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Original Article

On the class of p-absolutely summable sequence $\ell^i(p)$ of interval numbers

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Abstract

In this article we have studied on the class of p-absolutely summable sequence of interval numbers. We have discussed some important properties like linearity, completeness, solidity, symmetry and inclusion relation.

Keywords: interval number, solid, symmetric, convergence free, sequence algebra

1. Introduction

Most of the scientific and numerical computations are based on a mathematical model in which the variables range over the real numbers. Interval arithmetic serves as a methodology to analyze and control numerical errors in computers and is widely used in real-valued numerical calculations, particularly those related to rounded and truncated errors appear in machine computations. The idea of interval arithmetic was used by Dwyer (1951; 1953) at the initial stage. Ramon E. Moore has applied interval arithmetic as an approach to round errors in mathematical computation. Further development on interval arithmetic was done by Moore (1966; 1959), Moore and Yang (1958, 1962), Fischer (1958). The useful fact of interval analysis is that, rather than a machine-rounded approximation, the solutions obtained here are intervals that guarantee the enclosure of the desired solution. Today interval algorithms are being used to solve numerical analysis, global optimization, and several engineering and CAD problems.

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2. Preliminaries

An interval $\bar{x} = [a, b]$ is the set of real numbers between a and b, i.e. $\bar{x} = [a, b] = \{x: a \le x \le b\}$. Let R denote the set of all real valued closed intervals. An interval number is an element of R and a closed subset of real numbers, represented by $\bar{x} = [x_\ell, x_r]$, where x_ℓ and x_r are the left and right points of $\bar{x}$ respectively. Geometrically $\bar{x}$ represents a line segment on the real line. In particular if $x_\ell = x_r = x$, then $\bar{x}$ is reduced to a real number $x = [x, x]$, called point interval or singleton. Thus we can say that an interval number is the generalization of the point interval.

For $\bar{x}_1, \bar{x}_2 \in R$ we have the following fundamental arithmetical operations.

(i) $\bar{x}_1, \bar{x}_2 \Rightarrow x_{1\ell} = x_{2\ell}$ and $x_{1r} = x_{2r}$.

(ii) $\bar{x}_1 + \bar{x}_2 = [x_{1\ell} + x_{2\ell}, x_{1r} + x_{2r}]$.

(iii) Let $\alpha \ge 0$, then $\alpha \bar{x} = [\alpha x_\ell, \alpha x_r]$ and if $\alpha < 0$, then $\alpha \bar{x} = [a x_r, \alpha x_\ell]$.

(iv) $\bar{x}_1 . \bar{x}_2 = [\min\{x_{1\ell}.x_{2\ell}, x_{1\ell}.x_{2r}, x_{1r}.x_{2\ell}, x_{1r}.x_{2r}\}, \max\{x_{1\ell}.x_{2\ell}, x_{1\ell}.x_{2r}, x_{1r}.x_{2\ell}, x_{1r}.x_{2r}\}]$.

(v) $\dfrac{\bar{x}_1}{\bar{x}_2} = [x_{1\ell}, x_{1r}] \times \dfrac{1}{[x_{2_r}, x_{2_l}]}$.

$= [\min\{ x_{1\ell} \div x_{2\ell}, x_{1\ell} \div x_{2r}, x_{1r} \div x_{2\ell}, x_{1r} \div x_{2r}\},$

$\max\{ x_{1\ell} \div x_{2\ell}, x_{1\ell} \div x_{2r}, x_{1r} \div x_{2\ell}, x_{1r} \div x_{2r}\}],\ 0 \notin \bar{x}_2$.

(vi) If $\bar{x}_1 \subset \bar{x}_2$ i.e. $[x_{1\ell}, x_{1r}] \subset [x_{2\ell}, x_{2r}] \Rightarrow x_{2\ell} < x_{1\ell} < x_{1r} < x_{2r}$.

Remark: The above property can be generalized for more than two intervals and is often called as nesting property of intervals.

(vii) Let $\bar{x} = [x_\ell, x_r]$, then $-\bar{x} = -[x_\ell, x_r] = [-x_r, -x_\ell]$.

