ESSAY II

SIMULTANEOUS AUCTION WITH EXTERNALITY AND THE OPTIMAL NUMBER OF LICENSE

1. INTRODUCTION

The presence of incomplete information makes auction an efficient way to allocate the licenses. There are two issues related to auction of the spectrum licenses.

First, the cost of making an allocation of an additional spectrum license is costless for the auctioneer. Therefore, we must take into account the optimal number of licenses to allocate. The existing literature (for example, Jehiel & Moldovanu, 2000 and Das Varma, 2003) has considered the allocation of only one license. In this essay, we extend the literature by letting the auctioneer allocates multiple spectrum licenses.

Second, there is an issue of how much will the bidder pays to the auctioneer to ensure exclusivity. In this essay, we assume that each bidder can bid for at most one license. The presence of externalities reduces the willingness to pay of a bidder when the auctioneer increases the number of spectrum licenses. However, this may be offset by the fact that the auctioneer extracts payments from more bidders when it increases the number licenses. We analyze factors that determine which of this two effects dominate when the auctioneer increases the number of licenses.

This essay models the spectrum-license allocation base on simultaneous auction. We deviate from Das Varma (2003) in which our model has multiple and endogenously determine number of spectrum licenses. The licenses that we consider are the innovations that used to reduce the costs of bidders who have access to the licenses. The magnitude of these cost savings is bidder specific. Furthermore, we assume that the cost of a bidder that acquires a license is private information.

In this model, a bidder that wins a spectrum license imposes a negative externality on the other bidders. Moreover, the magnitude of externality depends on the characteristic of the bidder that wins a license, such as marginal cost (the bidder's "type"). We assume that each bidder knows her type and she also knows that the types of her competitors are independently drawn from a distribution (which is common knowledge). Therefore, we have interdependent values but independent types.

To analyze the equilibrium bidding strategy, we need to determine the value of a spectrum license to each type of bidder. We show that the way to define the value of a license when multiple licenses are being allocated is a generalization of the intrinsic value proposed by Das Varma (2003).

The intrinsic value of a spectrum license to a bidder with marginal cost c is the value to this bidder of winning a license rather than losing it to a competitor with the same cost. The intrinsic value plays a similar role in our analysis as the exogenously determined value of an object does in the Milgrom and Weber (1982)[1]. By substituting our intrinsic value for the exogenously specified value in Milgrom and Weber's model, we can exploit their analysis to derive some of our results.

We begin by considering the case in which the number of spectrum licenses is predetermined to be an integer less than the number of bidders. We construct a problem that replicates the outcome of well-known auction rule, the uniform price auction (the multi-unit generalization of a second-price auction). In the equilibrium of an auction of k spectrum licenses, the winners are the bidders with the lowest types. For each type of bidder, we determine the expected payment in the truth-telling equilibrium. We then use this result to characterize the equilibrium bidding strategy.

Finally, we consider how many spectrum licenses the auctioneer should offer to maximize its revenues. In order to address the issue, it is necessary to make specific assumptions about the structure of product market, the magnitude of externalities, and the distribution of types. By means of examples, we show that the allocation of a single license is optimal when the magnitude of the externalities is high or when bidders are more likely to have high costs. However, when either of these conditions does not hold, the allocation of multiple spectrum licenses turn to be optimal for the auctioneer.

The essay is consisted of 4 sections. Section 1 is introduction, section 2 is literature review, section 3 is the model, and the last section is conclusion. Most of the proofs are presented in appendix B.

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[1] The value of an object in Milgrom and Weber (1982) is the difference between a bidder's payoffs from winning the object instead of losing it.

2. RELATED LITERATURE

This essay is closely related to the literature on licensing under incomplete information and the optimal number of licenses. There are several related articles in the literature[2].

Katz and Shapiro (1986) and Hoppe, Jehiel, and Moldovanu (2004) have studied the problem of determining the optimal number of licenses when there is complete information. We, on the other hand, suppose that a bidder's marginal after acquiring a spectrum license is private information.

Recently, there has been an emerging literature on auctions with externalities in which there is incomplete information. Several of these studies discuss auctions of licenses. Jehiel, Moldovanu and Stachetti (1999) have considered the problem of optimal auction design in the presence of externalities. However, they have assumed that the auctioneer allocates only one license. Some other important works that deal with an auction of a single object (not necessarily a spectrum license) in the presence of externalities are Jehiel, Moldovanu and Stachetti (1996), Jehiel and Moldovanu (2000) and Moldovanu and Sela (2003).

