ESSAY III

AN OPTIMAL MECHANISM IN THE PRESENCE OF EXTERNALITY AND INCOMPLETE INFORMATION

1. INTRODUCTION

This essay considers the problem of an auctioneer who wants to earn revenue by allocating multiple spectrum licenses. The spectrum licenses are identical in the sense that they endow the spectrum licensee with the right to use the same cost-reducing technology. The bidders of the licenses are the firms that compete in the product market.

The payoff of each bidder depends on the marginal costs of every bidder in the industry. Prior to auction, each bidder produces at a marginal cost of one. The marginal cost of any bidder that obtains the spectrum license is at most one. However, it is assumed that different bidders have different abilities to exploit the technology from the spectrum license and, consequently, there is heterogeneity in the costs of the bidders that obtain the license. Each bidder receives a one-dimensional signal that is a measure of the cost reduction that bidder can achieve by obtaining the license. A bidder that receives a higher signal than another bidder achieves a lower cost than the other bidder and hence has a higher payoff. If a bidder cannot obtain a license, she produces at a marginal cost of one.

Furthermore, the payoff of a bidder also depends negatively on the signal of any other spectrum licensee. This has the implication that even though a bidder obtains the rights by obtaining just one license, she still has a positive value for the second license because, by obtaining the second license, it prevents a competitor from acquiring a license. Moreover, the payoff of a bidder depends on the signals of the bidders that obtain a license, not on how many licenses each of these bidders has. Hence, given a profile of signals, the auctioneer maximizes its revenue by optimally choosing the number of licensees. We provide a framework for the analysis of the optimal number of licensees. Because the payoff of a licensee depends on the nature of the product market, we also analyze the role of different product market factors in the determination of the optimal number of licensees.

In the essay, we consider the simplified problem of determining the auctioneer's revenue when the auctioneer allocates two identical spectrum licenses and there are three bidders. The analysis can be extended to the allocation of $k > 2$ licenses to $n > k$ bidders with its attendant combinatorial complexity. The payoffs of the bidders are taken to have the same properties as the payoffs of bidders that compete in quantities. First, we determine the auctioneer's revenue when the signal of each bidder is publicly observable. We find that in the optimal allocation mechanism when the signal of each bidder is publicly observable, the auctioneer selects the allocation rule that maximizes the industry gross payoff. The optimal mechanism of the auctioneer for the case in which all licensees have the same signal was solved in Kamien (1992, pp. 348-352) and in Kamien, Oren, and Tauman (1992). We extend their analysis to the case in which licensees can have different signals.

Next, we analyze the case in which the signal of each bidder is her private information. In order to analyze the problem, we define an allocation rule as a specification of the licensees, and consider the allocation in a truth-telling equilibrium of a direct mechanism. The auctioneer's problem is to choose the allocation rule that maximizes its revenue from the allocation of two licenses, provided the allocation rule satisfies the incentive compatibility and individual rationality constraints. Therefore, we first derive the necessary and sufficient conditions for incentive compatibility. We then show that some allocation rules satisfy the necessary and sufficient conditions for incentive compatibility. Because of the assumption of independence of signals, we prove that two allocation procedures that have the same allocation rule and the same payoff of a bidder with the worst possible signal yield the same revenue to the auctioneer. This is known as the *revenue equivalence theorem*, first established in a simpler context by Riley and Samuelson (1981).

We then determine the auctioneer's revenue in an incentive compatible mechanism (that is, in a mechanism in which the allocation rule satisfies the necessary and sufficient conditions for incentive compatibility). In order to determine the auctioneer's revenue, we define the concept of *industry virtual payoff*, which is the analog, for the situation in which there are externalities, of the concept of virtual value (see Myerson (1981), Bulow & Roberts (1989), and Bulow & Klemperer (1996)). The industry virtual payoff is the industry profit, less the information rents of the spectrum licensees; hence, the industry virtual payoff, given any profile of signals, depends on the allocation rule. The revenue of the auctioneer, given any allocation rule, is the expected industry virtual payoff, less the product of the payoff of a bidder with the lowest possible signal and the number of bidders in the industry. In models of allocations with externalities, the payoff of the bidder with the lowest possible signal depends on the mechanism (Kamien (1992), Jehiel, Moldovanu, & Stachetti (1996, 1999)). If the auctioneer can credibly commit to punish a bidder that refuses to participate in the mechanism by allocating a spectrum license to the other bidders, then the payoff of the bidder with the lowest possible signal is minimized and is independent of the equilibrium allocation. Hence, in the optimal mechanism, the auctioneer chooses the allocation mechanism that maximizes the industry virtual payoff for any arbitrary profile of signals, and threatens to punish a bidder that refuses to participate in the mechanism by allocating a license to the other bidders. We determine the number of licensees in the optimal mechanism, both when the signal of each bidder is publicly observable, and when the signal of each bidder is her private information.

We then illustrate the role of product market factors in the determination of the optimal number of licensees. First, we show that the presence of significant externalities along with incomplete information may cause the auctioneer to select a fewer number of licensees compared to the situation in which the signal of each bidder was publicly observable. Next, we show that an increase in the magnitude of externalities may lead to a decrease in the expected number of licensees.

The essay is consisted of 4 sections. Section 1 is introduction, section 2 is literature review, section 3 is the model, and the last section is conclusion. Most of the proofs are presented in appendix C.

2. RELATED LITERATURE

This essay is closely related to the literature on allocation problem with externalities. One of the earliest analyses is Katz and Shapiro (1986). Their analysis assumes that the signal of a bidder is publicly observable, and each bidder can obtain at most one license (not necessarily a spectrum license). Another work in the same spirit is Hoppe, Jehiel, and Moldovanu (2004). We relax both of the assumptions mentioned above.

There has been article on allocation with externalities, in which the signal of each bidder is private information. Jehiel, Moldovanu, and Stachetti (1996, 1999) can show that the payoff of the bidder with the worst signal is endogenous to the mechanism. We find such a result in our model. Other examples of articles that analyze allocation with externalities are Jehiel and Moldovanu (2000), Moldovanu and Sela (2003), DasVarma (2003), Katzman and Rhodes-Kropf (2002) and Goeree (2003). However, unlike our essay, none of these papers consider multiple licenses.

There has also been some work on allocations of multiple licenses in the presence of externalities. Jehiel and Moldovanu (2001, 2004) show the impossibility of implementing efficient allocations when the signals are multi-dimensional. In our model, the signals are unidimensional. This essay is also closely related to Dana and Spier (1994). In their article, Dana and Spier consider the problem of auctioning production rights to bidders in an industry. Dana and Spier assume that "a firm (bidder) earns zero profits if it is not awarded a production right" Dana and Spier (1994: 129) while we assume that if a bidder does not purchase a spectrum license, her payoff is lower (and depends on the signals of the other licensees) compared to her payoff before the auction. In Dana and Spier, the auctioneer (which is the government) maximizes social welfare which is a function of the revenue from the sale, profits of firms, and consumer surplus, while in our model; the auctioneer maximizes its revenue from the auction.

Schmitz (2002) has analyzed revenue-maximizing allocations from an auction of multiple licenses when the signal of each bidder is her private information. He has shown that the optimal number of licensees under private information can be two even when the optimal number of licensees under complete information is one. There are two major differences between Schmitz's model and ours. First, in Schmitz's model, each bidder can win at most one license but we impose no such restriction. Second, Schmitz assumes that with positive probability a licensee is not able to exploit the technology whereas, in our model, this is not the case. It can be shown that if this assumption is relaxed in Schmitz's model, it is always optimal for the auctioneer to allocate both licenses to one bidder. In contrast, we do not make such an assumption but show that it can be optimal for the auctioneer to choose multiple licensees.