(viii) Let $\bar{x}_1, \bar{x}_2, \bar{x}_3 \in R$, then $\bar{x}_3(\bar{x}_1 + \bar{x}_2) \subseteq \bar{x}_3\,\bar{x}_1 + \bar{x}_3\,\bar{x}_2$.

Remark: The equality $\bar{x}_3(\bar{x}_1 + \bar{x}_2) = \bar{x}_3\,\bar{x}_1 + \bar{x}_3\,\bar{x}_2$ holds with the condition that if $a \in [x_{1\ell}, x_{1r}] = \bar{x}_1$ and $b \in [x_{2\ell}, x_{2r}] = \bar{x}_2$ then $ab \geq 0$. It holds well with the point interval $a = [a, a]$, i.e. $a(\bar{x}_1 + \bar{x}_2) = a\,\bar{x}_1 + a\,\bar{x}_2$.

(ix) For $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4 \in R$ if $\bar{x}_1 \subset \bar{x}_2$ and $\bar{x}_3 \subset \bar{x}_4$ then

(a) $\bar{x}_1 + \bar{x}_3 \subset \bar{x}_2 + \bar{x}_4$

(b) $\bar{x}_1 - \bar{x}_3 \subset \bar{x}_2 - \bar{x}_4$

(c) $\bar{x}_1 . \bar{x}_3 \subset \bar{x}_2 . \bar{x}_4$

(d) $\bar{x}_1 / \bar{x}_3 \subset \bar{x}_2 / \bar{x}_4$, if $0 \notin \bar{x}_3, \bar{x}_4$.

The absolute value of $\bar{x} = [x_\ell, x_r]$ is defined by

$$|\bar{x}| = \begin{cases} [\min\{|x_\ell||x_r|\},\ \max\{|x_\ell||x_r|\}], & \text{if } x_\ell . x_r \geq 0, \\ [0, \max\{|x_\ell||x_r|\}], & \text{if } x_\ell . x_r < 0. \end{cases}$$

It is known that R is a complete metric space. It is easy to see that the set of all interval numbers is a complete metric space defined by

$$d(\bar{x}_1, \bar{x}_2) = \max\{|x_{1\ell} - x_{2\ell}|, |x_{1r} - x_{2r}|\}.$$

In the special case of $\bar{x}_1 = [a, a]$ and $\bar{x}_2 = [b, b]$, we obtain usual metric on R, $d(\bar{x}_1, \bar{x}_2) = |a - b|$.

Consider the transformation f from N to R defined by $k \rightarrow f(k) = \bar{x}$, then $(\bar{x}_n)$ is called the sequence of interval numbers, where $\bar{x}_n$ is the n^{th} term of the sequence $(\bar{x}_n)$. We denote the set of all sequences of interval number by w^i.

For sequences of interval numbers $(\bar{x}_n)$, $(\bar{y}_n) \in w^i$ the linear structure of w^i includes the addition of $(\bar{x}_n) + (\bar{y}_n)$ and scalar multiplication $(\alpha\,\bar{x}_n)$ defined by

$$(\bar{x}_n) + (\bar{y}_n) = [\bar{x}_{n_\ell} + \bar{y}_{n_\ell}, \bar{x}_{n_r} + \bar{y}_{n_r}].$$

If $\alpha \geq 0$ then $(\alpha\,\bar{x}_n) = [\alpha\,\bar{x}_{n_\ell}, \alpha\,\bar{x}_{n_r}]$ and if $\alpha < 0$ then $(\alpha\,\bar{x}_n) = [\alpha\,\bar{x}_{n_r}, \alpha\,\bar{x}_{n_\ell}]$.

Definition 2.1. An interval sequence $\bar{x} = (\bar{x}_n)$ is said to be convergent to the interval number $\bar{x}_0$ if for each $\varepsilon > 0$ there exists a positive integer n_0 such that $d(\bar{x}_n, \bar{x}_0) < \varepsilon$ for all $n \geq n_0$, and we write it as $\lim_n \bar{x}_n = \bar{x}_0$ which imply $\lim_n \bar{x}_{n_\ell} = \bar{x}_{0_\ell}$ and $\lim_n \bar{x}_{n_r} = \bar{x}_{0_r}$.