Jehiel and Moldovanu (2000) have studied the equilibrium in a second price auction of a single license. However, in their model, the externality may not be negative. In particular, they have allowed for the possibility that the winner may impose a positive externality on the losers (a likely possibility in network industries). These authors have identified the separating equilibrium in the presence of negative externalities. However, in the presence of positive externalities, they have proved that a separating equilibrium may fail to exist. In our model, we deal with negative externalities only and, therefore, a separating equilibrium exists. Jehiel and Moldovanu (2000) assume that after the auction, but before the bidders produce any output, the types become common knowledge. We retain this assumption here.

There is a related literature (for example, Das Varma (2003)) that does not assume that types will be revealed exogenously after the auction. In Das Varma's

[2] There has also been previous work on the efficacy of different mechanisms in allocation of licenses. This literature has been surveyed in Kamien (1992).

model, the winner of a license signals her type through her bid. Although with rational expectations, the other bidders correctly infer her type from her bid, the equilibrium in this model is different from that of Jehiel and Moldovanu (2000) because the signalling motive modifies the equilibrium bidding strategy. Here, the focus is on analyzing multi-unit auctions and, therefore, for the sake of tractability, we assume that a bidder's type is revealed exogenously after the auction. Consequently, the signalling motive will not influence the bidding function in our model.

There has been some work on auctions of multiple licenses under incomplete information. Jehiel and Moldovanu (2001; 2004) show the impossibility of implementing efficient outcomes when types are multi-dimensional. We assume that a bidder's type is of one dimension. Moreover, in our model, when the auctioneer chooses to allocate multiple spectrum licenses, each bidder is allowed to bid for only one license. Hence, the allocations we consider are not always the efficient allocations as defined in Jehiel and Moldovanu (2001).

Dana and Spier (1994) consider auctions of production rights in an industry, but in their model, the auctioneer (which is the government) maximizes social welfare. Moreover, the payoff of a bidder that does not win a license is assumed to be zero.

To our knowledge, Schmitz (2002) is the only article that has analyzed revenue maximizing allocations from an auction of multiple licenses with incomplete information. He has shown that the optimal number is not always one. There are two key differences between our model and his. First, Schmitz assumes that there is some probability that a bidder that wins a license may not be able to commercially exploit the license, whereas, in our model, this is not the case. Second, Schmitz assumes that a bidder's payoff is zero if she does not win a license or if one of the other bidders does. We do not restrict the payoff functions in this way.

It can be shown that if either of these assumptions is relaxed in Schmitz's model, the auctioneer will only want to allocate a single license. In our model, we do not adopt either of Schmitz's assumptions.

3. THE MODEL

This section constructs a deviate version of Das Varma (2003). The first section describes the environment of the model. The second section explains the concept of the intrinsic value. The third section solves for equilibrium bidding strategy. The last section is a study on the optimal number of licenses.

3.1 Environment

We consider the simultaneous auction with n bidders, $N = \{1,2,....,n\}$. Each bidder i produces a product at a constant marginal cost. The profit of any bidder i depends on her own cost, c_i and the costs of the other bidders, denote by c_{-i}.

We write the profit function of bidder i as $\pi(c_i;c_{-i})$. The function $\pi(.)$ is assumed to be twice continuously differentiable in all arguments. Moreover, the following relations are assumed to hold:

$$\frac{\partial \pi(c_i;c_{-i})}{\partial c_i} < 0, \qquad \frac{\partial \pi(c_i;c_{-i})}{\partial c_j} > 0 \qquad \textit{for } j \neq i$$

Let define what we mean by negative externality in context of our model;

Definition : A bidder j with type c_j is said to impose a *negative externality* on bidder i with type c_i, whenever $\frac{\partial \pi(c_i;c_{-i})}{\partial c_j} > 0$ for $j \neq i$.

Initially, all bidders have the marginal cost equal to 1. Now suppose that there is a limited spectrum licenses with property that a bidder that has access to the license is able to produce at a cost that does not exceed 1.