Brocas (2005) also analyzes a model of allocation of $k \geq 2$ licenses in which each bidder can win at most one license but her payoff functions are not motivated by standard models of market competition. In simultaneous but independent work, Figueroa and Skreta (2005) have considered a general model of auction of multiple objects in the presence of externalities whereas our model deals only with the auction of spectrum licenses. They derive the optimal mechanism both when the non-participation payoffs are own-type dependent and when they are own-type independent; we consider the optimal mechanism when the non-participation payoffs are own-type independent. In our model, the payoff function of bidders have the properties of payoff functions that arise in equilibrium when bidders compete in quantities using differentiated products; in Figueroa and Skreta the payoff functions are not derived from an oligopoly model. Because we model the nature of competition in the product market more explicitly, we show the relationship between the level of product differentiation and the optimal number of spectrum licensees. Moreover, because we assume that the marginal payoff of a bidder is decreasing in a competitor's signal (as in many oligopoly models), we derive a different regularity condition from the one in Figueroa and Skreta. Further, we analyze the problem both when the signal of every bidder is publicly observable and when they are private information. This allows us to highlight the effect of negative externalities alone and the effect of both negative externalities and private information

There is another related literature that analyzes auctions of heterogeneous objects. Palfrey (1983), Armstrong (2000) and Avery and Hendershott (2000) analyze auctions of heterogeneous objects when bidders have an exogenously specified private value for each object. In contrast, in our model, the spectrum licenses are identical and the value of winning a license is determined only after the auction. In Palfrey's model, the auctioneer, who is the owner of many heterogeneous objects, decides to partition the objects into separate bundles and sell each bundle separately using auctions. In comparison, in our model, the auctioneer makes the bundling decision *ex post*. Palfrey finds that the desirability of selling all the objects as a bundle depends on the number of bidders. Armstrong (2000) and Avery and Hendershott (2000) extend Palfrey's model to determine the revenue-maximizing auction, under different assumptions about the bidders.

3. THE MODEL

This section proposes a framework to analyze the allocation of multiple spectrum licenses. The first section describes auctioneer's problem in the complete information scenario. The second section considers the direct mechanism under incomplete information. The last section considers the role of product market factors in the determination of the optimal number of licensees.

3.1 Complete Information Scenario

Consider an auctioneer who wants to maximize revenue by allocating two spectrum licenses. There are three bidders for the licenses, labelled bidders 1, 2, and 3. Initially, the bidders produce with a marginal cost of 1. Before the auction occurs, each bidder i (i = 1, 2, 3) receives a signal $s_i \in [0,1]$ that determines her marginal cost c_i if she obtains a license, in the following way:

$$c_i = 1 - s_i \tag{1}$$

Otherwise, the bidder continues to produce at a marginal cost of 1.

The signals are assumed to be identically and independently distributed across bidders, with $G(s)$ (resp., $g(s)$) as the distribution function (resp., density function). In this section, it is assumed that the signal of each bidder is publicly observable. Let $s^3_{(k)}$ be the k^{th} highest statistic from a sample of size 3 where the sample includes all the potential bidders for the licenses and assume that $s^3_{(k)}$ is distributed as $F^3_k(.)$ with the associated density function $f^3_k(.)$. A typical bidder has two competitors and the expected payoff of a bidder depends on the signals of the two competitors. Therefore, when a sample refers to a bidder's competitors, we work with a sample size of 2 and in these cases, we replace 3 with 2 in the superscripts of the expressions above. In such a case, the sample consists of all the competitors of bidder i. We denote the joint density of $s^j_{(1)},...,s^j_{(k)}$ by $f^j_{1...k}(s_{(1)},...,s_{(k)})$ where j,k = 1, 2, 3 and $k \le j$.

A bidder's payoff depends upon her own signal as well as on the signals drawn by the other bidders. If bidder i obtains a spectrum license and, if the bidder with signal $s_{(p)}^2$ also obtains a license, then bidder i's payoff is given by $\pi\left(s_i; s_{(p)}^2, 0\right)$ where p is either 1 or 2. If bidder i does not obtain a license and, if the bidders with signals $s_{(1)}^2$ and $s_{(2)}^2$ obtain one license each, then bidder i' s payoff is $\pi\left(0; s_{(1)}^2, s_{(2)}^2\right)$. Finally, if bidder i is not a licensee and the bidder with signal $s_{(p)}^2$ obtains both licenses, then the payoff of bidder i is $\pi\left(0; s_{(p)}^2, 0\right)$; p = 1, 2.

The specification of $\pi(.;.,.)$ depends on the nature of competition among the bidders and other market parameters such as the demand function. However, regardless of the functional form, we assume that $\pi(.;.,.)$ is twice continuously differentiable and has the following properties:

$$\pi_1(s_i;.,.) > 0, \quad \pi_{11}(s_i;.,.) \geq 0 \tag{2}$$

$$\pi_j(.;.,.) < 0, \quad j = 2,3 \tag{3}$$

$$\pi_{12}(.;.,.) < 0, \quad \pi_{13}(s_i;.,.) < 0, \quad \pi_1(.;.,.) > -\pi_2(.;..) \tag{4}$$

where $\pi_j(.;.,.)$ refers to the partial derivative of $\pi(.;.,.)$ with respect to the j^{th} argument. The inequalities in (2) imply that the payoff of bidder i, when she obtains a spectrum license, is increasing and convex in her own signal. The inequality in (3) captures the effect of negative externalities because it implies that when a competitor of bidder i obtains a license, then the payoff of bidder i is strictly decreasing in that competitor's signal. The first two inequalities in (4) imply that the marginal payoff of a bidder's signal is decreasing in another bidder's signal, while the third inequality in (4) implies that the payoff of a bidder is more sensitive to her own signal than to another bidder's signal.

Below, we illustrate the payoff function for a market in which the bidders compete in quantities and show that the payoff function satisfies the properties described in (2), (3), and (4).

Example 1

Suppose an auctioneer wants to allocate two spectrum licenses and there are three potential firms, who are firms that compete in quantities producing differentiated products. The inverse demand function for firm i is given by

$$p_i = \tau - q_i - \mu\left(q_{(1)}^2 + q_{(2)}^2\right), \qquad where\ \mu \in [0,1]$$

In this demand function, $q_{(j)}^2$ is the output of the firm with cost $1 - s_{(j)}^2$. Also, μ is the externality parameter that captures the effect of the other firms' decisions on firm i's payoff. Given $\mu \in [0,1]$, the payoff function is given by:

$$\pi\left(s_i; s_{(1)}^2, s_{(2)}^2\right) = \left[\frac{(\tau - 1)(2 - \mu) + (2 + \mu)s_i - \mu\left(s_{(1)}^2 + s_{(2)}^2\right)}{2(1 + \mu)(2 - \mu)}\right]^2 \qquad (5)$$

Because we are considering the problem of allocating two licenses, at least one of s_i, $s_{(1)}^2$, or $s_{(2)}^2$ is 0. It can be verified that the payoff function (which is the reduced form equilibrium profit function) satisfies (2), (3), and (4).