Definition 2.2. An interval sequence $\bar{x} = (\bar{x}_n)$ is said to be interval Cauchy sequence if for every $\varepsilon > 0$ there exists $k_0 \in N$ such that $d(\bar{x}_n, \bar{x}_k) < \varepsilon$, for $n, k > k_0$.

We produce the following concepts for the classes of sequences of interval numbers.

Definition 2.3. An interval sequence $\bar{x} = (\bar{x}_n)$ is said to be bounded if $\sup_n d(\bar{x}_n, \theta) < \infty$, equivalently, if there exist $\mu \in R$ such that $|\bar{x}_n| \leq \mu$ for all $n \in N$.

Definition 2.4. An interval sequence space E^{i} is said to be solid if $(\bar{x}_n) \in E^{\text{i}}$ whenever $(\bar{y}_n) \in E^{\text{i}}$ and $(\bar{x}_n) \leq (\bar{y}_n)$, for all $n \in N$.

Definition 2.5. An interval sequence space E^{i} is said to be symmetric if $(\bar{x}_{\pi(n)}) \in E^{\text{i}}$, whenever $(\bar{x}_n) \in E^{\text{i}}$, where π is a permutation on N.

Definition 2.6. An interval sequence space E^{i} is said to be convergence free if $(\bar{y}_n) \in E^{\text{i}}$ whenever $(\bar{x}_n) \in E^{\text{i}}$ and $\bar{x}_n = \bar{0}$ implies that $\bar{y}_n = \bar{0}$.

Definition 2.7. An interval sequence space E^{i} is said to be sequence algebra if for $(\bar{x}_n), (\bar{y}_n) \in E^{\text{i}}$ $(\bar{x}_n \otimes \bar{y}_n) \in E^{\text{i}}$ whenever $(\bar{x}_n), (\bar{y}_n) \in E^{\text{i}}$.

Chiao (2002) introduced sequence of interval numbers and studied the usual convergence. Recently Esi (2011; 2011) has made several investigations on different classes of sequence of interval numbers. Şengönül and Eryılmaz (2010) introduced the following sequence spaces of interval numbers and proved their completeness.

$$c_0^i = \left\{\bar{x} = (\bar{x}_n) \in w^i : \lim_n \bar{x}_n = \theta, \text{ where } \theta = [0,\ 0]\right\}.$$

$$c^i = \left\{\bar{x} = (\bar{x}_n) \in w^i : \lim_n \bar{x}_n = \bar{x}_0,\ \bar{x}_0 \in R\right\}.$$

$$\ell_\infty^i \left\{\bar{x} = (\bar{x}_n) \in w^i : \sup_n\{|x_{n_\ell}|, |x_{n_r}|\} < \infty\right\}.$$

The class of p-absolutely summable sequence $\ell^i(p)$ of interval numbers is defined by

$$\ell^i(p) = \left\{\bar{x} = (\bar{x}_n) \in w^i : \sum_{n=1}^{\infty} \{d(\bar{x}_n, \theta)\}^{p_n} < \infty\right\},$$

where $\bar{x} = [x_\ell, x_r]$, and $p = (p_n)$ is a bounded sequence of +ve numbers so that $0 < p_n \leq \sup p_n < \infty$.

Consider the metric d on $\ell^i(p)$ is defined by

$$d(\bar{x}_n, \bar{y}_n) = \left\{\sum_k \{\max\{|x_{n_\ell} - y_{n_\ell}|, |x_{n_r} - y_{n_r}|\}\}^{p_n}\right\}^{\frac{1}{M}},$$

where $0 < p_n \leq \sup p_n < \infty$ and $M = \max(1, \sup p_n)$.

3. Main Results

Theorem 3.1. The class of sequence $\ell^i(p)$ is a linear space with the co-ordinate wise addition and scalar multiplication.

