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Full Paper

Some generalisations of analytic functions with respect to 2k-symmetric conjugate points

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Abstract: A new class of analytic functions with respect to 2k-symmetric conjugate points is introduced. This class combines the class of starlike functions and convex functions with respect to 2k-symmetric conjugate points. Some interesting properties such as subordinations, inclusion relationships, integral representations, convolution condition and inequalities are discussed in relation to the coefficients of this class of functions.

Keywords: analytic functions, symmetric point, conjugate point

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INTRODUCTION

Let A be the class of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$. Let f and g be two functions which are analytic in E. We say that the function f is subordinate to the function g (represented by $f \prec g$ or $f(z) \prec g(z)$) in E if there exists a function w analytic in E with $w(0) = 0$ and $|w(z)| < 1$ in E such that $f(z) = g(w(z))$. In particular, if g is univalent in E, then $f(0) = g(0)$ and $f(E) \subset g(E)$. The classes of starlike and convex univalent functions are defined respectively as

$$S^* = \left\{ f : f \in A \text{ and } Re\frac{zf'(z)}{f(z)} > 0,\ z \in E \right\},$$

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$$C=\left\{f: f\in A \text{ and } Re\frac{(zf'(z))'}{f'(z)}>0,\ z\in E\right\}.$$

The class of $M(\lambda)$ of λ-convex function introduced by Mocanu [1] is defined as: Let $f\in A$ and $z^{-1}f(z)f'(z)\neq 0$. Then $f\in M(\lambda)$ for $\lambda\in R$ if

$$\operatorname{Re}\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\frac{(zf'(z))'}{f'(z)}\right]>0,\ \ z\in E.$$

A function f which is analytic in the open unit disc E is said to be starlike with respect to the symmetric points [2] if it satisfies

$$\frac{zf'(z)}{f(z)-f(-z)}\prec \varphi(z),\ z\in E,$$

where $\varphi(z)\in P$, the class of functions with positive real part. Such class of functions is denoted by $S_s^*(\varphi)$. Also, a function $f\in A$ is said to be in the class $C_s(\varphi)$ if and only if

$$zf'\in S_s^*(\varphi).$$

The class $C_s(\varphi)$ was studied by Stankiewicz [3]. A function f which is analytic in the open unit disc E is said to be starlike with respect to the symmetric conjugate points [4] if it satisfies

$$\frac{zf'(z)}{f(z)-\overline{f(-\bar{z})}}\prec \varphi(z),\ z\in E,$$

where $\varphi(z)\in P$, the class of functions with positive real part. Such class of functions is denoted by $S_{sc}^*(\varphi)$. Also, a function $f\in A$ is said to be in the class $C_{sc}(\varphi)$ if and only if

$$zf'\in S_{sc}^*(\varphi).$$

The classes $S_{sc}^*(\varphi)$ and $C_{sc}(\varphi)$ were studied by Ravichandran [5].

Al-Amiri et al. [6] introduced and investigated a class of functions starlike with respect to 2k-symmetric conjugate points $S_{sc}^k(\varphi)$ which satisfies the relation

$$\frac{zf'(z)}{f_{2k}(z)}\prec \varphi(z), z\in E,$$

where $\varphi(z)\in P$, $k\geq 2$ is a fixed positive integer and f_{2k} is defined by

$$f_{2k}(z)=\frac{1}{2k}\sum_{\mu=0}^{k-1}\left(\varepsilon^{-\mu}f\left(\varepsilon^{\mu}z\right)+\varepsilon^{\mu}\overline{f\left(\varepsilon^{\mu}\bar{z}\right)}\right),\qquad \varepsilon=exp\frac{2\pi i}{k}. \tag{2}$$

It is clear that a function $f\in A$ is said to be in the class $C_{sc}^k(\varphi)$ if and only if

$$zf'\in S_{sc}^k(\varphi).$$

These classes, $S_{sc}^k(\varphi)$ of starlike functions with respect to $2k$-symmetric conjugate points and $C_{sc}^k(\varphi)$ of convex functions with respect to $2k$-symmetric conjugate points, were studied by Wang and Gao [7].

From (2) we have

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$$f_{2k}\left(x^{\mu}z\right)=x^{\mu}f_{2k}(z) \text{ and } \overline{f_{2k}\left(x^{\mu}\overline{z}\right)}=x^{-\mu}f_{2k}(z);$$

$$f'_{2k}\left(x^{\mu}z\right)=f'_{2k}(z) \text{ and } \overline{f'_{2k}\left(x^{\mu}\overline{z}\right)}=f'_{2k}(z);$$

$$f''_{2k}(z)=\frac{1}{2k}\sum_{\mu=0}^{k-1}\left(\varepsilon^{\mu}f''\left(\varepsilon^{\mu}z\right)+\varepsilon^{-\mu}\overline{f''\left(\varepsilon^{\mu}\overline{z}\right)}\right).$$

In terms of convolution,

$$f_{2k}(z)=z+\sum_{j=2}^{\infty}\frac{a_j+\overline{a_j}}{2}c_j z^j$$

$$=\frac{1}{2}\left((f*h)(z)+\overline{(f*h)(\overline{z})}\right) \text{ where } h(z)=\frac{1}{k}\sum_{\mu=2}^{k-1}z\left(1-x^{\mu}z\right)^{-1}.$$

Wang and Jiang [8] introduced and investigated the class $M_{sc}^{2k}(\lambda,\varphi)$ which is defined as: Let $f\in A$ and $z^{-1}f(z)f'(z)\neq 0$. Then $f\in M_{sc}^{2k}(\lambda,\varphi)$ for $\lambda\in R$ if

$$\operatorname{Re}\left[(1-\lambda)\frac{zf'(z)}{f_{2k}(z)}+\lambda\frac{(zf'(z))'}{f'_{2k}(z)}\right] \text{p } \varphi(z), z\in E,$$

where $\varphi(z)\in P$, $k\geq 2$ is a fixed positive integer and f_{2k} is defined by (2).