It is important to note the relationship between bidder i's payoff from obtaining a spectrum license when she has a signal of 0, and her payoff from not obtaining a license. First, consider the case that only one of bidder i's competitors—the bidder with signal $s_{(p)}^2$—obtains a license. Then bidder i's payoff is $\pi\left(0; s_{(p)}^2, 0\right)$ when either she obtains a license or when the bidder with signal $s_{(p)}^2$ obtains both the licenses. Second, in the case when both of bidder i's competitors with signals $s_{(1)}^2$ and $s_{(2)}^2$ obtain a license, bidder i's payoff is $\pi\left(0; s_{(1)}^2, s_{(2)}^2\right)$. However, if bidder i instead obtains a license by displacing one of her rivals (say the bidder with signal $s_{(1)}^2$), then her payoff is $\pi\left(0; s_{(2)}^2, 0\right)$ which is different from $\pi\left(0; s_{(1)}^2, s_{(2)}^2\right)$. It follows from (3) that bidder i with signal 0 may be better off by obtaining a license if it displaces a competitor; even though she cannot obtain a reduction in her own cost, she can prevent a rival from doing so.

3.1.1 Mechanism under Complete Information

In this section, we determine the revenue of the auctioneer in the optimal mechanism, when the signal of each bidder is publicly observable. In order to do so, we fix a profile of signals $\left(s_{(1)}^3, s_{(2)}^3, s_{(3)}^3\right)$. Each allocation is denoted by the vector

$a = (a1, a2, a3)$ where ai is the number of licenses that the bidder with signal $s^3_{(i)}$ obtains in equilibrium; i = 1, 2, 3. Suppose the auctioneer commits to allocate the two spectrum licenses according to a in exchange for a payment of $m^c_i(a)$ from bidder i; i = 1, 2, 3. If bidder i does not accept the auctioneer's offer (that is, if bidder i does not participate in the mechanism), the payoff of bidder i is denoted by π^c_i. Let $\chi^c(ai)$ be an indicator variable that takes a value 1 if ai > 0 and 0 otherwise. Therefore, the auctioneer's problem is the following:

$$\max_{a} \sum_{i=1}^{3} m^c_i(a)$$

$$s.t. \qquad \pi(\chi^c(ai)s_i;.,.) - m^c_i(a) \geq \pi^c_i, \quad for\ i = 1,2,3 \tag{6}$$

Observe that the auctioneer can extract a higher payment from bidder i if he can reduce the payoff of bidder i from not participating in the mechanism. In the lemma below, π^c_i has been determined for every bidder i in the optimal mechanism.

Lemma 3.1 (Outside option in the optimal mechanism)

In the optimal mechanism,

$$\pi^c_i = \pi(0; s^2_{(1)}, s^2_{(2)}), \qquad for\ i = 1,2,3 \tag{7}$$

where $s^2_{(1)}$ and $s^2_{(2)}$ are the signals of bidder i's competitors such that $s^2_{(1)} \geq s^2_{(2)}$.

proof : see appendix C.1.

If a bidder does not participate in the mechanism, then her payoff depends on how the spectrum licenses are allocated in such an eventuality. The maximum punishment that the auctioneer can credibly threaten to inflict on a non-participating bidder is to allocate the licenses to both of her competitors, and the payoff of bidder i in such a case is given by $\pi(0; s^2_{(1)}, s^2_{(2)})$; note that $\pi(0; s^2_{(1)}, s^2_{(2)})$ is less than $\pi(0;0,0)$ which is the profit a bidder makes before the auction is introduced. In the optimal mechanism, the auctioneer makes each bidder indifferent between participating and non-participating and hence we have the above lemma.

Under complete information, the auctioneer can extract the entire surplus of the three bidders and hence, in the optimal mechanism, all the constraints in (6) must be binding. Therefore, the auctioneer's problem can be re-stated as

$$\max_{a}\left\{\sum_{i=1}^{3}\pi\left(\chi^{c}(ai)s_{i};.,.\right)-\sum_{i=1}^{3}\pi_{i}^{c}\right\} \tag{8}$$

and it follows from (7) that

$$\sum_{i=1}^{3}\pi_{i}^{c}=\pi\left(0;s_{(2)}^{3},s_{(3)}^{3}\right)+\pi\left(0;s_{(1)}^{3},s_{(3)}^{3}\right)+\pi\left(0;s_{(1)}^{3},s_{(2)}^{3}\right) \tag{9}$$

Notice that $\sum_{i=1}^{3}\pi\left(\chi^{c}(ai)s_{i};.,.\right)$ is the industry gross payoff from the mechanism. Hence, the auctioneer's optimal allocation a^{*} maximizes the industry gross payoff. This is stated formally in the following proposition.

Proposition 3.1 (Optimal allocation under complete information)

Suppose the signal of each bidder is publicly observable. Then, in the optimal allocation a^{*}, the auctioneer maximizes the industry gross payoff, that is,

$$a^{*}\in\arg\max\sum_{i=1}^{3}\pi\left(\chi^{c}(ai)s_{i};.,.\right)$$

and the auctioneer's revenue in the optimal mechanism is

$$\sum_{i=1}^{3}\pi\left(\chi^{c}(ai^{*})s_{i};.,.\right)-\sum_{i=1}^{3}\pi_{i}^{c}\text{ , where }\sum_{i=1}^{3}\pi_{i}^{c}\text{ is given by (9).}$$

proof : follows from the discussion above.

In later sections, we show that when the signal of each bidder is private information, then the auctioneer cannot extract all the surplus of the bidders and hence the optimal allocation in such a case may not be the allocation that maximizes the industry gross payoff. In order to determine the auctioneer's revenue when the signal of each bidder is her private information, in the section below we describe the auctioneer's problem when the signal of each bidder is her private information.

3.2 Incomplete Information Scenario

From now on, unless otherwise mentioned, the signal of each bidder is assumed to be her private information and the auctioneer is assumed to have the power to credibly commit to a mechanism for the allocation of two licenses. By the revelation principle, there is no loss of generality in restricting our attention to direct mechanisms. In a direct mechanism, the auctioneer asks each bidder to give a report about her signal, and implements some outcome depending on the profile of reports.

3.2.1 The Direct Mechanism

Before we proceed to set up the direct mechanism, we define its building blocks. The first building block is the *signal space* of the bidders. We denote it by Ω. Because the signal of each bidder is distributed on the unit interval, Ω = [0, 1] × [0, 1] × [0, 1]. The second building block is the payment rule and it is defined below:

A *payment rule* is a mapping that specifies the payments of the bidders as a function of the profile of reports. Let $r = \{r_1, r_2, r_3\} \in \Omega$ be the reports of the three bidders about their signals. Then, the payment rule $M(r)$ is specified by the following vector:

$$M(r) = (M_1(r), M_2(r), M_3(r)),$$

where $M_j(r)$ is the payment of bidder j when the profile of reports is r.

The third building block is the allocation rule which is specified below:

An *allocation rule* is a mapping that prescribes the distribution of spectrum licenses among the bidders as a function of the bidders' reports $r = \{r_1, r_2, r_3\} \in \Omega$. Let $q_j(r)$ be the number of licenses bidder j obtains when the reported profile is r, where j = 1, 2, or 3. The allocation rule $Q(r)$ is then defined to be the vector $Q(r) = (q_1(r), q_2(r), q_3(r))$. Because the auctioneer commits to allocate two spectrum licenses, the following relations must also hold:

$$\text{For all } \; r \in \Omega, \; q_j(r) \in \{0,1,2\} \; \text{and} \; \sum_{j=1}^{3} q_j(r) = 2, \quad j = 1,2,3$$

When $r_1 = r_2 = r_3$, because there are only two licenses to be allocated, the auctioneer can allocate them in any random way.

