APPENDICES

APPENDIX A
A.1
PROOF OF LEMMA 1.1

Pick any bidder $i \in N$. If bidder i does not win one spectrum license in the first round, her value of license in the second round is $x_{2,i}$. Given that bidder i's competitors follow some strictly increasing strategy $b(.)$ where the maximal of $b(.)$ is $b(x_{2,(1)})$. The bidder i's problem is to choose b_i to maximize

$$E\left[\left(x_{2,i} - x_{2,(1)}\right)\chi_{b_i > x_{2,(1)}}\right] \qquad (A.1)$$

Where

$$\chi_{b_i > x_{2(1)}} = 1 \quad if\ b_i > x_{2,(1)}$$
$$= 0 \quad otherwise$$

By definition $x_{2,(1)}$ is the maximal among n-1 samples, hence the distribution function is $F(x)^{n-1}$.

Therefore, (A.1) can be re-write as follow:

$$\int_0^{b_1} (n-1)(x_{2,i} - x) f(x) F(x)^{n-2} dx \qquad (A.2)$$

Now, we can show that $b_i = x_{2,i}$ (truth-telling bid) maximize (A.2)

Suppose first that $b_i < x_{2,i}$, then letting $b_i \to x_{2,i}$ the integral in (A.2) increased by

$$\int_{b_i}^{x_{2,i}} (n-1)(x_{2,i} - x) f(x) F(x)^{n-2} dx \qquad (A.3)$$

This is true since if $b_i < x < x_{2,i}$, then $x_{2,i} - x > 0$. The reverse happens if $b_i > x_{2,i}$ since in the region $b_i > x > x_{2,i}$ and the integrand is negative.

For the case that bidder i does win one spectrum license in the first round, we apply the similar argument to obtain the result.

APPENDIX B

B.1

PROOF OF PROPOSITION 2.1

For the sake of exposition, we present the proof for $k = 2$. The other cases can be proved analogously.

Consider $k = 2$,

$$U(c_i) = \int_{c_i}^{1}\int_{0}^{c_{(2)}} \pi^W(c_i; c_{(1)}) f_{12}^{n-1}(c_{(1)}, c_{(2)}) dc_{(1)} dc_{(2)}$$

$$+\int_{0}^{c_i}\int_{0}^{c_{(2)}} \pi^L(1; c_{(1)}, c_{(2)}) f_{12}^{n-1}(c_{(1)}, c_{(2)}) dc_{(1)} dc_{(2)} - m_2(c_i) \qquad (B.1)$$

We also know that,

$$P(\tilde{c}, c_i) = \int_{\tilde{c}}^{1}\int_{0}^{c_{(2)}} \pi^W(c_i; c_{(1)}) f_{12}^{n-1}(c_{(1)}, c_{(2)}) dc_{(1)} dc_{(2)}$$

$$+\int_{0}^{\tilde{c}}\int_{0}^{c_{(2)}} \pi^L(1; c_{(1)}, c_{(2)}) f_{12}^{n-1}(c_{(1)}, c_{(2)}) dc_{(1)} dc_{(2)} - m_2(\tilde{c}) \qquad (B.2)$$

The expression for $U(\tilde{c})$ and $P(c_i, \tilde{c})$ are obtained from (B.1) and (B.2). From incentive compatibility, we have;

$$U(c_i) \geq P(\tilde{c}, c_i) \qquad (B.3)$$

and

$$U(\tilde{c}) \geq P(c_i, \tilde{c}) \qquad (B.4)$$

Furthermore,

$$P(\tilde{c}, c_i) = U(\tilde{c}) + \int_{\tilde{c}}^{1}\int_{0}^{c_{(2)}} [\pi^W(c_i; c_{(1)}) - \pi^W(\tilde{c}, c_{(1)})] f_{12}^{n-1}(c_{(1)}, c_{(2)}) dc_{(1)} dc_{(2)} \qquad (B.5)$$

and

$$P(c_i, \tilde{c}) = U(c_i) + \int_{c_i}^{1}\int_{0}^{c_{(2)}} [\pi^W(\tilde{c}; c_{(1)}) - \pi^W(c_i, c_{(1)})] f_{12}^{n-1}(c_{(1)}, c_{(2)}) dc_{(1)} dc_{(2)} \qquad (B.6)$$