THE CLASS $M_{sc}^{2k}(\alpha,\beta,\lambda)$

Keeping in view the above mentioned classes, we now define the following subclass of analytic functions with respect to $2k$-symmetric conjugate points.

Definition 1. Let $f\in A$ and f_{2k} be defined by (2). Then $f\in M_{sc}^{2k}(\alpha,\beta,\lambda)$ if and only if

$$\left|(1-\lambda)\frac{zf'(z)}{f_{2k}(z)}+\lambda\frac{(zf'(z))'}{f'_{2k}(z)}-1\right|\leq\beta\left|\alpha\left((1-\lambda)\frac{zf'(z)}{f_{2k}(z)}+\lambda\frac{(zf'(z))'}{f'_{2k}(z)}\right)+1\right|, \quad z\in E, \tag{3}$$

where $0\leq\alpha\leq 1$, $0<\beta\leq 1, \lambda>0$, $k\geq 2$.

Special Cases

(i) For $\lambda=1$, the class $M_{sc}^{2k}(\alpha,\beta,\lambda)$ yields the class $C_{sc}^{(k)}(\alpha,\beta)$, consisting of univalent functions satisfying the condition:

$$\left|\frac{(zf'(z))'}{f'_{2k}(z)}-1\right|\leq\beta\left|\alpha\left(\frac{(zf'(z))'}{f'_{2k}(z)}\right)+1\right|, \quad z\in E,$$

where $0\leq\alpha\leq 1$, $0<\beta\leq 1$, and $k\geq 2$.

(ii) For $\lambda=0$, the class $M_{sc}^{2k}(\alpha,\beta,\lambda)$ produces the class $S_{sc}^{(k)}(\alpha,\beta)$, satisfying the condition:

$$\left|\frac{zf'(z)}{f_{2k}(z)}-1\right|\leq\beta\left|\alpha\left(\frac{zf'(z)}{f_{2k}(z)}\right)+1\right|, \quad z\in E,$$

where $0\leq\alpha\leq 1$, $0<\beta\leq 1$, and $k\geq 2$.

(iii) When $k=2$, $\lambda=1$ and $\alpha=\beta=1$, we obtain the class C_{sc} [9].

(iv) For $k=2$, $\lambda=0$, $\alpha=\beta=1$, $M_{sc}^{2k}(\alpha,\beta,\lambda)$ reduces to the class S_{sc}^{*} [10].

(v) Taking $k=2$, $\lambda=1$, $M_{sc}^{2k}(\alpha,\beta,\lambda)$ reduces to the class $C_{sc}(\alpha,\beta)$.

(vi) For $k=2$, $\lambda=0$, $M_{sc}^{2k}(\alpha,\beta,\lambda)$ reduces to the class $S_{sc}^{*}(\alpha,\beta)$.

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PRELIMINARY RESULTS

Lemma 2 [11]**.** Suppose that the function ϕ is convex and univalent in E with $\phi(0)=1$ and

$$Re(\beta\phi(z)+\gamma)>0 \text{ for } \beta,\ \gamma\in\mathbb{C},\ z\in E.$$

If p is analytic in E with $p(0)=1$, then

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec\phi(z) \text{ implies } \ p(z)\prec\phi(z),\ z\in E.$$

Lemma 3 [12]**.** Let $\beta,\gamma\in\mathbb{C}$ and ϕ be a convex and univalent function with

$$Re(\beta\phi(z)+\gamma)>0,\ z\in E.$$

Also, let $h\in A: h(z)\prec\phi(z)$. If $p\in P$ and

$$p(z)+\frac{(zp'(z))}{\beta h(z)+\gamma}\prec\phi(z), \text{ then } p(z)\prec\phi(z).$$

Lemma 4 [13]**.** Let F be analytic and convex and univalent in E. If $f,\ g\in A$ and $f,\ g\prec F$, then

$$\sigma f+(1-\sigma)g\prec F,\ 0\le\sigma\le 1.$$

MAIN RESULTS

Theorem 1. A function $f\in M_{sc}^{2k}(\alpha,\beta,\lambda)$ if and only if

$$(1-\lambda)\frac{zf'(z)}{f_{2k}(z)}+\lambda\frac{(zf'(z))'}{f'_{2k}(z)}\prec\frac{1+\beta z}{1-\alpha\beta z}.$$

Proof. Let $f\in M_{sc}^{2k}(\alpha,\beta,\lambda)$. Then from (3) we have

$$\left|(1-\lambda)\frac{zf'(z)}{f_{2k}(z)}+\lambda\frac{(zf'(z))'}{f'_{2k}(z)}-1\right|<\beta\left|\alpha\left[(1-\lambda)\frac{zf'(z)}{f_{2k}(z)}+\lambda\frac{(zf'(z))'}{f'_{2k}(z)}\right]+1\right|.$$