Proof. Let $(\bar{x}_n), (\bar{y}_n) \in \ell^i(p)$ and α, β be scalars. Then

$$\sum_{n=1}^{\infty}\{d(\bar{x}_n,\theta)\}^{p_n} < \infty \text{ and } \sum_{n=1}^{\infty}\{d(\bar{y}_n,\theta)\}^{p_n} < \infty.$$

We have

$$\sum_{n=1}^{\infty}\{d[(\bar{\alpha} x_n + \beta \bar{y}_n),\theta]\}^{p_n} \le \sum_{n=1}^{\infty}\{d(\bar{\alpha} x_n,\theta)\}^{p_n} +$$

$$\sum_{n=1}^{\infty}\{d(\beta \bar{y}_n,\theta)\}^{p_n} = |\alpha|^{p_n}\sum_{n=1}^{\infty}\{d(\bar{x}_n,\theta)\}^{p_n} +$$

$$|\beta|^{p_n}\sum_{n=1}^{\infty}\{d(\bar{y}_n,\theta)\}^{p_n} < \infty.$$

This completes the proof.

Theorem 3.2. The class of sequence $\ell^i(p)$ is a complete metric space with respect to the metric defined by

$$\bar{d}(\bar{x},\bar{y}) = \left\{\sum_n [d(\bar{x}_n,\bar{y}_n)]^{p_n}\right\}^{\frac{1}{M}}$$

Proof. It is easy to verify that $\bar{d}$ is a metric on $\ell^i(p)$. Let $\bar{x}^j = (\bar{x}_n^j) = (\bar{x}_1^j, \bar{x}_2^j, \bar{x}_3^j, \ldots)$ be a Cauchy sequence in $\ell^i(p)$ for each j. Then for every $\varepsilon > 0$ there exist a $n_0 \in N$ such that

$$\bar{d}(\bar{x}_n^j, \bar{x}_n^k) = \left\{\sum_n [d(\bar{x}_n^j, \bar{y}_n^k)]^{p_n}\right\}^{\frac{1}{M}} < \varepsilon, \text{ for } j, k \ge n_0.$$

$$\Rightarrow d(\bar{x}_n^j, \bar{y}_n^k) < \varepsilon, \text{ for } j, k \ge n_0.$$

$$\Rightarrow \left\{\sum_k \{\max |x_{n_\ell}^j - y_{n_\ell}^k|, |x_{n_r}^j - y_{n_r}^k|\}^{p_n}\right\}^{\frac{1}{M}} < \varepsilon.$$

This implies $|x_{n_\ell}^j - y_{n_\ell}^k| < \varepsilon$ and $|x_{n_r}^j - y_{n_r}^k| < \varepsilon$. This shows that $(\bar{x}^j)$ is a Cauchy sequence in R. Since R is complete, $(\bar{x}_n^j)$ is convergent. Let $\lim_{j\to\infty} \bar{x}_n^j = \bar{x}_n$ for each $n \in N$. Thus for each $\varepsilon > 0$, there exists n_0 such that $\bar{d}(\bar{x}_n^j, \bar{x}_n) < \varepsilon$, for $j \ge n_0$. The proof will be complete once we show that $\bar{x}_n \in \ell^i(p)$. We have

$$\bar{d}(\bar{x}_n, \theta) \le \bar{d}(\bar{x}_n, \bar{x}_n^j) + \bar{d}(\bar{x}_n^j, \theta)$$

$$< \varepsilon + K < \infty.$$

This completes the proof.

Theorem 3.3. The class of sequence $\ell^i(p)$ is solid and hence monotone.

Proof. Let $(\bar{x}_n)$ and $(\bar{y}_n)$ be two sequences of interval numbers such that $|\bar{x}_n| \le |\bar{y}_n|$, for all $k \in N$. Let $(\bar{x}_n) \in \ell^i(p)$ then $\sum_{n=1}^{\infty}\left\{d(\bar{x}_n,\theta)\right\}^{p_n} = \sum_n \left[\max\{|x_{n_\ell}|, |x_{n_r}|\}\right]^{p_n} < \infty$. Now we have

$$\sum_{n=1}^{\infty}\left\{d(\bar{y}_n,\theta)\right\}^{p_n} \le \sum_{n=1}^{\infty}\left\{d(\bar{x}_n,\theta)\right\}^{p_n} < \infty.$$

This implies $(\bar{y}_n) \in \ell^i(p)$.

Theorem 3.4. The class of sequence $\ell^i(p)$ is a sequence algebra.