Therefore, from (B.3), (B.4), (B.5) and (B.6), we obtain the following inequality

$$\int_{\tilde{c}}^{1}\int_{0}^{c_{(2)}} [\pi^W(c_i; c_{(1)}) - \pi^W(\tilde{c}, c_{(1)})] f_{12}^{n-1}(c_{(1)}, c_{(2)}) dc_{(1)} dc_{(2)}$$

$$\le U(c_i) - U(\tilde{c})$$

$$\le \int_{c_i}^{1}\int_{0}^{c_{(2)}} [\pi^W(c_i;c_{(1)}) - \pi^W(\tilde{c}, c_{(1)})] f_{12}^{n-1}(c_{(1)}, c_{(2)}) dc_{(1)} dc_{(2)} \qquad (B.7)$$

By letting $\tilde{c} \to c_i$ it follow from (B.7) that

$$U'(c_i) = \int_{c_i}^{1}\int_{0}^{c_{(2)}} \frac{\partial \pi^W(c_i;c_{(1)})}{\partial c_i} f_{12}^{n-1}(c_{(1)}, c_{(2)}) dc_{(1)} dc_{(2)} \qquad (B.8)$$

Thus,

$$U(1) - U(c_i) = \int_{c_i}^{1}\int_{c}^{1}\int_{0}^{c_{(2)}} \frac{\partial \pi^W(c;c_{(1)})}{\partial c} f_{12}^{n-1}(c_{(1)}, c_{(2)}) dc_{(1)} dc_{(2)} dc$$

$$= \int_{c_i}^{1}\int_{0}^{c_{(2)}}\int_{c_i}^{c_{(2)}} \frac{\partial \pi^W(c;c_{(1)})}{\partial c} f_{12}^{n-1}(c_{(1)}, c_{(2)}) dc dc_{(1)} dc_{(2)}$$

$$= \int_{c_i}^{1}\int_{0}^{c_{(2)}} [\pi^W(c_{(2)};c_{(1)}) - \pi^W(c_i, c_{(1)})] f_{12}^{n-1}(c_{(1)}, c_{(2)}) dc_{(1)} dc_{(2)} \qquad (B.9)$$

Notice that, (4) and (5) imply that

$$U(1) = \int_{0_i}^{1}\int_{0}^{c_{(2)}} \pi^L(1; c_{(1)}, c_{(2)}) f_{12}^{n-1}(c_{(1)}, c_{(2)}) dc_{(1)} dc_{(2)} - m_2(1) \qquad (B.10)$$

Hence, from (B.9) and (B.10),

$$U(c_i) = \int_{0_i}^{1}\int_{0}^{c_{(2)}} \pi^L(1; c_{(1)}, c_{(2)}) f_{12}^{n-1}(c_{(1)}, c_{(2)}) dc_{(1)} dc_{(2)}$$

$$- \int_{c_i}^{1}\int_{0}^{c_{(2)}} [\pi^W(c_{(2)};c_{(1)}) - \pi^W(c_i, c_{(1)})] f_{12}^{n-1}(c_{(1)}, c_{(2)}) dc_{(1)} dc_{(2)} - m_2(1) \qquad (B.11)$$