By taking $F_{2k}(z)=(1-\lambda)\dfrac{zf'(z)}{f_{2k}(z)}+\lambda\dfrac{(zf'(z))'}{f'_{2k}(z)}$, we have

$$|F_{2k}(z)-1|^2<\beta^2|\alpha F_{2k}(z)+1|^2$$

or

$$(1-\alpha^2\beta^2)|F_{2k}(z)|^2-2(1+\alpha\beta^2)\operatorname{Re}F_{2k}(z)<\beta^2-1.$$

If $\alpha\neq 1$ or $\beta\neq 1$, then we have

$$|F_{2k}(z)|^2-2\left(\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right)\operatorname{Re}F_{2k}(z)+\left(\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right)^2<\frac{\beta^2-1}{1-\alpha^2\beta^2}+\left(\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right)^2$$

or

$$\left|F_{2k}(z)-\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right|^2<\frac{\beta^2(1+\alpha)^2}{(1-\alpha^2\beta^2)^2},$$

which represents the disk with centre at $\dfrac{1+\alpha\beta^2}{1-\alpha^2\beta^2}$ and radius $\dfrac{\beta(1+\alpha)}{1-\alpha^2\beta^2}$. The function

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$$\omega(z) \prec \phi(z) = \frac{1+\beta z}{1-\alpha\beta z}$$

maps the unit disk onto the disk

$$\left|\omega - \frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right| < \frac{\beta(1+\alpha)}{1-\alpha^2\beta^2}$$

and we notice that $F_{2k}(E) \subset \phi(E)$, $F_{2k}(0) = \phi(0)$ and ϕ is univalent in E. Therefore, we get

$$F_{2k}(z) \prec \phi(z) = \frac{1+\beta z}{1-\alpha\beta z}.$$

Conversely, suppose that $F_{2k}(z) \prec \dfrac{1+\beta z}{1-\alpha\beta z}$. Then using subordination, we write

$$F_{2k}(z) = \frac{1+\beta w(z)}{1-\alpha\beta w(z)} \tag{4}$$

where $|w(z)|<1$. From (4) we have

$$\left|F_{2k}(z)-1\right| = \left|\frac{1+\beta w(z)}{1-\alpha\beta w(z)} - 1\right| = \left|\frac{\beta w(z)+\alpha\beta w(z)}{1-\alpha\beta w(z)}\right| = \beta\left|\frac{(1+\alpha)w(z)}{1-\alpha\beta w(z)}\right|. \tag{5}$$

Also,

$$\left|\alpha F_{2k}(z)+1\right| = \left|\frac{\alpha+\alpha\beta w(z)}{1-\alpha\beta w(z)} + 1\right| = \beta\left|\frac{1+\alpha}{1-\alpha\beta w(z)}\right|. \tag{6}$$

By using (6) in (5), we obtain

$$\left|F_{2k}(z)-1\right| = \beta\left|(\alpha F_{2k}(z)+1)w(z)\right| < \beta\left|\alpha F_{2k}(z)+1\right|,$$

where $|w(z)|<1$ for all $z \in E$. This implies that

$$\left|(1-\lambda)\frac{zf'(z)}{f_{2k}(z)} + \lambda\frac{(zf'(z))'}{f'_{2k}(z)} - 1\right| < \beta\left|\alpha\left[(1-\lambda)\frac{zf'(z)}{f_{2k}(z)} + \lambda\frac{(zf'(z))'}{f'_{2k}(z)}\right] + 1\right|.$$

Hence $f \in M_{sc}^{2k}(\alpha,\beta,\lambda)$.

Theorem 2. Let $f \in M_{sc}^{2k}(\alpha,\beta,\lambda)$. Then $f_{2k} \in M_{sc}(\alpha,\beta,\lambda)$. Furthermore, $f_{2k} \in S_{sc}^{*}(\alpha,\beta,\lambda)$.

Proof. Let $f \in M_{sc}^{2k}(\alpha,\beta,\lambda)$. Then by Theorem 1, we write

$$(1-\lambda)\frac{zf'(z)}{f_{2k}(z)} + \lambda\frac{(zf'(z))'}{f'_{2k}(z)} \prec \frac{1+\beta z}{1-\alpha\beta z},$$

where f_{2k} is defined by (2), $0 \le \alpha \le 1$, $0 < \beta \le 1$, $k \ge 2$ is fixed positive integer, $\lambda > 0$ and $z \in E$. Using subordination, we write

$$(1-\lambda)\frac{zf'(z)}{f_{2k}(z)} + \lambda\frac{(zf'(z))'}{f'_{2k}(z)} = \frac{1+\beta w(z)}{1-\alpha\beta w(z)}. \tag{7}$$

On replacing z by $x^m z$ in (7) for $m = 0,1,2...,k-1, x^{2k} = 1$, we have

$$(1-\lambda)\frac{x^m zf'(x^m z)}{f_{2k}(x^m z)} + \lambda\frac{f'(x^m z) + x^m zf''(x^m z)}{f'_{2k}(x^m z)} = \frac{1+\beta w(x^m z)}{1-\alpha\beta w(x^m z)}. \tag{8}$$

