0, \quad (31)$$

we get

$$G^{(v)}(p) = 0. \quad (32)$$

Also, with the help of following results:

$$\left(\frac{g(y(p))}{[g'(y(p))f(y(p)) + g(y(p))f'(y(p))]}\right)' = 0 \quad (33)$$

and

$$(z(p))^{(v)} - [f(z(p))]^{(v)}\left(\frac{g(y(p))}{[g'(y(p))f(y(p)) + g(y(p))f'(y(p))]}\right) = 0, \quad (34)$$

we also obtain

$$G^{(vii)}(p) = 45f''(p)\left(\frac{2g'(p)f'(p)+f''(p)g(p)}{f'^5(p)g(p)}\right) - \frac{45f''(p)g'(p)}{g(p)f'^4(p)} \neq 0. \quad (35)$$

From this it follows that the discussed relation gives at least a sixth-order convergence relation and all the methods deduced from this main relation are also sixth-order convergent.

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NUMERICAL RESULTS

In this section we present some numerical examples (Table 1) to illustrate the efficiency [27] and performance of the newly developed methods. We compare Newton's method (NM) [1], Kou's method (KM) [11] and Ham et al.'s method (HM) [4] with Algorithm 1, Algorithm 2 and Algorithm 3, which are introduced in this presentation. We note that the newly derived methods do not require the computation of a second derivative to carry out the iterations. All computations were done with MAPLE using 64-digit floating-point arithmetics (Digits=64) and $\varepsilon = 10^{-25}$. The following stopping criteria were used for computer programs:

$$\text{(i) } |x_{n+1} - x_n| < \varepsilon \qquad \text{(ii) } |f(x_{n+1})| < \varepsilon.$$

The following examples [4] are considered for the numerical results. All the methods are implemented and the results of comparison are shown in the Table 1.

$f_1(x) = x^3 + 4x^2 - 10,$ $\quad f_2(x) = x^2 - e^x - 3x + 2,$

$f_3(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5,$ $\quad f_4(x) = \sin x.e^x + \ln(x^2 + 1),$

$f_5(x) = (x-1)^3 - 2,$ $\quad f_6(x) = (x+2)e^x - 1,$

$f_7(x) = \sin^2 x - x^2 + 1.$

Table 1. Comparison of various iterative methods for $\alpha = 1$ (numbers of iterations are displayed.)

$f(x)$	NM	KM	HM	Algorithm 1	Algorithm 2	Algorithm 3
$f_1, x_0 = -.5$	98	Div*	32	13	14	7
$f_1, x_0 = 1$	6	2	2	3	3	3
$f_2, x_0 = 0$	5	2	2	2	2	2
$f_2, x_0 = 1$	5	2	2	2	2	2
$f_3, x_0 = -1$	6	2	2	3	3	3
$f_3, x_0 = -2$	9	4	4	4	4	4
$f_4, x_0 = 2$	6	3	3	3	3	3
$f_4, x_0 = -5$	7	3	3	3	3	3
$f_5, x_0 = 3$	7	3	3	3	3	3
$f_5, x_0 = 4$	8	3	4	3	3	4
$f_6, x_0 = 2$	9	4	5	3	4	4
$f_6, x_0 = 4$	12	6	5	4	5	5
$f_7, x_0 = 1$	6	3	3	3	3	3
$f_7, x_0 = 2.5$	7	3	3	3	3	3

* Divergence of method

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CONCLUSIONS

All the described methods have the sixth-order convergence and are free from the 2^{nd} and higher-order derivatives. If we consider the definition of efficiency index as $p^{\frac{1}{m}}$, where p is the order of the method and m is the number of functional evaluations per iteration required by the method, it is clear that Algorithms 1–3 have the efficiency index equal to $6^{\frac{1}{5}} \approx 1.430969081$, which is better than the one from the Newton's method ($2^{\frac{1}{2}} \approx 1.414$).