Combing (B.1) and (B.11) and rearranging, we obtain:

$$m_2(c_i) = m_2(1) + \int_{c_i}^{1} \int_{0}^{c_{(2)}} [\pi^W(c_{(2)}; c_{(1)}) - \pi^L(1; c_{(1)}, c_{(2)})] f_{12}^{n-1}(c_{(1)}, c_{(2)}) dc_{(1)} dc_{(2)}$$

$$= m_2(1) + \int_{c_i}^{1} \{ \int_{0}^{c_{(2)}} [\pi^W(c_{(2)}; c_{(1)}) - \pi^L(1; c_{(1)}, c)] \frac{f_{12}^{n-1}(c_{(1)}, c_{(2)} = c)}{f_2^{n-1}(c)} dc_{(1)} \} f_2^{n-1}(c) dc$$

$$= m_2(1) + \int_{c_i}^{1} V_2(c) f_2^{n-1}(c) dc \qquad (B.12)$$

B.2

PROOF OF PROPOSITION 2.2

By definition,

$$R_k = n\int_0^1 m_k(c)g(c)dc \qquad (B.13)$$

Substituting (8) in (B.13) and integrating by parts, we obtain,

$$R_k = n\int_0^1 V_k(c)f_k^{n-1}(c)G(c)dc \qquad (B.14)$$

Note that,

$$F_k^{n-1}(c) = 1 - \sum_{j=0}^{k-1}\binom{n-1}{j}[G(c)]^j[1-G(c)]^{n-j-1} \qquad (B.15)$$

It can be shown that (B.15) implies,

$$f_k^{n-1}(c) = \binom{n-1}{k-1}(n-k)[G(c)]^{k-1}[1-G(c)]^{n-k-1}g(c) \qquad (B.16)$$

From which it follows that

$$nf_k^{n-1}(c)G(c) = kf_{k+1}^{n}(c) \qquad (B.17)$$

Substituting (B.17) into (B.14) yield

$$R_k = k\int_0^1 V_k(c)f_{k+1}^{n}(c)dc = kEV_k\left(c_{(k+1)}^{n}\right)$$

APPENDIX C

C.1

PROOF OF LEMMA 3.1

If bidder i does not participate in the mechanism, then her payoff π_i^c depends on how the auctioneer allocates the spectrum licenses in such an eventuality. Notice that, if bidder i does not participate in the mechanism, then the auctioneer can choose to allocate a license to each of her competitors, or a license to one of her competitors, or to not allocate any license. Hence, bidder i's payoff if she does not participate in the mechanism is either $\pi\left(0; s_{(1)}^2, s_{(2)}^2\right)$, $\pi\left(0; s_{(1)}^2, 0\right)$, $\pi\left(0; s_{(2)}^2, 0\right)$, or $\pi(0;0,0)$. Next, notice that it follows from (2)–(4) that

$$\pi\left(0; s_{(1)}^2, s_{(2)}^2\right) = \min\{\pi\left(0; s_{(1)}^2, s_{(2)}^2\right),\ \pi\left(0; s_{(1)}^2, 0\right),\ \pi\left(0; s_{(2)}^2, 0\right),\ \pi(0;0,0)\} \qquad (C.1)$$

Notice that (C.1) holds regardless of the auctioneer's choice of the allocation rule a in the mechanism. It also follows from (6) that in the optimal mechanism, the payoff of every nonparticipating bidder has to be minimized. Therefore, we obtain the result.

C.2
PROOF OF PROPOSITION 3.2

We denote the payoff of bidder i in the truth-telling equilibrium of the direct mechanism as follows:

$$\Psi_{iQ}(s_i) = V_{iQ}(s_i, s_i)$$

First, we prove the necessity part. Note that incentive compatibility implies that

$$\Psi_{iQ}(s_i) \geq V_{iQ}(r_i, s_i) \quad \textit{for all } r_i, s_i \in [0,1] \qquad (C.2)$$

Moreover, we can re-write $V_{iQ}(r_i, s_i)$ as follows:

$$V_{iQ}(r_i, s_i) = \Psi_{iQ}(r_i) + \sum_{k=1}^{2} \Phi_k(r_i)[\Pi(r_i, s_i \mid k) - \Pi(r_i, r_i \mid k)] \qquad (C.3)$$

$$= \Psi_{iQ}(r_i) + \int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(r_i)\Pi_2(r_i, s \mid k)ds \qquad (C.4)$$