Taking conjugate, we write

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$$(1-\lambda)\frac{\overline{x^m zf'\left(x^m\overline{z}\right)}}{\overline{f_{2k}\left(x^m\overline{z}\right)}}+\lambda\frac{\overline{f'\left(x^m\overline{z}\right)}+\overline{x^m\overline{z}f''\left(x^m\overline{z}\right)}}{\overline{f'_{2k}\left(x^m\overline{z}\right)}}=\frac{1+\beta\overline{w\left(x^m\overline{z}\right)}}{1-\alpha\beta\overline{w\left(x^m\overline{z}\right)}}. \tag{9}$$

Adding (8) and (9), we obtain

$$(1-\lambda)\left\{\frac{zf'\left(x^m z\right)}{f_{2k}\left(x^m z\right)}+\frac{\overline{x^m\overline{z}f'\left(x^m\overline{z}\right)}}{\overline{f_{2k}\left(x^m\overline{z}\right)}}\right\}$$

$$+\lambda\left\{\frac{f'\left(x^m z\right)+x^m zf''\left(x^m z\right)}{f'_{2k}\left(x^m z\right)}+\frac{\overline{f'\left(x^m\overline{z}\right)}+\overline{x^m\overline{z}f''\left(x^m\overline{z}\right)}}{\overline{f'_{2k}\left(x^m\overline{z}\right)}}\right\}=\frac{1+\beta w\left(x^m z\right)}{1-\alpha\beta w\left(x^m z\right)}+\frac{1+\beta\overline{w\left(x^m\overline{z}\right)}}{1-\alpha\beta\overline{w\left(x^m\overline{z}\right)}}.$$

By using (2) and applying summation for $m=0,1,2...,k-1$ in the above equation, we have

$$\frac{(1-\lambda)}{2k}\sum_{m=0}^{k-1}\left\{\frac{zf'\left(x^m z\right)}{f_{2k}\left(x^m z\right)}+\frac{\overline{x^m\overline{z}f'\left(x^m\overline{z}\right)}}{\overline{f_{2k}\left(x^m\overline{z}\right)}}\right\}$$

$$+\frac{\lambda}{2k}\sum_{m=0}^{k-1}\left\{\frac{f'\left(x^m z\right)+x^m zf''\left(x^m z\right)}{f'_{2k}\left(x^m z\right)}+\frac{\overline{f'\left(x^m\overline{z}\right)}+\overline{x^m\overline{z}f''\left(x^m\overline{z}\right)}}{\overline{f'_{2k}\left(x^m\overline{z}\right)}}\right\}$$

$$=\frac{1}{2k}\sum_{m=0}^{k-1}\left\{\frac{1+\beta w\left(x^m z\right)}{1-\alpha\beta w\left(x^m z\right)}+\frac{1+\beta\overline{w\left(x^m\overline{z}\right)}}{1-\alpha\beta\overline{w\left(x^m\overline{z}\right)}}\right\}.$$

Thus,

$$(1-\lambda)\frac{zf'_{2k}(z)}{f_{2k}(z)}+\lambda\frac{\left(zf'_{2k}(z)\right)'}{f'_{2k}(z)}=\frac{1}{2k}\sum_{m=0}^{k-1}\left\{\frac{1+\beta w\left(x^m z\right)}{1-\alpha\beta w\left(x^m z\right)}+\frac{1+\beta\overline{w\left(x^m\overline{z}\right)}}{1-\alpha\beta\overline{w\left(x^m\overline{z}\right)}}\right\}$$

or

$$(1-\lambda)\frac{zf'_{2k}(z)}{f_{2k}(z)}+\lambda\frac{\left(zf'_{2k}(z)\right)'}{f'_{2k}(z)}=\frac{1}{2k}\sum_{m=0}^{k-1}\left\{\frac{1+\beta w\left(x^m z\right)}{1-\alpha\beta w\left(x^m z\right)}+\frac{1+\beta\overline{w\left(x^m\overline{z}\right)}}{1-\alpha\beta\overline{w\left(x^m\overline{z}\right)}}\right\}\in P[\alpha,\beta],$$

where $P[\alpha,\beta]$ is a convex set and $p(z)\prec\frac{1+\beta z}{1-\alpha\beta z}$. This implies that

$$(1-\lambda)\frac{zf'_{2k}(z)}{f_{2k}(z)}+\lambda\frac{\left(zf'_{2k}(z)\right)'}{f'_{2k}(z)}\prec\frac{1+\beta z}{1-\alpha\beta z}, \tag{10}$$

which further implies that $f_{2k}\in M_{sc}(\alpha,\beta,\lambda)$.

Now, let $p(z)=\frac{zf'_{2k}(z)}{f_{2k}(z)}$. After some manipulation, we have

$$(1-\lambda)\frac{zf'_{2k}(z)}{f_{2k}(z)}+\lambda\frac{\left(zf'_{2k}(z)\right)'}{f'_{2k}(z)}=p(z)+\lambda\frac{zp'(z)}{p(z)}.$$

From (10), we obtain

$$p(z)+\lambda\frac{zp'(z)}{p(z)}\prec\frac{1+\beta z}{1-\alpha\beta z}.$$

Using Lemma 2, we get

$$p(z)=\frac{zf'_{2k}(z)}{f_{2k}(z)}\prec\frac{1+\beta z}{1-\alpha\beta z}.$$

Hence $f_{2k} \in S_{sc}(\alpha,\beta,\lambda)$.