We now define the direct mechanism as follows:

A direct mechanism is a triple $\Gamma = \{\Omega, Q(.), M(.)\}$

We denote the expected revenue of the auctioneer when he commits to implement an allocation rule $Q(r)$ by R_Q. The auctioneer's problem is to choose the allocation rule that maximizes his expected revenue. In an allocation rule, all that matters for the allocation is the signal of a bidder and not her identity. Hence, from now on, only the non-decreasing reports $\hat{r} \equiv \left(r_{(1)}^3, r_{(2)}^3, r_{(3)}^3\right)$ is considered, where $r_{(1)}^3 \geq r_{(2)}^3 \geq r_{(3)}^3$.

3.2.2 Expected Payoff of a Bidder

A revenue-maximizing auctioneer extracts the maximum possible amount from each bidder, under the condition that each bidder's signal is her private information and the auctioneer has two spectrum licenses to allocate. The amount that the auctioneer extracts from each bidder depends on the expected payoff of each bidder, which in turn depends on the auctioneer's allocation rule. In the rest of this section, we show the relationship between the auctioneer's allocation rule and the expected payoff of a bidder.

In order to do so, first notice that given any allocation rule Q, the signal space can be partitioned into six sets

$$A(\hat{r} \mid Q) \equiv \{A1(\hat{r} \mid Q), A2(\hat{r} \mid Q), A3(\hat{r} \mid Q), A4(\hat{r} \mid Q), A5(\hat{r} \mid Q), A6(\hat{r} \mid Q)\}$$

such that each set maps an ordered profile of reports $\hat{r}$ to a particular allocation of licenses. The allocation corresponding to each set is presented in the table below. As an example, suppose that under an allocation rule Q, a particular ordered profile of reports $\hat{r}$ belongs to the set $A6(\hat{r} \mid Q)$. Then, the bidder reporting $r_{(1)}^3$ obtains two licenses and the others obtain nothing. Similarly, depending on other profile of reports, a different allocation of licenses is obtained. Also notice that any allocation rule is associated with a unique partition of the signal space and vice versa. It is assumed that the auctioneer can commit to allocate the licenses according to the allocation rule Q.

Table 3.1

Partition of the Report Space induced by an Allocation Rule

Set	Allocation of $r^3_{(1)}$	Allocation of $r^3_{(2)}$	Allocation of $r^3_{(3)}$
$A1(\hat{r} \mid Q)$	0	2	0
$A2(\hat{r} \mid Q)$	0	0	2
$A3(\hat{r} \mid Q)$	0	1	1
$A4(\hat{r} \mid Q)$	1	1	0
$A5(\hat{r} \mid Q)$	1	0	1
$A6(\hat{r} \mid Q)$	2	0	0

Next, in order to relate the allocation rule to bidder i's expected payoff, we determine the distribution of spectrum licenses from bidder i's perspective, when the auctioneer commits to the allocation rule Q, bidder i reports r_i, and the other bidders report truthfully. Let $s^2_{(1)}$ (resp., $s^2_{(2)}$) be the highest (resp., lowest) of the signals of bidder i's competitors and denote

$$\hat{s}_{-i} = \left(s^2_{(1)}, s^2_{(2)}\right)$$

as the non-decreasing signals of bidder i's competitors. Then, given the profile of reports $(r_i, \hat{s}_{-i})$ and the partition $A(\hat{r} \mid Q)$, there also exists a partition

$$B(r_i) \equiv \{B1(r_i), B2(r_i), B3(r_i), B4(r_i), B5(r_i), B6(r_i)\}$$

of the signal space of bidder i's competitors, as a function of bidder i's report r_i. Each subset of the partition $B(r_i)$, given r_i, is the set of ordered signals of bidder i's competitors $\hat{s}_{-i}$ that result in the same allocation. In the table below, we list the allocation corresponding to each subset of the partition $B(r_i)$. For example, it follows from the table that $B3(r_i)$ is defined as follows:

$B3(r_i) = \{\hat{s}_{-i} \mid bidder\ i\ is\ not\ a\ licensee\ and\ her\ competitors\ obtain\ one\ license\ each\}$

Notice that there are six subsets of $B(r_i)$ because there are six ways of allocating the two licenses.

Table 3.2

Allocation of Licenses as a Function of Bidder i's Report

Set	Bidder i	Bidder with signal $s^2_{(1)}$	Bidder with signal $s^2_{(2)}$
$B1(r_i)$	0	2	0
$B2(r_i)$	0	0	2
$B3(r_i)$	0	1	1
$B4(r_i)$	1	1	0
$B5(r_i)$	1	0	1
$B6(r_i)$	2	0	0

The partition $B(r_i)$ can be used to determine the probability $\Phi_k(r_i)$ that bidder i obtains k spectrum licenses (k = 0, 1, 2) and her expected payoff conditional on obtaining k licenses $\Pi(r_i, s_i \mid k)$ when bidder i reports r_i and the others report truthfully. Hence, we can determine the expected payoff of bidder i from reporting r_i when her competitors report truthfully, and this is discussed below.

Let $\phi_{Bk}(r_i)$ be the probability that the ordered profile of signals of bidder i's competitors belong to the set $Bk(r_i)$ when bidder i reports her signal as r_i, the rivals report truthfully, and the auctioneer commits to the allocation rule Q, that is,

$$\phi_{Bk}(r_i) \equiv P\{\hat{s}_{-i} \in Bk(r_i)\}, \qquad k = 1,2,\ldots,6$$

The expected payment of bidder i, denoted by $m_i(r_i)$, can be similarly defined when bidder i reports r_i, the other bidders report truthfully, and the auctioneer commits to the allocation rule Q. Therefore,

$$m_i(r_i) = \int_{\hat{s}_{-i}} M_i(r_i, \hat{s}_{-i}) f^2_{12}(.) d\hat{s}_{-i}$$

Using the definitions of $\phi_{Bk}(r_i)$ and $m_i(r_i)$, the expected payoff of bidder i, denoted by $V_{iQ}(r_i, s_i)$, is defined below, when bidder i with a signal of s_i reports r_i, the others report truthfully, and the auctioneer chooses Q.

$$\begin{aligned} V_{iQ}(r_i, s_i) = {} & \phi_{B1}(r_i) E\{\pi(0; s_{(1)}^2, 0) \mid \hat{s}_{-i} \in B1(r_i)\} \\ & + \phi_{B2}(r_i) E\{\pi(0; s_{(2)}^2, 0) \mid \hat{s}_{-i} \in B2(r_i)\} \\ & + \phi_{B3}(r_i) E\{\pi(0; s_{(1)}^2, s_{(2)}^2) \mid \hat{s}_{-i} \in B3(r_i)\} \\ & + \phi_{B4}(r_i) E\{\pi(s_i; s_{(1)}^2, 0) \mid \hat{s}_{-i} \in B4(r_i)\} \\ & + \phi_{B5}(r_i) E\{\pi(s_i; s_{(2)}^2, 0) \mid \hat{s}_{-i} \in B5(r_i)\} \\ & + \phi_{B6}(r_i) \pi(s_i; 0, 0) - m_i(r_i) \end{aligned} \tag{10}$$