Therefore, from (C.2) and (C.4), we find that incentive compatibility implies the following condition:

$$\Psi_{iQ}(s_i) - \Psi_{iQ}(r_i) \geq \int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(r_i)\Pi_2(r_i, s \mid k)ds \qquad (C.5)$$

and, by interchanging the variables, we find that,

$$\Psi_{iQ}(r_i) - \Psi_{iQ}(s_i) \geq \int_{s_i}^{r_i} \sum_{k=1}^{2} \Phi_k(s_i)\Pi_2(s_i, s \mid k)ds \qquad (C.6)$$

Combining (C.5) and (C.6), we obtain the following inequality:

$$\int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(s_i)\Pi_2(s_i, s \mid k)ds \geq \Psi_{iQ}(s_i) - \Psi_{iQ}(r_i) \geq \int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(r_i)\Pi_2(r_i, s \mid k)ds \qquad (C.7)$$

Notice that, the above inequality implies (21). Next, we divide all the terms in (C.7) and let $r_i \to s_i$ to obtain the result that

$$\Psi'_{iQ}(s_i) = \sum_{k=1}^{2} \Phi_k(s_i)\Pi_2(s_i, s_i \mid k) \qquad (C.8)$$

and hence,

$$\Psi_{iQ}(s_i) = \Psi_{iQ}(0) + \int_{0}^{s_i} \sum_{k=1}^{2} \Phi_k(s)\Pi_2(s, s \mid k)ds \qquad (C.9)$$

which is (20).

Next, we prove the sufficiency part. Suppose, (20) and (21) are satisfied, but the mechanism is not incentive compatible. Then, there exists s_i and r_i such that

$$V_{iQ}(r_i, s_i) > \Psi_{iQ}(s_i) \tag{C.10}$$

and substituting (C.4) in (C.10), we obtain that,

$$\int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(r_i) \Pi_2(r_i, s \mid k) ds > \Psi_{iQ}(s_i) - \Psi_{iQ}(r_i) \tag{C.11}$$

Furthermore, using (20) in (C.11), we obtain

$$\int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(r_i) \Pi_2(r_i, s \mid k) ds > \int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(s_i) \Pi_2(s_i, s \mid k) ds \tag{C.12}$$

Notice that (C.12) contradicts (21).

C.3
PROOF OF COROLLARY 3.1

Pick any two values of s_i in $[0,1]$, say $s_i^{'}$ and $s_i^{''}$. Without loss of generality, let $s_i^{'} < s_i^{''}$. Using (21), we obtain that

$$\Phi_1\left(s_i^{'}\right)\Pi_2\left(s_i^{'}, s_i^{'} \mid 1\right) + \Phi_2\left(s_i^{'}\right)\Pi_2\left(s_i^{'}, s_i^{'} \mid 2\right)$$
$$\leq \Phi_1\left(s_i^{''}\right)\Pi_2\left(s_i^{''}, s_i^{'} \mid 1\right) + \Phi_2\left(s_i^{''}\right)\Pi_2\left(s_i^{''}, s_i^{'} \mid 2\right) \qquad (C.13)$$

Further, because the payoffs are convex in the signal s, therefore,

$$\Pi_2\left(s_i^{''}, s_i^{'} \mid 1\right) \leq \Pi_2\left(s_i^{''}, s_i^{''} \mid 1\right) \qquad (C.14)$$

and

$$\Pi_2\left(s_i^{''}, s_i^{'} \mid 2\right) \leq \Pi_2\left(s_i^{''}, s_i^{''} \mid 2\right) \qquad (C.15)$$

Combining (C.13), (C.14) and (C.15), we obtain the result.