Given the auction rule, the auctioneer's problem is to choose the number of spectrum licenses to maximize its revenue subject to the constraint that no bidder can obtain more than one license. Once the auction has been completed and the licenses have been transferred, the bidders produce and earn profits that depend on their post-auction marginal costs.

Decision tree of the model is as follows;

$t=0$	$t=1$	$t=2$	$t=3$
Bidders receive private information $c \in [0,1]$	Bidders simultaneously submit bid	The spectrum licenses are transferred	Bidders produce and earn profit

We identify a bidder's "type" with her marginal cost if she has access to the license. At the time of the auction, each bidder knows what her cost will be if she wins a license. However, she only knows that the marginal costs of all of the other bidders who win a license are independently drawn from a distribution $G(.)$ with support $[0,1]$.

We assume that $G(.)$ is continuously differentiable. Let $g(.)$ be the corresponding density function, where $g(c) > 0$ for all $c \in (0,1)$. Let $F_k^n(.)$ denote the distribution function of $c_{(k)}^n$, where $c_{(k)}^n$ is the k^{th} lowest cost realization among the n bidders, and let $f_k^n(.)$ denote the associated density function. Let $F_{1k}^n(c_{(1)}, c_{(2)}, ..., c_{(k)})$ denote the joint distribution of $c_{(1)}^n, c_{(2)}^n, ..., c_{(k)}^n$ and let $f_{1k}^n(c_{(1)}, c_{(2)}, ..., c_{(k)})$ denote its corresponding density function.

If bidder i wins a spectrum license, her cost will be $c_i \leq 1$. Thus, if k spectrum licenses are being auctioned and bidder i wins one of them, then $(k-1)$ of her competitors will also have a license. Therefore, in this model, at most k bidders will have costs less than 1 and at least $(n-k)$ bidders will have costs equal to 1.

3.2 The Intrinsic Value

In the subsequent analysis, we consider an auction of $k < n$ spectrum licenses in which the licenses are allocated to bidders with the k lowest marginal costs. This allocation is an extension of the ones considered in Jehiel and Moldovanu (2000) and Das Varma (2003).

If k licenses are being auctioned and bidder i wins one of them, then bidder i's profit will be $\pi(c_i; c_{(1)}^{n-1}, c_{(2)}^{n-1}, ..., c_{(k-1)}^{n-1}, e_{n-k})$ where $c_{(j)}^{n-1}$ $(j = 1,2,...(k-1))$ are the marginal costs of the other winning bidders and e_{n-k} is a (n − k) vector of ones. If bidder i does not win, then her profit will be $\pi(1; c_{(1)}^{n-1}, c_{(2)}^{n-1}, ..., c_{(k)}^{n-1}, e_{n-k-1})$, where e_{n-k-1} is a (n − k − 1) vector of ones.

In the following discussion, we drop the superscripts $n-1$ or n from the order statistics if it is clear from the context whether $n-1$ or n draws are being made from the distribution. Furthermore, we define the following:

$$\pi^W(c_i; c_{(1)}, c_{(2)}, ..., c_{(k-1)}) \equiv \pi(c_i; c_{(1)}^{n-1}, c_{(2)}^{n-1}, ..., c_{(k-1)}^{n-1}, e_{n-k})$$

$$\pi^L(1; c_{(1)}, c_{(2)}, ..., c_{(k)}) \equiv \pi(1; c_{(1)}^{n-1}, c_{(2)}^{n-1}, ..., c_{(k)}^{n-1}, e_{n-k-1})$$

and

$$\pi^0 \equiv \pi(1; e_{n-1})$$

Consider the situation in which one bidder has marginal cost of c if she is awarded a spectrum license and the k^{th} lowest of the marginal costs of her competitors is also c. In this case, not all bidders with cost c can win a spectrum license. The intrinsic value, $V_k(c)$, of a spectrum license to a bidder with cost c when $k < n$ licenses are being auctioned and each bidder can bid for only one license is the value to this bidder of winning a license rather than losing it to a competitor with the same cost. Formally,

$$V_k(c) \equiv E[\, \pi^W(c; c_{(1)}, c_{(2)}, ..., c_{(k-1)}) - \pi^L(1; c_{(1)}, c_{(2)}, ..., c_{(k)}) \mid c_{(k)} = c] \qquad (1)$$