As mentioned above, let $\Phi_k(r_i)$ be the probability that bidder i obtains k licenses when she reports r_i, the other bidders report truthfully, and the auctioneer commits to the allocation rule Q. In particular, the probabilities $\Phi_k(r_i)$ for k = 0, 1, 2 are related to the probabilities $\phi_{Bj}(r_i)$ for j = 1, 2, . . . , 6 as follows:

$$\begin{aligned} & \Phi_0(r_i) \equiv \phi_{B1}(r_i) + \phi_{B2}(r_i) + \phi_{B3}(r_i) \\ & \Phi_1(r_i) \equiv \phi_{B4}(r_i) + \phi_{B5}(r_i) \\ & \textit{and} \ \ \Phi_2(r_i) \equiv \phi_{B6}(r_i) \end{aligned} \tag{11}$$

Observe that the probability of obtaining k licenses depends on bidder i's reported signal r_i and not on her true signal s_i. Moreover, the expected gross payoff of bidder i from obtaining k licenses (when her true signal is s_i and her reported signal is r_i) denoted by $\Pi(r_i, s_i \mid k)$ is described below for k = 0, 1,and 2.

$$\begin{aligned} \Pi(r_i, s_i \mid 0) \equiv {} & \frac{\phi_{B1}(r_i)}{\Phi_0(r_i)} E\{\pi(0; s_{(1)}^2, 0) \mid \hat{s}_{-i} \in B1(r_i)\} \\ & + \frac{\phi_{B2}(r_i)}{\Phi_0(r_i)} E\{\pi(0; s_{(2)}^2, 0) \mid \hat{s}_{-i} \in B2(r_i)\} \\ & + \frac{\phi_{B3}(r_i)}{\Phi_0(r_i)} E\{\pi(0; s_{(1)}^2, s_{(2)}^2) \mid \hat{s}_{-i} \in B3(r_i)\} \end{aligned} \tag{12}$$

$$\Pi(r_i, s_i \mid 1) \equiv \frac{\phi_{B4}(r_i)}{\Phi_1(r_i)} E\{\pi(s_i ; s_{(1)}^2, 0) \mid \hat{s}_{-i} \in B4(r_i)\}$$
$$+ \frac{\phi_{B5}(r_i)}{\Phi_1(r_i)} E\{\pi(s_i ; s_{(2)}^2, 0) \mid \hat{s}_{-i} \in B5(r_i)\} \qquad (13)$$

and

$$\Pi(r_i, s_i \mid 2) \equiv \pi(s_i ; 0,0) \qquad (14)$$

Furthermore, $\Pi_j(r_i, s_i \mid k)$ denotes the partial derivative of $\Pi(r_i, s_i \mid k)$ with respect to the j^{th} argument where j is either 1 or 2.

3.2.3 The Auctioneer's Problem

We now state the auctioneer's problem formally. In order to do so, we use (12), (13) and (14) to rewrite the expected payoff of bidder i as given in (10) as follows:

$$V_{iQ}(r_i, s_i) = \Phi_0(r_i)\Pi(r_i, s_i \mid 0) + \Phi_1(r_i)\Pi(r_i, s_i \mid 1) + \Phi_2(r_i)\Pi(r_i, s_i \mid 2) - m_i(r_i) \qquad (15)$$

In the truth-telling equilibrium, it is the best response of bidder i to report her signal truthfully, given that other bidders report truthfully. This is known as incentive compatibility and is defined formally below:

The allocation and payment rule satisfy Incentive Compatibility (IC) if, for every bidder i,

$$V_{iQ}(s_i, s_i) \geq V_{iQ}(r_i, s_i) \qquad \textit{for all } r_i \textit{ and } s_i \in [0,1] \qquad (16)$$

Moreover, no bidder can be forced to participate in the mechanism. This can be ensured if no bidder becomes worse off participating in the mechanism than by staying out. This is known as individual rationality. Suppose that if bidder i stays out of the mechanism, she gets a payoff of $\underline{\pi}_i$. Then, the individual rationality constraint is formally as follows.

The allocation and payment rule satisfy Individual Rationality (IR) if, for every bidder i,

$$V_{iQ}(s_i, s_i) \geq \underline{\pi}_i \qquad (17)$$

We now define the auctioneer's problem as follows :

$$Select\ \{B1(.),...,B6(.),m_1(.),m_2(.),m_3(.)\}\ to \tag{18}$$
$$Max \sum_{i=1}^{3}\int_0^1 m_i(s_i)g(s_i)ds_i$$
$$s.t.\ IC\ and\ IR$$

The optimal mechanism solves the problem for all possible $\{Q, M_i\}$. In principle, this problem can be solved in two steps. First, fix the allocation rule $Q(.)$ arbitrarily. Then, determine $m_i(.)$ to satisfy IC and IR and use this function to obtain R_Q where

$$R_Q = \sum_{i=1}^{3}\int_0^1 m_i(s_i)g(s_i)ds_i \tag{19}$$

In the second step, select $Q(.)$ to maximize R_Q.

In the next section, we use incentive compatibility to narrow down the search to a subset of all possible allocation and payment rules.

3.3 Incentive Compatible Mechanism

In this section, we determine the auctioneer's revenue in an incentive compatible direct mechanism, as a function of the allocation rule. In order to do so, we first provide an alternative characterization of incentive compatibility in the following proposition:

Proposition 3.2 (Incentive compatible direct mechanism)

The direct mechanism is incentive compatible, if and only if,

$$V_{iQ}(s_i, s_i) = V_{iQ}(0,0) + \int_0^{s_i} [\Phi_1(s)\Pi_2(s, s \mid 1) + \Phi_2(s)\Pi_2(s, s \mid 2)]ds \tag{20}$$

and

$$\Phi_1(r_i)\Pi_2(r_i, s_i \mid 1) + \Phi_2(r_i)\Pi_2(r_i, s_i \mid 2)\ is\ nondecreasing\ in\ r_i\ \ for\ all\ r_i \in [0,1] \tag{21}$$

proof : see appendix C.2.

From (20) it follows that the marginal change in the equilibrium payoff $V_{iQ}(s_i, s_i)$ with respect to the signal s_i is given by

$$\Phi_1(s_i)\Pi_2(s_i, s_i \mid 1) + \Phi_2(s_i)\Pi_2(s_i, s_i \mid 2) \tag{22}$$

Notice that under the assumptions of the model, the expression in (22) is positive. In addition, we also show that, under the assumptions of the model, the expression in (22) is non-decreasing in the signal s_i.

Corollary 3.1 (Marginal change of $V_{iQ}(s_i, s_i)$)

Suppose the payoff of a bidder is strictly convex in her signal s_i. Then (21) implies that the marginal change in the equilibrium payoff (given in (22)) with respect to s_i is non-decreasing in s_i.

proof : see appendix C.3.

The above corollary and (22) implies that, in our model, the equilibrium payoff function $V_{iQ}(s_i, s_i)$ is positively sloped and convex in s_i. Hence, incentive compatibility ensures that if the IR constraint is satisfied for a bidder with signal 0, then it is satisfied for a bidder with any arbitrary signal.

3.3.1 Incentive Compatible Allocations

It is instructive at this point to consider the kind of allocations that satisfy (21) and hence are implementable. First, we consider allocations in which the number of licensees is known with certainty when the bidders report their types. There are six possible allocation rules with such a feature and they are the following: (i) The bidder with the highest report obtains both the licenses, (ii) The bidder with the second highest report obtains both the licenses, (iii) The bidder with the third highest report obtains both the licenses, (iv) The bidders with the highest and second highest reports obtain a license each, (v) The bidders with the second highest and third highest reports obtain a license each, and (vi) The bidders with the highest and third highest reports obtain a license each.