Theorem 3. Let $0 \le \alpha \le 1,\ 0 < \beta \le 1,\ k \ge 2$ (fixed positive integer) and $\lambda > 0$. Then

$$M_{sc}^{2k}(\alpha,\beta,\lambda) \subset S_{sc}^{(k)}(\alpha,\beta).$$

Proof. Let $f \in M_{sc}^{2k}(\alpha,\beta,\lambda)$. Then by Theorem 1, we write

$$(1-\lambda)\frac{zf_{2k}'(z)}{f_{2k}(z)} + \lambda\frac{(zf_{2k}'(z))'}{f_{2k}'(z)} \prec \frac{1+\beta z}{1-\alpha\beta z},$$

where $f_{2k}(z)$ is defined by (2). Now we let

$$p(z) = \frac{zf'(z)}{f_{2k}(z)} \quad \text{and} \quad h(z) = \frac{zf_{2k}'(z)}{f_{2k}(z)},$$

where h and p satisfy the conditions of Lemma 3. This implies that

$$(1-\lambda)\frac{zf_{2k}'(z)}{f_{2k}(z)} + \lambda\frac{(zf_{2k}'(z))'}{f_{2k}'(z)} = p(z) + \lambda\frac{zp'(z)}{h(z)} \prec \frac{1+\beta z}{1-\alpha\beta z}. \tag{11}$$

From (11) and by using Lemma 3, we obtain

$$p(z) = \frac{zf'(z)}{f_{2k}(z)} \prec \frac{1+\beta z}{1-\alpha\beta z},$$

which implies that $f \in S_{sc}^{(k)}(\alpha,\beta)$. Hence

$$M_{sc}^{2k}(\alpha,\beta,\lambda) \subset S_{sc}^{(k)}(\alpha,\beta).$$

Theorem 4. Let $0 \le \alpha \le 1$ and $0 < \beta \le 1,\ 0 \le \lambda_1 < \lambda_2$. Then

$$M_{sc}^{2k}(\alpha,\beta,\lambda_2) \subset M_{sc}^{2k}(\alpha,\beta,\lambda_1).$$

Proof. Suppose that $f \in M_{sc}^{2k}(\alpha,\beta,\lambda_2)$. Then by Theorem 1 we have

$$h_1(z) = (1-\lambda_2)\frac{zf_{2k}'(z)}{f_{2k}(z)} + \lambda_2\frac{(zf_{2k}'(z))'}{f_{2k}'(z)} \prec \frac{1+\beta z}{1-\alpha\beta z}.$$

Also from Theorem 3, we write

$$h_2(z) = \frac{zf'(z)}{f_{2k}(z)} \prec \frac{1+\beta z}{1-\alpha\beta z}.$$

Now

$$(1-\lambda_1)\frac{zf_{2k}'(z)}{f_{2k}(z)} + \lambda_1\frac{(zf_{2k}'(z))'}{f_{2k}'(z)} = \left(1-\frac{\lambda_1}{\lambda_2}\right)\frac{zf'(z)}{f_{2k}(z)} + \frac{\lambda_1}{\lambda_2}\left\{(1-\lambda_2)\frac{zf_{2k}'(z)}{f_{2k}(z)} + \lambda_2\frac{(zf_{2k}'(z))'}{f_{2k}'(z)}\right\}$$

$$= \left(1-\frac{\lambda_1}{\lambda_2}\right)h_1(z) + \frac{\lambda_1}{\lambda_2}h_2(z).$$

Since $\frac{1+\beta z}{1-\alpha\beta z}$ is a convex set, therefore by using Lemma 4 we get the required result.

Theorem 5. Let $f \in M_{sc}^{2k}(\alpha,\beta,\lambda)$. Then we have

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$$f_{2k}(z)=\left[\frac{1}{\lambda}\int_0^z\frac{1}{u}\left[u\exp\int_0^u\frac{(1+\alpha)\beta}{\varsigma}\frac{1}{2k}\sum_{m=0}^{k-1}\left(\frac{w(x^m\varsigma)}{1-\alpha\beta w(x^m\varsigma)}+\frac{\overline{w(x^m\overline{\varsigma})}}{1-\alpha\beta\overline{w(x^m\overline{\varsigma})}}\right)d\varsigma\right]^{\frac{1}{\lambda}}du\right]^{\lambda},$$

where f_{2k} is defined by equality (2), $\lambda\neq 0$, w is analytic in E with $w(0)=0$ and $|w(z)|<1$.

Proof. Let $f\in M_{sc}^{2k}(\alpha,\beta,\lambda)$. Then from Theorem 1, we have

$$(1-\lambda)\frac{zf'(z)}{f_{2k}(z)}+\lambda\frac{(zf'(z))'}{f'_{2k}(z)}\prec\frac{1+\beta z}{1-\alpha\beta z}.$$