where the expectation in (1) is taken over $c_{(1)}, c_{(2)}, ..., c_{(k-1)}$ conditional on $c_{(k)} = c$. Equivalently,

$$V_k(c) = \int_0^c \int_0^{c_{(k-1)}} .... \int_0^{c_{(2)}} [\pi^W(c\ ; c_{(1)}, c_{(2)}, ..., c_{(k-1)}) - \pi^L(1; c_{(1)}, c_{(2)}, ..., c_{(k-1)}, c)]$$

$$\times \frac{f_{1k}^{n-1}(c_{(1)}, c_{(2)}, ..., c_{(k-1)}, c_{(k)} = c)}{f_k^{n-1}(c)} dc_{(1)} .... dc_{(k-1)} \qquad (2)$$

To understand (2), notice the following inequalities which obtain directly from the order statistics;

$$0 \le c_{(1)} \le .... \le c_{(k-1)} \le c_{(k)} = c \le 1$$

Hence, $c_{(k-1)}$ can take any value from 0 to c, and given any value of $c_{(k-1)}$ in this range, $c_{(k-2)}$ can take any value from 0 to $c_{(k-1)}$, and so on. This explains the specification of the integral limits.

The concept of the intrinsic value was first introduced by Das Varma (2003) for the special case of one license is being auctioned. Below, we show that $V_k(c)$ is the price that a bidder is willing to pay in a uniform price auction.

We assume that the profit function and the distribution of costs are such that

$$V_k'(c) < 0 \tag{3}$$

This assumption is sufficient for the existence of a separating equilibrium in standard auctions. In general, this inequality may not be satisfied because the inequality depends both on the profit function and on the distribution of costs.

3.3 Equilibrium Bidding Strategy

In this section, we consider an auctioneer that uses a standard auction to allocate a fixed number of spectrum licenses k, where $k \in K = \{1,2,...,(n-1)\}$. In a standard auction, the k highest bidders win a license each (Krishna, 2002, p. 168).

By the revelation principle, associated with standard auction there is an equivalent direct mechanism whose truth-telling equilibrium has the same outcome as the standard auction. In a direct mechanism, the auctioneer announces the probability that a bidder wins a spectrum license (the allocation rule) and her expected payment (the payment rule) as a function of the bidders' reports about their types. It then implements an outcome based on the reports.

We consider symmetric monotonic equilibria of standard auctions, that is, equilibria in which all bidders follow the same bidding strategy and the bids are strictly decreasing in types. Analogous to the definition of a standard auction, we call a direct mechanism standard if the mechanism allocates one spectrum license each to the k bidders with the lowest reported types.

The allocation rule that we consider is one in which the auctioneer awards the licenses to the k bidders with the lowest reported types. In this mechanism, in a truth-telling equilibrium, bidder i wins the auction if her type c_i is less than the k^{th} lowest of her competitors' types, that is, if $c_i < c_{(k)}$.

Therefore, if bidder i wins a spectrum license, her expected profit in equilibrium is:

$$\int_{c_i}^{1}\int_{0}^{c_{(k)}}\cdots\int_{0}^{c_{(2)}} \pi^W(c_i;c_{(1)},c_{(2)},\ldots,c_{(k-1)})\; f_{1k}^{n-1}(c_{(1)},c_{(2)},\ldots,c_{(k)})\; dc_{(1)}\ldots.dc_{(k)}$$

Similarly, if $c_i > c_{(k)}$ this bidder does not win a license, her expected profit is:

$$\int_{0}^{c_i}\int_{0}^{c_{(k)}}\cdots\int_{0}^{c_{(2)}} \pi^L(1;c_{(1)},c_{(2)},\ldots,c_{(k)})\; f_{1k}^{n-1}(c_{(1)},c_{(2)},\ldots,c_{(k)})\; dc_{(1)}\ldots.dc_{(k)}$$

We define $m_k(c_i)$ to be the expected payment of a bidder that reports type c_i when k licenses are being auctioned.