Suppose the auctioneer commits to allocate both the licenses to the bidder with the highest report, that is, it commits to implement the first allocation rule mentioned above. In this case,

$$\Phi_1(r_i)\Pi_2(r_i, s_i \mid 1) + \Phi_2(r_i)\Pi_2(r_i, s_i \mid 2)$$
$$= \int_0^{r_i} \pi_1(s_i;0,0) f_1^2\left(s_{(1)}^2\right) ds_{(1)}^2$$

and the above expression is non-decreasing in the report r_i. Hence, the allocation in which the bidder with the highest report obtains both licenses is implementable. One can check that the allocations in (ii) and (iii) are not implementable (see extended appendix). Next consider (iv), that is, the allocation in which the auctioneer commits to allocate one license each to the bidders with the highest two reports. In this case,

$$\Phi_1(r_i)\Pi_2(r_i, s_i \mid 1) + \Phi_2(r_i)\Pi_2(r_i, s_i \mid 2)$$
$$= \int_0^{r_i} \int_{s_{(2)}^2}^{1} \pi_1\left(s_i; s_{(1)}^2, 0\right) f_1^2\left(s_{(1)}^2 \mid s_{(2)}^2\right) f_2^2\left(s_{(2)}^2\right) ds_{(1)}^2 ds_{(2)}^2$$

and this is also non-decreasing in r_i. Hence, the allocation in which the two bidders with the highest signals obtain one license each is implementable. One can check that the allocations in (v) and (vi) are not implementable.

Next, we consider allocations that have the feature that the number of licensees is uncertain before the auction. There are potentially many allocation rules that have this feature. Below, we consider a particular class of such allocation rules. We call each rule in this class the *Non-decreasing cutoff (NDC) rule*. These allocation rules have the feature that if the reports $r_{(1)}^3$ and $r_{(2)}^3$ are "close" to each other, then the bidders with the reports $r_{(1)}^3$ and $r_{(2)}^3$ obtain one license each; else, the bidder with the report $r_{(1)}^3$ obtains both licenses. In particular, corresponding to each allocation Q, there is a non-decreasing function $\underline{s}(r;Q) \le r$, such that if $r_{(2)}^3 \ge \underline{s}\left(r_{(1)}^3;Q\right)$, then $\left(r_{(1)}^3, r_{(2)}^3, r_{(3)}^3\right) \in A4(\hat{r} \mid Q)$ and the bidders with the highest and second highest reports obtain one license each, while if $r_{(2)}^3 < \underline{s}\left(r_{(1)}^3;Q\right)$, then $\left(r_{(1)}^3, r_{(2)}^3, r_{(3)}^3\right) \in A6\left(\hat{r} \mid Q\right)$ and the bidder with the highest report obtains both licenses[1].

Formally,

$$\left(r_{(1)}^3, r_{(2)}^3, r_{(3)}^3\right) \in \begin{cases} A4\left(\hat{r} \mid Q\right) \; if \;\; 0 \leq \max\left\{r_{(3)}^3, \underline{s}\left(r_{(1)}^3;Q\right)\right\} \leq r_{(2)}^3 \leq r_{(1)}^3 \leq 1 \\ A6\left(\hat{r} \mid Q\right) \; if \;\; 0 \leq r_{(3)}^3 \leq r_{(2)}^3 \leq \underline{s}\left(r_{(1)}^3;Q\right) \leq r_{(1)}^3 \leq 1 \end{cases} \tag{23}$$

We now determine the expected payoff of bidder i under the NDC rule, when she reports r_i. It follows from (23) that under any allocation rule that belongs to the NDC class, if $r_{(1)}^2 < \underline{s}\left(r_i;Q\right)$, then bidder i obtains both licenses, and if $\underline{s}\left(r_i;Q\right) \leq r_{(1)}^2 < r_i$, bidder i obtains one license. In order to specify bidder i's allocation under the NDC rule in the case that $r_{(1)}^2$ is greater than r_i, define

$$\bar{s}\left(r_i;Q\right) := \max\left\{r_{(1)}^2 \mid \underline{s}\left(r_{(1)}^2;Q\right) \leq r_i\right\}$$

Given bidder i's report r_i, $\bar{s}\left(r_i;Q\right)$ is, by construction, the maximum value of $r_{(1)}^2$ such that bidder i can obtain a license. The above statement implies that if $r_{(2)}^2 < r_i < r_{(1)}^2 \leq \bar{s}\left(r_i;Q\right)$, then bidder i can obtain exactly one license while, if either $r_{(2)}^2 > r_i$ or if $r_{(1)}^2 > \bar{s}\left(r_i;Q\right)$, then bidder i cannot obtain any license. Observe that $\bar{s}\left(r_i;Q\right)$ is non-decreasing in r_i because of the fact that $\underline{s}\left(r_i;Q\right)$is non-decreasing in r_i.

Suppose bidder i reports r_i and the other bidders report truthfully. Under an allocation rule that belongs to the NDC class described above,

$$\Phi_1\left(r_i\right)\Pi\;\left(r_i, s_i \mid 1\right) = \int_{\underline{s}\left(r_i;Q\right)}^{\bar{s}\left(r_i;Q\right)} \int_0^{\min\left\{r_i, s_{(1)}^2\right\}} \pi\left(s_i; s_{(1)}^2, 0\right) f_{12}^2\left(s_{(1)}^2, s_{(2)}^2\right) ds_{(2)}^2 ds_{(1)}^2 \tag{24}$$

and $$\Phi_2\left(r_i\right)\Pi\;\left(r_i, s_i \mid 2\right) = \pi\left(s_i;0,0\right) \int_0^{\underline{s}\left(r_i;Q\right)} \int_0^{s_{(1)}^2} f_{12}^2\left(s_{(1)}^2, s_{(2)}^2\right) ds_{(2)}^2 ds_{(1)}^2 \tag{25}$$

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[1] The optimal allocation rule for fixed externalities belongs to this class, see Dana and Spier (1994) and Schmitz (2002).

We now check whether these classes of allocations satisfy (21). Notice that, from (24) and (25),

$$\frac{\partial}{\partial r_i}[\Phi_1(r_i)\Pi_2(r_i,s_i \mid 1)+\Phi_2(r_i)\Pi_2(r_i,s_i \mid 2)]$$
$$=\bar{s}'(r_i;Q)\pi_1\left(s_i;\bar{s}(r_i;Q),0\right)F_2^2\left(r_i \mid s_{(1)}^2=\bar{s}(r_i;Q)\right)f_1^2\left(\bar{s}(r_i;Q)\right)$$
$$+\underline{s}'(r_i;Q)\left\{\int_0^{\underline{s}(r_i;Q)}-\pi_{12}(s_i;s,0)ds\right\}f_1^2\left(\underline{s}(r_i;Q)\right) \qquad (26)$$

It follows that, given (2) and (4), an allocation rule in this class satisfies (21), and is hence implementable.

3.3.2 Revenue under Incentive Compatible Mechanism

We now determine the expected payment of a bidder in an incentive compatible direct mechanism. Below, we prove a version of the revenue equivalence theorem by showing that the expected payments of bidder i with signal s_i in two mechanisms that have the same allocation rule and the same net payoff for a bidder with signal 0 are equal.