Proof. Let $(\bar{x}_n)$ and $(\bar{y}_n)$ be two sequences of interval numbers taken from $\ell^i(p)$. Then we have

$$\sum_{n=1}^{\infty}\left\{d(\bar{x}_n,\theta)\right\}^{p_n} < \infty \text{ and } \sum_{n=1}^{\infty}\left\{d(\bar{y}_n,\theta)\right\}^{p_n} < \infty, \text{ for all } n \in N.$$

We have

$$\sum_{n=1}^{\infty}\left\{d(\bar{x}_n \otimes \bar{y}_n,\theta)\right\}^{p_n} \le \sum_{n=1}^{\infty}\left\{d(\bar{x}_n,\theta).d(\bar{y}_n,\theta)\right\}^{p_n}$$

$$\le \left(\sum_{n=1}^{\infty}\left\{d(\bar{x}_n,\theta)\right\}^{p_n}\right)\left(\sum_{n=1}^{\infty}\left\{d(\bar{y}_n,\theta)\right\}^{p_n}\right) < \infty.$$

Thus $(\bar{x}_n \otimes \bar{y}_n) \in \ell^i(p)$.

This completes the proof.

Theorem 3.5. The class of sequence $\ell^i(p)$ is not convergence free.

Proof. We provide the following example in support of the proof.

Example 1. Consider the interval sequence $(\bar{x}_n)$ defined by $\bar{x}_n = \left[\frac{1}{n+1}, \frac{1}{n}\right]$, for $n \in \mathbb{N}$ and take $p_n \ge 1$, for all $k \in N$. Then we have

$$\sum_{n=1}^{\infty}\left\{d(\bar{x}_n,\theta)\right\}^{p_n} = \sum_{n=1}^{\infty}\left[\max\left\{\left|\frac{1}{n+1}\right|, \left|\frac{1}{n}\right|\right\}\right]^{p_n} < \infty.$$

Now consider the interval sequence $(\bar{y}_n)$ defined by $\bar{y}_n = [n, n+1]$, for $n \in N$ and take $p_n \ge 1$, for all $k \in N$. Then we have

$$\sum_{n=1}^{\infty}\left\{d(\bar{y}_n,\theta)\right\}^{p_n} = \sum_{n=1}^{\infty}\left[\max\{|n|, |n+1|\}\right]^{p_n} \to \infty.$$

Thus we conclude that is not convergence free.

This completes the proof.

Theorem 3.6. The class of sequence $\ell^i(p)$ is not symmetric.

Proof. The proof follows from the following example.

Example 2. Consider the interval sequence $(\bar{x}_n)$ defined by

$$\bar{x}_n = \begin{cases} \left[\frac{1}{(n+1)^2}, \frac{1}{n^2}\right], & \text{for } n \text{ odd,} \\ \left[\frac{1}{n+1}, \frac{1}{n}\right], & \text{for } n \text{ even.} \end{cases}$$

Consider $p_n = 1$, for all n odd and $p_n = 2$, for all n even.
Then we have

$$\sum_{n=1}^{\infty} \left\{ d(\bar{x}_n, \theta) \right\}^{p_n} = \sum_{n=1}^{\infty} \left[\max \left\{ \left| \frac{1}{(n+1)^2} \right|, \left| \frac{1}{n^2} \right| \right\} \right] < \infty.$$

Now consider the rearrangement $(\bar{y}_n)$ of $(\bar{x}_n)$ defined by

$$(\bar{y}_n) = (\bar{x}_2, \bar{x}_1, \bar{x}_4, \bar{x}_3, \bar{x}_6, \ldots\ldots\ldots)$$

Then we have

$$\sum_{n=1}^{\infty} \left\{ d(\bar{y}_n, \theta) \right\}^{p_n} > \sum_{n \text{ odd}} \left[\max \left\{ \left| \frac{1}{(n+1)} \right|, \left| \frac{1}{n} \right| \right\} \right] = \infty.$$

Thus $(\bar{y}_n) \notin \ell^i(p)$.
This completes the proof.

Theorem 3.7. For $0 < p < q$, $\ell^i(p) \subset \ell^i(q)$.

Proof. The proof is easy, so omitted.

4. Conclusions

In this article we have discussed some properties of an important class of sequences of interval numbers with some easy and suitable examples. The results so obtained may link to some new ideas and help in studying some other properties of the class of sequences and some other classes of sequences.

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