C.4
PROOF OF PROPOSITION 3.3

From (20), it follows that:

$$V_{iQ}\left(s_i, s_i\right) = V_{iQ}\left(0,0\right) + \int_0^{s_i} \sum_{k=1}^{6} \beta_k\left(s, s\right) ds \qquad (C.17)$$

Combining (27) and (C.17), we obtain:

$$m_i\left(s_i\right) = \sum_{k=1}^{6}\left[\alpha_k\left(s_i\right) - \int_0^{s_i} \beta_k\left(s, s\right) ds\right] - V_{iQ}\left(0,0\right)$$

C.5

PROOF OF PROPOSITION 3.4

Substituting (28) in (19), we obtain the following relation:

$$R_Q = 3\int_0^1 \sum_{k=1}^{6}\left[\alpha_k(s_i) - \int_0^{s_i}\beta_k(s,s)ds\right]g(s_i)ds_i - 3V_Q(0,0)$$

$$= 3\sum_{k=1}^{6}\left[\int_0^1 \alpha_k(s_i)g(s_i)ds_i - \int_0^1\int_0^{s_i}\beta_k(s,s)g(s_i)dsds_i\right] - 3V_Q(0,0) \qquad (C.18)$$

Furthermore, we integrate by parts the second expression in the right hand side of (C.18), and obtain the following expression for the auctioneer's revenue:

$$R_Q = 3\sum_{k=1}^{6}\left[\int_0^1 \gamma_k(s_i)g(s_i)ds_i\right] - 3V_Q(0,0) \qquad (C.19)$$

Where $\gamma_k(s) := \alpha_k(s) - \beta_k(s,s)\dfrac{1-G(s)}{g(s)}; \qquad k = 1,\ldots,6.$

We now simplify the expression in (C.19). For the sake of exposition, we only present the simplification for the case in which the auctioneer commits to allocate both the licenses to the bidder with the highest signal for all possible profile of reports; one can use the same technique for any arbitrary allocation rule. Under such a commitment,

$$\left(s_{(1)}^3, s_{(2)}^3, s_{(3)}^3\right) \in A6(\hat{r} \mid Q) \text{ for any possible value of } \left(s_{(1)}^3, s_{(2)}^3, s_{(3)}^3\right)$$

Pick any arbitrary s_i in $[0,1]$, Notice that if bidder i with signal s_i has the highest signal among the three bidders, then the set $B6(s_i)$ is realized. Moreover, if $s_i = s_{(2)}^3 \, or \, s_{(3)}^3$, then the set $B1(s_i)$ is realized. From (C.19), it follows that

$$R_Q = 3\sum_{k=1}^{6}\left[\int_0^1 \gamma_k(s_i)g(s_i)ds_i\right] - 3V_Q(0,0) \qquad (C.20)$$

Therefore, we first evaluate the expression

$$\sum_{k=1}^{6}\gamma_k(s_i) = \gamma_6(s_i) + \gamma_1(s_i) + \gamma_1(s_i) \qquad (C.21)$$

By expanding the expression in the right hand side of (C.20), we obtain the following:

$$\sum_{k=1}^{6}\gamma_k(s_i)=\int_0^{s_i}\int_0^{s_{(1)}^2}\left[\pi(s_i;0,0)-\frac{1-G(s_i)}{g(s_i)}\frac{\partial\pi(s_i;0,0)}{\partial s_i}\right]f_{12}^2(.)ds_{(2)}^2ds_{(1)}^2$$

$$+\int_{s_i}^{1}\int_0^{s_i}\pi(0;s_{(1)}^2,0)f_{12}^2(.)ds_{(2)}^2ds_{(1)}^2+\int_{s_i}^{1}\int_{s_i}^{s_{(1)}^2}\pi(0;s_{(1)}^2,0)f_{12}^2(.)ds_{(2)}^2ds_{(1)}^2 \qquad (C.22)$$