We define $\alpha_k(r_i)$ for $k=1,...,6$ as follows:

$$\alpha_k(r_i)=\begin{cases}E\{\pi(.;.,.)\} & if\ \ \hat{s}_{-i}\in Bk(r_i)\\ 0 & if\ \ \hat{s}_{-i}\notin Bk(r_i)\end{cases}$$

In particular, $\alpha_k(r_i)$ is the expected gross payoff to bidder i when she reports a signal r_i, the others report truthfully and the profile of reports belong to the set $Bk(r_i)$ for $k=1,...,6$. Observe that by construction, the following equality must hold:

$$V_{iQ}(s_i,s_i)=\sum_{k=1}^{6}\alpha_k(s_i)-m(s_i) \qquad (27)$$

Analogously, we define $\beta_k(r_i,s_i)$ for k = 4, 5, or 6 as follows:

$$\beta_k(r_i)=\begin{cases}E\{\pi_1(.;.,.)\} & if\ \ \hat{s}_{-i}\in Bk(r_i)\\ 0 & if\ \ \hat{s}_{-i}\notin Bk(r_i)\end{cases}$$

It follows from the above definition that $\beta_k(r_i, s_i)$ is the expected value of the marginal gross payoff to bidder i when she reports r_i, the others report truthfully and the profile of reports belong to the set $Bk(r_i)$ for $k = 1,...,6$. The expected payment of a bidder in an incentive compatible direct mechanism is presented in the following proposition.

Proposition 3.3 (Expected payment under incentive compatible mechanism)

In the truth-telling equilibrium of an incentive compatible and individually rational direct mechanism in which a bidder with signal 0 obtains a net payoff of $V_Q(0,0)$, the expected payment of bidder i with signal s_i is given by

$$m(s_i) = \sum_{k=1}^{6} \left[\alpha_k(s_i) - \int_0^{s_i} \beta_k(s,s)ds \right] - V_Q(0,0) \quad provided \;\; V_Q(0,0) \geq \underline{\pi} \qquad (28)$$

proof : see appendix C.4.

The above proposition shows that the expected payment of a bidder in two mechanisms are the same whenever these mechanisms have the same partition $\{Bk(r_i)\}_{k=1}^{6}$, and the same net payoff of the bidder with signal 0. Also, notice that, under the condition that a bidder with signal 0 earns a payoff of $V_Q(0,0)$, the equilibrium payment function $m_i(.)$ is the same for all the bidders. Consequently, the subscript i from the expected payment function $m_i(.)$ has been dropped.

We now use proposition 3.3 to determine the revenue of the auctioneer in the truth-telling equilibrium of an incentive compatible direct mechanism. Below, we define the *industry virtual payoff* and show that the expected revenue of the auctioneer is the expected value of the industry virtual payoff. The industry virtual payoff depends on the allocation rule and hence, the auctioneer maximizes expected revenue by choosing the allocation rule that maximizes the expected industry virtual payoff.

Definition : Given the profile of signals $\left(s_{(1)}^{3}, s_{(2)}^{3}, s_{(3)}^{3}\right)$ and the allocation rule Q, such that

$$\left(s_{(1)}^{3}, s_{(2)}^{3}, s_{(3)}^{3}\right) \in Ak(\hat{r} \mid Q);\ k = 1,\ldots,6$$

the *industry virtual payoff*, denoted by $\lambda_{Ak(\hat{r}|Q)}\left(s_{(1)}^{3}, s_{(2)}^{3}, s_{(3)}^{3}\right)$ is given by

$$\lambda_{Ak(\hat{r}|Q)}\left(s_{(1)}^{3}, s_{(2)}^{3}, s_{(3)}^{3}\right) = \sum_{j=1}^{3}\left[\pi(.;.,.) - \chi_{w}\left(s_{(j)}^{3}\right)\frac{1-G\left(s_{(j)}^{3}\right)}{g\left(s_{(j)}^{3}\right)}\pi_{1}(.;.,.) \mid \left(s_{(1)}^{3}, s_{(2)}^{3}, s_{(3)}^{3}\right) \in Ak(\hat{r} \mid Q)\right];$$
$$k = 1,\ldots,6 \tag{29}$$

where $\chi_{w}\left(s_{(j)}^{3}\right)$ is an indicator variable that takes value 1 if the bidder with type $s_{(j)}^{3}$ obtains a spectrum license.

As an example, suppose the profile of signals $\left(s_{(1)}^{3}, s_{(2)}^{3}, s_{(3)}^{3}\right)$ belongs to the set $A4(\hat{r} \mid Q)$ under the auctioneer's allocation rule. Then,

$$\lambda_{A4(\hat{r}|Q)}\left(s_{(1)}^{3}, s_{(2)}^{3}, s_{(3)}^{3}\right) = \pi\left(s_{(1)}^{3}; s_{(2)}^{3}, 0\right) + \pi\left(s_{(2)}^{3}; s_{(1)}^{3}, 0\right) + \pi\left(0; s_{(1)}^{3}, s_{(2)}^{3}\right) - \frac{1-G\left(s_{(1)}^{3}\right)}{g\left(s_{(1)}^{3}\right)}\pi_{1}\left(s_{(1)}^{3}; s_{(2)}^{3}, 0\right) - \frac{1-G\left(s_{(2)}^{3}\right)}{g\left(s_{(2)}^{3}\right)}\pi_{1}\left(s_{(2)}^{3}; s_{(1)}^{3}, 0\right).$$

The industry virtual payoff is the gross industry payoff less the information rents of the licensees. The information rent depends on the distribution function of the signals and the signals of the licensees. It also follows from (4) that $\pi_{12}(.;.,.) < 0$ and hence, the information rent of a licensee is non-increasing in the signal of another licensee. Below, we use (28) and (29) to determine the revenue of the auctioneer in the truth-telling equilibrium of any incentive compatible direct mechanism.

Proposition 3.4 (Expected revenue under incentive compatible mechanism)

The expected revenue of the auctioneer in the truth-telling equilibrium of an incentive compatible direct mechanism is

$$R_{Q} = \int_{0}^{1}\int_{0}^{s_{(1)}^{3}}\int_{0}^{s_{(2)}^{3}}\lambda_{Ak(\hat{r}|Q)}\left(s_{(1)}^{3}, s_{(2)}^{3}, s_{(3)}^{3}\right)f_{123}^{3}(.)\,ds_{(3)}^{3}ds_{(2)}^{3}ds_{(1)}^{3} - 3V_{Q}(0,0) \tag{30}$$

Where

$$\left(s_{(1)}^{3}, s_{(2)}^{3}, s_{(3)}^{3}\right) \in Ak(\hat{r} \mid Q);\ k = 1,\ldots,6$$

proof : see appendix C.5.

This proposition states that the revenue of the auctioneer in the truth-telling equilibrium of a direct mechanism is the expected industry virtual payoff, less the product of the number of bidders and the payoff of the bidder with signal zero. Notice that the industry virtual payoff in (30) given any profile of signals depends on the auctioneer's allocation rule.

3.3.3 The Optimal Allocation

The allocation rule that maximizes (30) is defined to be the optimal allocation rule and the associated mechanism is said to be the optimal mechanism. Below, we use proposition 3.4 to determine the optimal mechanism when the payoff function exhibits signal-dependent externalities. It follows from (30) that in the optimal mechanism, the expected industry virtual payoff is maximized and the payoff of the bidder with signal zero is minimized. In order to describe the allocation that maximizes the expected industry virtual payoff, we define

$$\lambda^*\left(s_{(1)}^3, s_{(2)}^3, s_{(3)}^3\right) = \max\left\{\lambda_{A1(\hat{r}|Q)}(.),..., \lambda_{A6(\hat{r}|Q)}(.)\right\}$$

Further, one has to check that the allocation that maximizes the expected industry virtual payoff satisfies the conditions for incentive compatibility given by (20) and (21). Notice that the expression in (30) relies on (20) only and therefore, there is no guarantee that the optimal allocation derived by maximizing the expression in (30) satisfies (21). In models of auctions without externalities, this problem is usually solved by assuming a regularity condition-in such cases, the regularity condition states that the virtual value is an increasing function of the signal.