Let $P(\tilde{c},c_i)$ denote the payoff of bidder i when her true type is c_i but she reports her type as being $\tilde{c}$. From the preceding discussion, $P(\tilde{c},c_i)$ takes the following form:

$$P(\tilde{c},c_i)=\int_{\tilde{c}}^{1}\int_{0}^{c_{(k)}}\cdots\int_{0}^{c_{(2)}} \pi^W(c_i;c_{(1)},c_{(2)},\ldots,c_{(k-1)})\; f_{1k}^{n-1}(c_{(1)},c_{(2)},\ldots,c_{(k)})\; dc_{(1)}\ldots.dc_{(k)}$$

$$+\int_{0}^{\tilde{c}}\int_{0}^{c_{(k)}}\cdots\int_{0}^{c_{(2)}} \pi^L(1;c_{(1)},c_{(2)},\ldots,c_{(k)})\; f_{1k}^{n-1}(c_{(1)},c_{(2)},\ldots,c_{(k)})\; dc_{(1)}\ldots.dc_{(k)} - m_k(\tilde{c}) \qquad (4)$$

The first term on the right hand side of (4) is the expected gross profit to bidder i from winning a spectrum license when her true type is c_i and her reported type is $\tilde{c}$. The second term is the expected gross profit to bidder i from not winning the license. Finally, the third term is the expected payment of a bidder with reported type $\tilde{c}$.

Expressions analogous to the first term appear in other models of auctions with interdependent values, such as those of Schmitz (2002), Dana and Spier (1994), Jehiel and Moldovanu (2004) and Milgrom and Weber (1982).

The second term is what distinguishes our model from the above class of models. If bidder i does not win a spectrum license, then her profit is higher, the higher the cost of a winner. Because the winners have marginal costs at most 1, the profit of bidder i if she does not win a license is lower than her profit before the auction. In other words, the winners impose a negative externality on bidder i.

In many models in the literature (including the ones cited above), the second term is assumed to be a constant. To distinguish these models from our own, the former models are said to have interdependencies and fixed externalities, whereas our model has interdependencies and type-dependent externalities.

When bidder i reports her true type, her expected payoff is

$$U(c_i) \equiv P(c_i, c_i) \tag{5}$$

In the truth-telling equilibrium, $U(c_i)$ is the payoff of a bidder with type c_i. In this equilibrium, the payment rule must be such that it is a best response for a bidder to report her true type, given that the competitors also reveal their types. This is the Incentive Compatibility (IC) constraint and implies that the following inequalities are satisfied:

$$U(c_i) \geq P(\tilde{c}, c_i) \text{ for all } i \in N \text{ and all } \tilde{c}, c_i \in [0,1] \tag{6}$$

Moreover, because participation in the mechanism is voluntary, no bidder should be made worse off by participating in the mechanism. This is the Individual Rationality (IR) constraint and implies that the following inequalities are satisfied:

$$U(c_i) \geq \int_0^1 \int_0^{c_{(k)}} \cdots \int_0^{c_{(2)}} \pi^L(1; c_{(1)}, c_{(2)}, \ldots, c_{(k)}) \; f_{1k}^{n-1}(c_{(1)}, c_{(2)}, \ldots, c_{(k)}) \; dc_{(1)} \ldots dc_{(k)}$$

$$\text{for all } c_i \in [0,1] \tag{7}$$

By substituting $c_i = 1$ in the left hand side of (7), we can check that the following inequality must be satisfied:

$$m_k(1) \leq 0$$

When types are independent, it is a well-known fact that there is revenue equivalence for any standard auctions that have the same allocation rule and that extract equal expected payments from the worst type of bidders. Proposition 2.1 is a version of the revenue equivalence theorem for the allocation of k spectrum licenses by a standard direct mechanism.

Proposition 2.1 (Expected payment)

In the truth-telling equilibrium of any standard direct mechanism for the allocation of $k(< n)$ spectrum licenses, the expected payment of a bidder with type c_i is given by

$$m_k(c_i) = m_k(1) + \int_{c_i}^{1} V_k(c) f_k^{n-1}(c) dc \tag{8}$$

proof : see appendix B.1

We have argued above that $m_k(1)$ must be non-positive in order to satisfy the individual rationality constraint and hence, we choose

$$m_k(1) = 0 \tag{9}$$

to ensure that the revenue of the auctioneer is maximized. Notice that it follows from (B.8) in the appendix B.1 that

$$U'(c_i) < 0 \text{ for all } c_i \in [0,1) \tag{10}$$

hence, the equilibrium payoff is decreasing in type.