By using subordination, we have

$$(1-\lambda)\frac{zf'(z)}{f_{2k}(z)}+\lambda\frac{(zf'(z))'}{f'_{2k}(z)}=\frac{1+\beta w(z)}{1-\alpha\beta w(z)},$$

where w is analytic with $w(0)=0$ and $|w(z)|<1$. Replacing z by $x^m z$ for $m=0,1,2...,k-1$, $w=exp\frac{2\pi j}{k}$ and using (2), we write

$$(1-\lambda)\frac{x^m zf'(x^m z)}{f_{2k}(x^m z)}+\lambda\frac{f'(x^m z)+x^m zf''(x^m z)}{f'_{2k}(x^m z)}=\frac{1+\beta w(x^m z)}{1-\alpha\beta w(x^m z)} \tag{12}$$

and

$$(1-\lambda)\frac{\overline{x^m zf'(x^m\overline{z})}}{\overline{f_{2k}(x^m\overline{z})}}+\lambda\frac{\overline{f'(x^m\overline{z})}+\overline{x^m zf''(x^m\overline{z})}}{\overline{f'_{2k}(x^m\overline{z})}}=\frac{1+\beta\overline{w(x^m\overline{z})}}{1-\alpha\beta\overline{w(x^m\overline{z})}}. \tag{13}$$

Adding (12) and (13), we obtain

$$(1-\lambda)\left\{\frac{zf'(x^m z)}{f_{2k}(x^m z)}+\frac{\overline{x^m\overline{z}f'(x^m\overline{z})}}{\overline{f_{2k}(x^m\overline{z})}}\right\}$$
$$+\lambda\left\{\frac{f'(x^m z)+x^m zf''(x^m z)}{f'_{2k}(x^m z)}+\frac{\overline{f'(x^m\overline{z})}+\overline{x^m\overline{z}f''(x^m\overline{z})}}{\overline{f'_{2k}(x^m\overline{z})}}\right\}=\frac{1+\beta w(x^m z)}{1-\alpha\beta w(x^m z)}+\frac{1+\beta\overline{w(x^m\overline{z})}}{1-\alpha\beta\overline{w(x^m\overline{z})}}.$$

Again using (2) and applying summation for $m=0,1,2...,k-1$ in the above equation, we get

$$\frac{(1-\lambda)}{2k}\sum_{m=0}^{k-1}\left\{\frac{zf'(x^m z)}{f_{2k}(x^m z)}+\frac{\overline{x^m\overline{z}f'(x^m\overline{z})}}{\overline{f_{2k}(x^m\overline{z})}}\right\}$$
$$+\frac{\lambda}{2k}\sum_{m=0}^{k-1}\left\{\frac{f'(x^m z)+x^m zf''(x^m z)}{f'_{2k}(x^m z)}+\frac{\overline{f'(x^m\overline{z})}+\overline{x^m\overline{z}f''(x^m\overline{z})}}{\overline{f'_{2k}(x^m\overline{z})}}\right\}$$
$$=\frac{1}{2k}\sum_{m=0}^{k-1}\left\{\frac{1+\beta w(x^m z)}{1-\alpha\beta w(x^m z)}+\frac{1+\beta\overline{w(x^m\overline{z})}}{1-\alpha\beta\overline{w(x^m\overline{z})}}\right\}.$$

Therefore,

$$(1-\lambda)\frac{zf'_{2k}(z)}{f_{2k}(z)}+\lambda\frac{(zf'_{2k}(z))'}{f'_{2k}(z)}=\frac{1}{2k}\sum_{m=0}^{k-1}\left\{\frac{1+\beta w(x^m z)}{1-\alpha\beta w(x^m z)}+\frac{1+\beta\overline{w(x^m\overline{z})}}{1-\alpha\beta\overline{w(x^m\overline{z})}}\right\}. \tag{14}$$

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From (13), we obtain

$$(1-\lambda)\frac{zf_{2k}'(z)}{f_{2k}(z)}+\lambda\frac{(zf_{2k}'(z))'}{f_{2k}'(z)}-\frac{1}{z}=\frac{1}{2k}\sum_{m=0}^{k-1}\left\{\frac{1+\beta w(x^m z)}{1-\alpha\beta w(x^m z)}+\frac{1+\beta\overline{w(x^m\overline{z})}}{1-\alpha\beta\overline{w(x^m\overline{z})}}\right\}-\frac{1}{z}$$

or

$$(1-\lambda)\frac{zf_{2k}'(z)}{f_{2k}(z)}+\lambda\frac{(zf_{2k}'(z))'}{f_{2k}'(z)}-\frac{1}{z}=\sum_{m=0}^{k-1}\frac{(1+\alpha)\beta}{z}\left(\frac{w(x^m z)}{1-\alpha\beta w(x^m z)}+\frac{\overline{w(x^m\overline{z})}}{1-\alpha\beta\overline{w(x^m\overline{z})}}\right).$$

On integration, we have

$$\log\left\{\frac{(f_{2k}(z))^{1-\lambda}(zf_{2k}'(z))^{\lambda}}{z}\right\}=\int_0^z\frac{(1+\alpha)\beta}{\varsigma}\frac{1}{2k}\sum_{m=0}^{k-1}\left(\frac{w(x^m\varsigma)}{1-\alpha\beta w(x^m\varsigma)}+\frac{\overline{w(x^m\overline{\varsigma})}}{1-\alpha\beta\overline{w(x^m\overline{\varsigma})}}\right)d\varsigma$$

or

$$\left[\frac{zf_{2k}'(z)}{f_{2k}(z)}\right]^{\lambda}f_{2k}(z)=z\exp\int_0^z\frac{(1+\alpha)\beta}{\varsigma}\frac{1}{2k}\sum_{m=0}^{k-1}\left(\frac{w(x^m\varsigma)}{1-\alpha\beta w(x^m\varsigma)}+\frac{\overline{w(x^m\overline{\varsigma})}}{1-\alpha\beta\overline{w(x^m\overline{\varsigma})}}\right)d\varsigma\,. \tag{15}$$