In our model, we can solve the problem using an appropriate regularity condition and this is described below. We define a problem to be *regular* if the following conditions are true: *(i)* Given any profile of signals $\left(s_{(1)}^3, s_{(2)}^3, s_{(3)}^3\right)$, either $\lambda^*(.) = \lambda_{A6(\hat{r}|Q)}(.)$ or $\lambda^*(.) = \lambda_{A4(\hat{r}|Q)}(.)$; hence, in the optimal allocation, the auctioneer either allocates both the licenses to the bidder with the highest signal (when $\lambda^*(.) = \lambda_{A6(\hat{r}|Q)}(.)$) or to the two bidders with the two highest signals (when

$\lambda*(.)=\lambda_{A4(\hat{r}lQ)}(.)$), *(ii)* The allocation induced by $\lambda*(.)$ belongs to the NDC class. Notice that if the problem is regular, then the allocation that maximizes the industry virtual payoff is incentive compatible, because allocations that belong to the NDC class satisfy (21).

In the proposition below, it has been assumed that if one of the bidders decides to stay out of the mechanism, then the auctioneer can credibly commit to allocate a license to each of the bidders who participate in the mechanism. Under such a commitment, the payoff to a bidder that decides not to participate in the mechanism is

$$\underline{\pi}=\int_0^1\int_0^{s_{(1)}^2}\pi\left(0;s_{(1)}^2,s_{(2)}^2\right)f_{12}^2(.)ds_{(2)}^2ds_{(1)}^2 \tag{31}$$

The value of $\underline{\pi}$ in (31) is used in the description of the optimal mechanism in Proposition 3.5.

Proposition 3.5 (Expected Revenue under the Optimal Allocation)

Suppose that the problem is regular, and if a bidder does not participate in the mechanism, the auctioneer can credibly commit to allocate one license each to the two other bidders. Moreover, suppose that the payoff function exhibits signal-dependent externalities. Then the revenue of the auctioneer in the truth-telling equilibrium of the optimal mechanism is given by

$$R_Q^*=\int_0^1\int_0^{s_{(1)}^3}\int_0^{s_{(2)}^3}\lambda*\left(s_{(1)}^3,s_{(2)}^3,s_{(3)}^3\right)f_{123}^3(.)ds_{(3)}^3ds_{(2)}^3ds_{(1)}^3-3\underline{\pi} \tag{32}$$

Where $\underline{\pi}$ is defined by (31).

proof : see appendix C.6.

The above proposition states that in a regular problem, the auctioneer can achieve her optimal revenue by choosing the allocation rule that maximizes the industry virtual payoff for every profile of reports. It follows from the definition of the industry virtual payoff that the following three factors affect the auctioneer's revenue when its selects one licensee instead of two licensees:

(1) The industry gross profit may increase or decrease, and hence, the effect of this factor on the auctioneer's revenue is ambiguous.

(2) If the auctioneer selects only one licensee, then the bidder with the second highest signal cannot earn any information rent and this increases the revenue of the auctioneer. Notice that this factor will be present even in a model with no externalities; however, the magnitude of this effect depends on the externality parameter. This factor reduces the number of licensees.

(3) If the auctioneer selects only one licensee, then the information rent of the bidder with the highest signal increases because $\pi_{12}(.;.,.) < 0$. This factor therefore reduces the revenue of the auctioneer. Notice that this factor emerges only in the presence of externalities.

The choice of the optimal number of spectrum licensees therefore depends on the relative strength of each of the above-mentioned factors which in turn depends on the nature of the product market. Below, we show the role of channels through which the product market influences the optimal number of licensees.

3.4 Role of the Product Market in Determining the Optimal Allocation

In order to show the role of product market factors in determining the optimal number of licensees, we need to make more explicit assumptions about the product market. For the purpose of the discussion below, we consider the scenario described in Example 1 when $\tau = 4$. In the example, bidders compete in quantites and a winner imposes a higher level of externalities on the others when the level of product differentiation is low.

Remark 1: Significant externalities may cause the auctioneer to select a fewer number of licensees in the presence of private information compared to what he would have done if bidders had no private information[2].

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[2] By stimulating more cases in the profile of signals, one observes that if the signals $s^3_{(1)}$ and $s^3_{(2)}$ are "close" to each other, then it is optimal for the case of 2 licensees; otherwise, it is optimal for the case of single licensee. Hence, one makes a conjecture that the "closeness" between $s^3_{(1)}$ and $s^3_{(2)}$ is also a factor in determining the optimal number of licensees.

Suppose the signals follow the uniform distribution. In the table below, we present the optimal number of licensees when the profile of signals is

$$\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) = (0.9, 0.8, 0.3)$$

Table 3.3

The Optimal Number of Licensees under Different Information Structure as a Function of the Externality Parameter

Information structure	Externality parameter		
	$\mu = 0$	$\mu = 0.5$	$\mu = 1$
Signals are publicly observable	2	2	2
Signals are private information	2	2	1

Notice that the optimal number of licensees is always two when the signal of each bidder is publicly observable. Furthermore, the optimal number of licensees remains at two even when the signal of each bidder is its private information, as long as the externality parameter is 0 or 0.5. However, when the externality parameter is 1, the auctioneer selects only one licensee in the optimal allocation when the signal of each bidder is her private information. In this example, private information by itself does not induce the auctioneer to select a fewer number of licensees. Instead, private information combined with the presence of significant externalities induces the auctioneer to allocate spectrum licenses to a fewer number of bidders.

Remark 2: When bidders have private information, a higher level of externality leads to a decrease in the expected number of licensees.

Consider the scenario described in the previous remark. For each possible profile of signals, we compute the optimal number of licensees and use this to compute the expected number of licensees. In the table below, we show how the expected number of licensees vary with the externality parameter (level of product differentiation in the industry).

Table 3.4

The Expected Number of Licensees

as a Function of the Externality Parameter

	Externality parameter		
	$\mu = 0$	$\mu = 0.5$	$\mu = 1$
Expected Number of Licensees	1.47	1.27	1.1

Observe that an increase in the externality parameter (or, a reduction in the level of product differentiation) leads to a decrease in the expected number of licensees.

4. CONCLUSION

This essay analyzes the auctioneer's revenue from the auction of two identical spectrum licenses, both when the signal of each bidder is publicly observable, and when the signal of each bidder is her private information. It is assumed that if a bidder refuses to participate in the mechanism, then the auctioneer can credibly commit to allocate a license to her competitors. When the signal of each bidder is publicly observable, the auctioneer's optimal allocation is the one that maximizes the industry payoff. In contrast, when the signal of each bidder is her private information, the auctioneer selects the allocation rule that maximizes the industry virtual payoff. Such an allocation may or may not be different from the allocation that maximizes the industry payoff. We find that the presence of private information leads the auctioneer to sometimes allocate licenses to a smaller number of bidders.

An important assumption in this model is that the private information of each bidder is one-dimensional. It follows from Jehiel and Moldovanu (2001) that an optimal allocation cannot be implemented when the signals are multi-dimensional. An interesting extension of this model is to analyze revenue-maximizing allocations when the signals are multi-dimensional.