Consider common form of standard auction; a uniform price auction. In these auctions, the bidders submit a bid and the bidders with the k highest bids win a spectrum license each and the winners pay the $(k+1)^{th}$ highest bid.

We denote the bid function in the equilibrium in which bids are decreasing in types by $B_{kU}(.)$. By the revelation principle, proposition 2.1 can be used to determine the functional form of these bid function.

Corollary 2.1 (Equilibrium bidding strategy)

In a uniform price auction of $k(< n)$ spectrum licenses, the equilibrium bidding strategy in a equilibrium in which bids are decreasing in type is given by:

$$B_{kU}(c_i) = V_k(c) \text{ for all } c_i \in [0,1] \tag{11}$$

proof : Because, by assumption, bids are decreasing in type in the equilibrium, in a uniform price auction of k licenses,

$$m_k(c_i) = \int_{c_i}^{1} B_{kU}(c) f_k^{n-1}(c) dc \qquad (12)$$

Because this is true for all c_i, it follows from (8), (9), (12) and the revelation principle that (11) holds.

In corollary 2.1, the bid function is determined indirectly using properties of the equivalent direct mechanism. However, one can directly check that the bid function in corollary 2.1 are indeed the equilibrium bids in the uniform price auctions, and assuming (3) is satisfied, that this bid is monotonically decreasing in type.

The analysis of Milgrom and Weber (1982) can be used to derive the bidding functions for a number of well-known auction formats for the allocation of a single object when there are interdependencies and fixed externalities; Weber (1983) is an extension of the analysis of Milgrom and Weber for the allocation of multiple objects without externalities. The bidding function in our model is identical to the ones derived by Weber except that the intrinsic value of a license replaces the "actual value of the object" (Weber, 1983, pp. 168-169) in the expressions for the equilibrium bidding function.

Milgrom and Weber have provided an intuitive explanation for the equilibrium bids observed in the uniform price (second-price) auction for a single object (Migrom and Weber, 1982, pp. 1101). We can use the same intuition to explain the bids observed in the uniform price auction of multiple spectrum licenses when there are externalities. Each bidder bids so that it is indifferent between winning and losing, which in the case considered here is the value of the license to this bidder conditional on the k^{th} lowest competitor having the same cost.

Notice that because there is a cost of losing the auction in our model, bidders bid more aggressively than they would have in the absence of the type-dependent negative externality.

Suppose, in the counterfactual case of no externality between bidder in the model, $\frac{\partial}{\partial c_{(j)}} \pi^L(1; c_{(1)}, \ldots, c_{(k)}) = 0 \; for \; j = 1,2,\ldots,k$, a bidder earns a profit of π^0 if it does not win a spectrum license and she bids $[\pi^W(c; c_{(1)}, c_{(2)}, \ldots, c_{(k-1)}) - \pi^0]$ in the uniform price auction. When there are negative externalities (that is, when $\frac{\partial \pi(c_i;c_{-i})}{\partial c_j} > 0$ $for\ j \neq i$) the equilibrium bid is the intrinsic value $V_k(c)$ which is larger than $[\pi^W(c;c_{(1)},c_{(2)},...,c_{(k-1)}) - \pi^0]$. Thus, the bidding is more aggressive in the presence of a type-dependent negative externality.

In proposition 2.1, we have identified the expected payment of a bidder in the equilibrium of a standard direct mechanism. Using this expected payment, we can compute the expected revenue of the auctioneer.

Proposition 2.2 (Expected revenue)

The expected revenue of the auctioneer in a standard direct mechanism for the allocation of $k < n$ spectrum licenses is given by

$$R_k = k\int_0^1 V_k(c) f_{k+1}^n(c)dc = kEV_k\left(c_{(k+1)}^n\right) \tag{13}$$

proof : see appendix B.2

The expression for R_k in (13) can be interpreted easily. Given any profile of types, the price in the uniform price auction is $V_k\left(c_{(k+1)}^n\right)$ because each type bids $V_k(.)$ and the winners pay the $(k+1)^{th}$ highest bid. Because k licenses are being allocated, the expected revenue is therefore as stated in proposition 2.2.