Let

$$F_{2k}(z)=\left[\frac{zf_{2k}'(z)}{f_{2k}(z)}\right]^{\lambda}f_{2k}(z),\ F_{2k}(0)=0,\ F_{2k}'(0)=1. \tag{16}$$

Since f_{2k} is λ-convex and if λ is not an integer, we can select a suitable branch so that $F_{2k}(z)$ is analytic in E. Logarithmic differentiation of (16) gives

$$\frac{zF_{2k}'(z)}{F_{2k}(z)}=(1-\lambda)\frac{zf_{2k}'(z)}{f_{2k}(z)}+\lambda\frac{(zf_{2k}'(z))'}{f_{2k}'(z)}.$$

Since f_{2k} is λ-convex in E, F_{2k} is starlike in E. Now we solve (15) for f_{2k} by assuming that $\lambda\neq 0$. (The case when $\lambda=0$ gives $f_{2k}(z)=F_{2k}(z)$). A formal manipulation leads to the solution:

$$f_{2k}(z)=\left[\frac{1}{\lambda}\int_0^z\frac{(F_{2k}(\varsigma))^{\frac{1}{\lambda}}}{\varsigma}d\varsigma\right]^{\lambda}.$$

By using (15), we have

$$F_{2k}(z)=\left[\frac{zf_{2k}'(z)}{f_{2k}(z)}\right]^{\lambda}f_{2k}(z)=z\exp\int_0^z\frac{(1+\alpha)\beta}{\varsigma}\frac{1}{2k}\sum_{m=0}^{k-1}\left(\frac{w(x^m\varsigma)}{1-\alpha\beta w(x^m\varsigma)}+\frac{\overline{w(x^m\overline{\varsigma})}}{1-\alpha\beta\overline{w(x^m\overline{\varsigma})}}\right)d\varsigma$$

or

$$f_{2k}(z)=\left[\frac{1}{\lambda}\int_0^z\frac{1}{u}\left[u\exp\int_0^u\frac{(1+\alpha)\beta}{\varsigma}\frac{1}{2k}\sum_{m=0}^{k-1}\left(\frac{w(x^m\varsigma)}{1-\alpha\beta w(x^m\varsigma)}+\frac{\overline{w(x^m\overline{\varsigma})}}{1-\alpha\beta\overline{w(x^m\overline{\varsigma})}}\right)d\varsigma\right]^{\frac{1}{\lambda}}du\right]^{\lambda},$$

which is the required integral representation for f_{2k} when $f\in M_{sc}^{2k}(\alpha,\beta,\lambda)$.

Theorem 6. Let $f\in M_{sc}^{2k}(\alpha,\beta,\lambda)$. Then

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$$f(z)=\int_0^z \frac{1+c}{u\left[f_{2k}(u)\right]^c}\left[\int_0^u [f_{2k}(t)]^c f'_{2k}(t)\frac{1+\beta w(t)}{1-\alpha\beta w(t)}dt\right]du,$$

where f_{2k} is defined by (2), w is analytic in E with $w(0)=0,\ |w(z)|<1$ and $c=\frac{1}{\lambda}$. If $\lambda=0$, then we have

$$f(z)=\int_0^z \frac{f_{2k}(u)}{u}\frac{1+\beta w(u)}{1-\alpha\beta w(u)}du.$$

Proof. Let $f\in M_{sc}^{2k}(\alpha,\beta,\lambda)$. Then by Theorem 1, we have

$$(1-\lambda)\frac{zf'(z)}{f_{2k}(z)}+\lambda\frac{(zf'(z))'}{f'_{2k}(z)}=\frac{1+\beta w(z)}{1-\alpha\beta w(z)},$$

where w is analytic in E with $w(0)=0$ and $|w(z)|<1$. For $\lambda\neq 0$, multiplying both sides by $\frac{1}{\lambda}[f_{2k}(z)]^{\frac{1}{\lambda}-1}f'_{2k}(z)$, we get

$$czf'(z)[f_{2k}(z)]^{c-1}f'_{2k}(z)+[f_{2k}(z)]^c(zf'(z))'=(1+c)[f_{2k}(z)]^c f'_{2k}(z)\frac{1+\beta w(z)}{1-\alpha\beta w(z)},$$

where $c=1/\lambda$. The left-hand side of the above equation is the exact differential of $zf'(z)[f_{2k}(z)]^c$. Therefore, on integrating both sides with respect to z we obtain

$$f'(z)=\frac{1+c}{z\left[f_{2k}(z)\right]^c}\int_0^z [f_{2k}(\varsigma)]^c f'_{2k}(t)\frac{1+\beta w(\varsigma)}{1-\alpha\beta w(\varsigma)}d\varsigma.$$

Therefore,

$$f(z)=\int_0^z \frac{1+c}{u\left[f_{2k}(u)\right]^c}\left[\int_0^u [f_{2k}(t)]^c f'_{2k}(t)\frac{1+\beta w(t)}{1-\alpha\beta w(t)}dt\right]du.$$