Notice that, in the first term on the right hand side of (C.22), $s_i=s_{(1)}^3$, and hence,

$$s_{(1)}^2=s_{(2)}^3 \text{ and } s_{(2)}^2=s_{(3)}^3$$

Similarly, in the second term $s_i=s_{(2)}^3$, and in the third term, $s_i=s_{(3)}^3$. Also, the following relation holds:

$$3g(s_i)f_{12}^2(s_{(1)},s_{(2)})=f_{123}^3(s_i,s_{(1)},s_{(2)})$$

Therefore, we can re-write the first term (on the right hand side) of (C.22) as follows:

$$3\int_0^1\gamma_6(s_i)g(s_i)ds_i$$

$$=3\int_0^1\int_0^{s_i}\int_0^{s_{(1)}^2}\left[\pi(s_i;0,0)-\frac{1-G(s_i)}{g(s_i)}\frac{\partial\pi(s_i;0,0)}{\partial s_i}\right]g(s_i)f_{12}^2(.)ds_{(2)}^2ds_{(1)}^2ds_i$$

$$=3\int_0^1\int_0^{s_{(1)}^3}\int_0^{s_{(2)}^3}\left[\pi(s_{(1)}^3;0,0)-\frac{1-G(s_{(1)}^3)}{g(s_{(1)}^3)}\frac{\partial\pi(s_{(1)}^3;0,0)}{\partial s_{(1)}^3}\right]g(s_{(1)}^3)f_{12}^2(.)ds_{(3)}^3ds_{(2)}^3ds_{(1)}^3$$

$$=\int_0^1\int_0^{s_{(1)}^3}\int_0^{s_{(2)}^3}\left[\pi(s_{(1)}^3;0,0)-\frac{1-G(s_{(1)}^3)}{g(s_{(1)}^3)}\frac{\partial\pi(s_{(1)}^3;0,0)}{\partial s_{(1)}^3}\right]f_{123}^3(.)ds_{(3)}^3ds_{(2)}^3ds_{(1)}^3$$

Similarly, we can expand the other terms to obtain the auctioneer's revenue as follows:

$$R_Q=\int_0^1\int_0^{s_{(1)}^3}\int_0^{s_{(2)}^3}\left[\pi(s_{(1)}^3;0,0)+2\pi(0;s_{(1)}^3,0)-\frac{1-G(s_{(1)}^3)}{g(s_{(1)}^3)}\frac{\partial\pi(s_{(1)}^3 0,0)}{\partial s_{(1)}^3}\right]$$

$$\times f_{123}^3(.)ds_{(3)}^3ds_{(2)}^3ds_{(1)}^3-3V_Q(0,0)$$

$$=\int_0^1\int_0^{s_{(1)}^3}\int_0^{s_{(2)}^3}\lambda_{A6(\hat{r}|Q)}(s_{(1)}^3,s_{(2)}^3,s_{(3)}^3)f_{123}^3(.)ds_{(3)}^3ds_{(2)}^3ds_{(1)}^3-3V_Q(0,0)$$

Analogously, the revenue of the auctioneer under any arbitrary allocation rule can be determined.

C.6
PROOF OF PROPOSITION 3.5

It follows from inspection of (30) that in the optimal mechanism, the payoff of a bidder with signal 0, given by $V_Q(0,0)$, has to be minimized, subject to

$$V_Q(0,0) \geq \underline{\pi}$$

Hence, in the optimal mechanism, we must have

$$V_Q(0,0) = \underline{\pi}$$

where $\underline{\pi}$ is defined in (31). Moreover, by construction, the maximum value of

$$\int_0^1 \int_0^{s^3_{(1)}} \int_0^{s^3_{(2)}} \lambda_{Ak(\hat{r}|Q)}\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) f^3_{123}(.) ds^3_{(3)} ds^3_{(2)} ds^3_{(1)} \qquad (C.23)$$

is given by

$$\int_0^1 \int_0^{s^3_{(1)}} \int_0^{s^3_{(2)}} \lambda^*\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) f^3_{123}(.) ds^3_{(3)} ds^3_{(2)} ds^3_{(1)} \qquad (C.24)$$

Notice that maximizing the expression in (C.23) and minimizing $V_Q(0,0)$ are independent of each other. Hence, we have the result.