It is our interest to know whether it is optimal to allocate multiple spectrum licenses. Suppose the auctioneer increases the number of licenses from 1 to k where $k = 2,3,...,n-1$. Then its revenue changes because of the following three effects:

(1) The number of licenses goes up, which increases the auctioneer's revenue.

(2) When one license is being allocated, the winners pay the second highest bid while in the case of k licenses, the winners pay the $(k+1)^{th}$ highest bid, which reduces the revenue.

(3) Bidder i of type c_i bids $V_1(c_i)$ when one license is auctioned and $V_k(c_i)$ when k licenses are auctioned. If $V_1(c_i) \geq V_k(c_i)$ for all $c_i \in [0,1]$ each bidder bids less

aggressively when there are more licenses being auctioned, and this may reduce the revenue. Similarly, if each bidder bids more aggressively, the revenue may increase.

The first two effects are present even in models without externalities. However, in the presence of type-dependent negative externalities, the third factor also applies. We are interested in knowing if the three effects simply counteract each other; that is, is it always optimal to allocate only one spectrum license?

Another point of interest is to determine the revenue when the types of the bidders become common knowledge after the auctioneer has announced the number of spectrum licenses k. In this case, the auctioneer can perfectly price discriminate among bidders. If bidder i with type c_i has one of the k lowest types (that is, if $c_i \le c^n_{(k)}$), then the auctioneer makes a take-it-or-leave-it offer to bidder i of

$$\pi^W(c_i; c_{(1)}, c_{(2)}, \ldots, c_{(k-1)}) - \pi^L(1; c_{(1)}, c_{(2)}, \ldots, c_{(k)}) \tag{14}$$

The first term is the profit of bidder i when bidder i and her competitors with the $(k-1)$ lowest costs win a license each. The second term is the profit of bidder i when she does not win a license and her competitors with the k lowest costs win a license each. In effect, if bidder i refuses to accept the offer, the auctioneer inflicts the maximum punishment on bidder i by allocating the licenses in a manner that minimizes bidder i's profits.

A similar strategy of the auctioneer has been used in Jehiel, Moldovanu, and Stacchetti (1996, p. 820). Notice that, it is the best response for each bidder to accept the offer. It follows immediately that the expected revenue, in this case, is given by

$$\begin{aligned} R_k^{com} = E[\ & \pi^W(c_{(1)}; c_{(2)}, \ldots, c_{(k-1)}) + \ldots. + \pi^W(c_{(k)}; c_{(1)}, \ldots, c_{(k-1)}) \\ & - \pi^L(1; c_{(2)}, \ldots, c_{(k+1)}) - \ldots - \pi^L\left(1; c_{(1)}, \ldots c_{(k-1)}, c_{(k+1)}\right)] \end{aligned} \tag{15}$$

3.4 The Optimal Number of Licenses

In this section, we investigate the auctioneer's choice of the optimal number of licenses to allocate both when a bidder's type is her private information and when a bidder's type is common knowledge. However, in order to use our model to find the optimal number of licenses, we need to make assumptions about the product market.

We assume that the bidders are profit-maximizing firms in Cournot industry. The following examples are distinguished by the magnitude of the externalities bidders impose on each other and the distribution of costs. Each of these factors affects the optimal number of licenses through its effect on the intrinsic values.

We show that the optimal number of spectrum licenses may or may not be one. Hence, it is restrictive to assume, as in Jehiel and Moldovanu (2000) and Das Varma (2003), that the auctioneer only allocates one license.

Another common feature of our examples is that the distribution of types (marginal costs) is given by the beta distribution with parameters λ_1 and λ_2. That is,

$$g(c) = \frac{1}{\beta(\lambda_1, \lambda_2)} c^{\lambda_1 - 1}(1-c)^{\lambda_2 - 1} \qquad \forall c \in [0,1]$$

For concreteness, (λ_1 , λ_2) is chosen to be either (1, 1) or (1, 2). When λ_1 = λ_2= 1, the distribution is uniform. When λ_1 = 1 and λ_2 = 2, the distribution is beta (the first-order stochastically dominated by the uniform distribution). The density functions for the order statistics can be computed from the density function for the distribution of types $g(c)$ (see David (1969, p. 9)).