REFERENCES

1. R. L. Burden and J. D. Faires, "Numerical Analysis", 9th Edn., PWS Publishing, Boston, **2001**.
2. C. Chun, "On the construction of iterative methods with at least cubic convergence", *Appl. Math. Comput.*, **2007**, *189*, 1384-1392.
3. C. Chun, "Some variants of Chebyshev-Halley methods free from second derivative", *Appl. Math. Comput.*, **2007**, *191*, 193-198.
4. Y. M. Ham, C. Chun and S. G. Lee, "Some higher-order modifications of Newton's method for solving nonlinear equations", *J. Comput. Appl. Math*, **2008**, *222*, 477-486.
5. C. Chun and Y. Kim, "Several new third-order iterative methods for solving nonlinear equations", *Acta Appl. Math.*, **2010**, *109*, 1053-1063.
6. M. Grau and M. Noguera, "A variant of Cauchy's method with accelerated fifth-order convergence", *Appl. Math. Lett.*, **2004**, *17*, 509-517.
7. J. H. He, "Homotpy perturbation technique", *Comput. Math. Appl. Mech. Eng.*, **1999**, *178*, 257-262.
8. J. H. He, "Variational iteration method–some recent results and new interpretations", *J. Comp. Appl. Math.*, **2007**, *207*, 3-17.
9. J. H. He, "Variational iteration method–a kind of nonlinear analytical technique: Some examples", *Int. J. Nonlinear Mech.*, **1999**, *34*, 699-708.
10. M. Inokuti, H. Sekine and T. Mura, "General use of the Lagrange multiplier in nonlinear mathematical physics", in "Variational Methods in the Mechanics of Solids" (Ed. S. Nemat-Nasser), Pergamon Press, Oxford, **1978**, pp.156-162.
11. J. Kou, "The improvements of modified Newton's method", *Appl. Math. Comput.*, **2007**, *189*, 602-609.
12. M. A. Noor, "New classes of iterative methods for nonlinear equations", *Appl. Math. Comput.*, **2007**, *191*, 128-131.
13. M. A. Noor, K. I. Noor and M. Waseem, "Fourth-order iterative methods for solving nonlinear equations", *Int. J. Appl. Math. Eng. Sci.*, **2010**, *4*, 43-52.
14. M. A. Noor and F. A. Shah, "Variational iteration technique for solving nonlinear equations", *J. Appl. Math. Comput.*, **2009**, *31*, 247-254.
15. M. A. Noor, F. A. Shah, K. I. Noor and E. Al-Said, "Variational iteration technique for finding multiple roots of nonlinear equations", *Sci. Res. Essays*, **2011**, *6*, 1344-1350.
16. M. A. Noor and K. I. Noor, "Iterative schemes for nonlinear equations", *Appl. Math. Comput.*, **2006**, *183*, 774-779.
17. M. A. Noor and F. A. Shah, "A family of iterative schemes for finding zeros of nonlinear

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equations having unknown multiplicity", *Appl. Math. Inform. Sci*., **2014**, *8*, 2367-2373.

18. M. A. Noor, K. I. Noor, E. Al-Said and M. Waseem, "Some new iterative methods for nonlinear equations", *Math. Problems. Eng.*, **2010**, *2010*, Art. ID 198943.
19. M. Waseem, M. A. Noor and K. I. Noor, "Efficient method for solving a system of nonlinear equations", *Appl. Math. Comput*., **2016**, *275*, 134-146.
20. F. A. Shah, M. A. Noor and M. Batool, "Derivative-free iterative methods for solving nonlinear equations", *Appl. Math. Inform. Sci.*, **2014**, *8*, 2189-2193.
21. F. A. Shah and M. A. Noor, "Variational iteration technique and some methods for the approximate solution of nonlinear equations", *Appl. Math. Inform. Sci. Lett.*, **2014**, *2*, 85-93.
22. F. A. Shah and M. A. Noor, "Some numerical methods for solving nonlinear equations by using decomposition technique", *Appl. Math. Comput.*, **2015**, *251*, 378-386.
23. F. A. Shah and M. A. Noor, "Higher order iterative schemes for nonlinear equations using decomposition technique", *Appl. Math. Comput.*, **2015**, *266*, 414-423.
24. M. A. Noor, M. Waseem, K. I. Noor and E. Al-Said, "Variational iteration technique for solving a system of nonlinear equations", *Optim. Lett.*, **2013**, *7*, 991-1007.
25. M. A. Noor, M. Waseem, K. I. Noor and M. A. Ali, "New iterative technique for solving nonlinear equations", *Appl. Math. Comput.*, **2015**, *265*, 1115-1125.
26. A. M. Ostrowski, "Solution of Equations and Systems of Equations", Academic Press, New York, **1967**.
27. J. F. Traub, "Iterative Methods for the Solution of Equations", Prentice–Hall, Englewood Cliffs, (NJ), **1964**.