If $\lambda=0$, then we have

$$f'(z)=\frac{f_{2k}(z)}{z}\frac{1+\beta w(z)}{1-\alpha\beta w(z)}.$$

Hence

$$f(z)=\int_0^z \frac{f_{2k}(u)}{u}\frac{1+\beta w(u)}{1-\alpha\beta w(u)}du.$$

Theorem 7. Let $f\in M_{sc}^{2k}(\alpha,\beta,\lambda)$. Then

$$\frac{1}{z}\left\{f(z)*\left(\frac{z}{(1-z)^2}-\frac{1+\beta e^{i\theta}}{2\left(1-\alpha\beta e^{i\theta}\right)}h(z)\right)\right\}-\frac{1+\beta e^{i\theta}}{2\left(1-\alpha\beta e^{i\theta}\right)}\overline{(f*h)(\bar{z})}\neq 0,$$

where $0\le\theta\le 2\pi,\ 0\le\alpha\le 1,\ 0<\beta\le 1, \lambda>0$ and $z\in E$.

Proof. Let $f\in M_{sc}^{2k}(\alpha,\beta,\lambda)$. Then by using Theorem 2 we have

$$f\in S_{sc}^{(k)}(\alpha,\beta),$$

which implies that, for $0\le\theta\le 2\pi$, we can write

$$\frac{zf'(z)}{f_{2k}(z)}\prec\frac{1+\beta z}{1-\alpha\beta z},$$

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and $\dfrac{zf'(z)}{f_{2k}(z)} \neq \dfrac{1+\beta e^{i\theta}}{1-\alpha\beta e^{i\theta}}$ implies that $zf'(z)-\left(\dfrac{1+\beta e^{i\theta}}{1-\alpha\beta e^{i\theta}}\right)f_{2k}(z)\neq 0.$

Therefore,

$$\frac{1}{z}\left\{zf'(z)-\left(\frac{1+\beta e^{i\theta}}{1-\alpha\beta e^{i\theta}}\right)f_{2k}(z)\right\}\neq 0. \tag{17}$$

For $zf'(z)=f(z)*\dfrac{z}{(1-z)^2}$ and $f_{2k}(z)=z+\sum_{j=0}^{\infty}\dfrac{a_j+\overline{a_j}}{2}c_j z^j=\dfrac{1}{2}\left\{(f*h)(z)+\overline{(f*h)(\overline{z})}\right\}$, where $h(z)=\dfrac{1}{k}\sum_{m=0}^{k-1}\dfrac{z}{1-w^m(z)}$, from (17) we obtain

$$\frac{1}{z}\left\{f(z)*\frac{z}{(1-z)^2}-\left(\frac{1+\beta e^{i\theta}}{1-\alpha\beta e^{i\theta}}\right)\frac{1}{2}\left\{(f*h)(z)+\overline{(f*h)(\overline{z})}\right\}\right\}\neq 0$$

or

$$\frac{1}{z}\left\{f(z)*\frac{z}{(1-z)^2}-\left(\frac{1+\beta e^{i\theta}}{1-\alpha\beta e^{i\theta}}\right)(f*h)(z)-\frac{1}{2}\left(\frac{1+\beta e^{i\theta}}{1-\alpha\beta e^{i\theta}}\right)\overline{(f*h)(\overline{z})}\right\}\neq 0$$

or

$$\frac{1}{z}\left\{f(z)*\left(\frac{z}{(1-z)^2}-\frac{1+\beta e^{i\theta}}{2\left(1-\alpha\beta e^{i\theta}\right)}h(z)\right)\right\}-\frac{1+\beta e^{i\theta}}{2\left(1-\alpha\beta e^{i\theta}\right)}\overline{(f*h)(\overline{z})}\neq 0.$$

Theorem 8. Let $f(z)=z+\sum_{j=2}^{\infty}a_j z^j$ be analytic in E and $0\leq\alpha\leq 1,\ 0<\beta\leq 1,\ k\geq 2, \lambda\geq 0$ with

$$\begin{aligned}&\sum_{j=2,\, j\neq lk+1}^{\infty}((1-\lambda)(1-\alpha\beta)j+\lambda(1-\alpha\beta)j^2)\,|a_j|+\\&\sum_{j=1}^{\infty}\left[\{(1-\lambda)(jk+1)+\lambda\}(1-\alpha\beta)+(1-\beta)(2+jk)\right]|Re(a_{jk+1})a_{jk+1}|+\\&\sum_{j=1}^{\infty}(1-\alpha\beta)(jk+1)^2\,|Re(a_{jk+1})a_{jk+1}|+(1-\beta)\sum_{j=1}^{\infty}(jk+1)\,|Re(a_{jk+1})|^2<\beta(1+\alpha)-2,\end{aligned} \tag{18}$$

where $l=0,1,2,\ldots$. Then $f\in M_{sc}^{2k}(\alpha,\beta,\lambda)$.

Proof. For the proof of this theorem, the desired result can be obtained by using series representation (1) and (2) of f and f_{2k} respectively in

$$\begin{aligned}M=&|(1-\lambda)zf'(z)f_{2k}'(z)+\lambda\left(zf'(z)\right)'f_{2k}(z)-f_{2k}(z)f_{2k}'(z)|-\\&\beta\,|\alpha\left\{(1-\lambda)zf'(z)f_{2k}'(z)+\lambda\left(zf'(z)\right)'f_{2k}(z)\right\}+f_{2k}(z)f_{2k}'(z)|\end{aligned}$$

and then applying the condition given above in (18).

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