Let the inverse demand take the following form:

$$p_i = 3 - q_i - \xi \sum_{j \neq i} q_j \quad \textit{where } \xi \in [0,1]$$

The inverse demand function has been chosen to ensure that all the assumptions on the profit function hold. In the demand function, the parameter ξ measures the magnitude of the externalities that competitors impose on each other. In the following examples, we consider several specifications for ξ and λ_2 and, in each case, determine the optimal number of spectrum licenses.

Example 1

We first choose ξ= 1, λ_1 = 1 and λ_2 = 1. In this case, the intrinsic value functions for the one and two spectrum licenses are

$$V_1(c) = \frac{1}{2}(3-c)(1-c)$$

$$V_2(c) = \frac{1}{4}(4-c)(1-c)$$

We now compute the expected revenue of the auctioneer. The results are tabulated below. In the table, k denotes the number of spectrum licenses, R_k^{com} denotes the revenue when the types are common knowledge, while R_k denotes the revenue under private information.

Table 2.1

The Externality Parameter is 1 and

Types are drawn from the Uniform Distribution

k	R_k^{com}	R_k
1	1	0.65
2	1.21	0.43
3	1.13	not defined

Note that the optimal number of spectrum licenses under private information is 1. However, the optimal number of licenses when the types are common knowledge is 2.

Example 2

For the parameters characterizing the type distribution in example 1, when the externality parameter is changed to ξ = 0.8, that is, when the bidders produce more differentiated products compared to the previous example. Table 2.2 is the analogue to table 2.1 for the new value of ξ.

Table 2.2

The Externality Parameter is 0.8 and

Types are drawn from the Uniform Distribution

k	R_k^{com}	R_k
1	0.88	0.58
2	1.16	0.43
3	1.14	not defined

Note that compared to example 1, the revenue under private information is lower in this example for any number of spectrum licenses. Because we have a smaller value of the externality parameter, the cost of losing the auction is also smaller for the bidders, and this reduces the revenues of the auctioneer. In this example, under private information, the optimal number of licenses is 1. Observe that, the optimal number of licenses when the types are common knowledge is still 2.

Example 3

We now consider another interesting specification of the parameter values. Suppose that ξ = 0.7, λ_1 = 1 and λ_2 = 2. Compared to example 2, the magnitude of the externalities that bidders impose on each other has been reduced (which tends to reduce the auctioneer's revenue), but at the same time, the probability that a bidder has a better (lower) draw from the cost distribution increases (which tends to increase the auctioneer's revenue).

Table 2.3

The Externality Parameter is 0.7 and

Types are drawn from the Beta (1, 2) Distribution

k	R_k^{com}	R_k
1	1.01	0.79
2	1.46	0.82
3	1.48	not defined

In this example, the optimal number of licenses is 3, if the auctioneer can observe the bidders' types.

Compared to example 2, the auctioneer's revenues are increased in this example under private information. Furthermore, the optimal number of spectrum licenses is 2. This example shows that, under private information, the optimal number of spectrum licenses need not be 1.

4. CONCLUSION

In this essay, we have analyzed a simultaneous auction of spectrum licenses. Our model deviates from Das Varma (2003) in which the auctioneer have multiple and endogenously determine number of licenses. Bidders who win a license impose a negative externality on their competitors. More generally, this model can be viewed as extending the analysis of Milgrom and Weber (1982) to allow for the possibility that the acquisition of the object generates a negative externality to the other bidders. We have shown that the value of a spectrum license to a bidder of a given type is her intrinsic value; that is, this bidder's value of winning a license rather than having a competitor with the same marginal cost win it instead. We have found that the amount that each type will bid in a uniform price auction is her intrinsic value.

Our main conclusion is that it is not always optimal to allocate a single license. This conclusion demonstrates the restrictiveness of the assumption used in the previous literature that the auctioneer only auctions one license.

Because of the possibility that a winner of a spectrum license imposes a negative externality on other bidders, each bidder tries to preempt her competitors from winning a license by raising her bid. Failure to recognize this preemption motive may lead the auctioneer to allocate a non-optimal number of licenses. In our model, we have assumed that each bidder can bid for only one license and, therefore, it can prevent at most one competitor from winning a license. It may be more realistic to suppose that a bidder would try to prevent as many competitors as possible from winning licenses. To capture the full force of the bid-raising effect described above, we need to relax the assumption that bidders can only bid for one license. This issue is the subject of ongoing research.