ESSAY I
SEQUENTIAL AUCTION WITH SYNERGY STRUCTURE OF BIDDER'S PREFERENCES

1. INTRODUCTION

The spectrum-licenses allocation problem is a challenge. The objective of this problem is to allocate the licenses to the bidders[1] best able to turn spectrum into valuable services. In academic area, there is an agreement that auction is the most efficient way to allocate spectrum licenses. Base on Shy (2001, p.149), licensing spectrum by other means has been proved to be socially wasteful.

Alternative allocation methods to auction are administrative process and lotteries (McMillan, 1994, pp. 150-157). Both methods were rejected in developed countries. An administrative process has those that are interested in the spectrum license make a proposal for how they intend to use it. This approach is referred as a "beauty contest". After observing all the proposals, the regulator allocates the licenses to those with the most attractive proposal.

A beauty contest suffers from several problems. First, it is slow and wasteful since the firms spend a huge sum on trying to influence the regulator's decision. Evidently, it took the Federal Communications Commission (FCC) an average of two years to award thirty licenses. Second, beauty contest lacks transparency. It is difficult to identify why one proposal won out over another. Moreover, under incomplete information, the ability of the regulator to choose the best proposal is limited.

The delays in an allocation of spectrum licenses led the FCC to switch to lotteries. With a lottery, the FCC randomly selects the license winners among those that apply. The problem is that since the licenses are valuable, so there is an incentive for large numbers to apply. The large number of applications wasted resources in processing the potential applications. Moreover, the winners were not those best

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[1] Throughout, we use female pronounces for bidder and we use the term "bidders" and "firms" interchangeably.

suited to provide a service. It took years for the licenses to be transferred via market transactions. Lotteries were abandoned in favor of auction.

The major advantage of auction is its tendency to allocate the spectrum licenses to the most efficient bidder. This is accomplished by bidding competition among bidders. The bidders with the highest value for the licenses are willing to bid higher than the others, and tend to win the licenses. Second advantage is the competition is not wasteful since it leads to auction revenues. Finally, auction is transparent means of assigning licenses. All parties can see who won the auction and why.

Evidently, in the early 1990s only New Zealand and the United States used auction to allocate the spectrum licenses. By 2000, 12 countries used or were planning to use an auction to allocate the licenses. However, there is no consensus on the auction rules (Klemperer, 2004, pp. 207-209). In recent auctions for 3G mobile licenses, the different results that occurred in industries, have led to a number of economists refocusing attention on the auction design.

In Thailand, the spectrum licenses are allocated through an administration process. The allocation process is governed by the Telecommunications Business Act (2001). According to the Act, the National Telecommunication Commission (NTC) shall allocate the licenses to the potential firms wishing to operate in telecommunications industry. Hypothetically, the allocation of spectrum licenses which take place without formal allocation procedures such as auction is inefficient.

Hence, the objective of the essays is to develop the auction models as the alternative choices to tackle the spectrum-license allocation problem.

An auction model in the first essay is based on two frameworks; we consider multiple-objects auction and impose synergy structure of bidder’s preferences into our model. Although previous authors have recognized that both multiple-objects auction (Weber, 1983, Ausubel & Milgrom, 2002, and Parke, 2001; 2006) and synergy structure of bidder’s preferences (Menezes & Monteiro, 1999) are useful to model the spectrum-license allocation problem, to our knowledge, this essay makes the difference in the sense that we incorporate synergy structure into the multiple-objects auction model.

Our model deviates from Parke (2006) and Menezes and Monteiro (1999) from the following aspects. First, Parke assumes that the auctioneer has a large number of objects (not necessary spectrum licenses) to allocate without synergy. Whereas, our model has small but multiple spectrum licenses and the bidder has preferences exhibits positive synergies; two spectrum licenses are worth more as a bundle than as separate licenses. Second, Menezes and Monteiro assume complete information of object's value and consider both positive and negative synergy. This is not the case of our model; our model is incomplete information of license's value and considers only the case of positive synergy.

We begin by considering the synergy structure of bidder's preferences in small and exogenously set of objects. By backward induction, we solve for the equilibrium bidding strategy at the terminal date, then solves backward round by round. We show that both synergy structures and independent second drawn are jointly determine the equilibrium bidding strategy in our model. More importantly, we show that it is not always efficient to allocate the spectrum licenses in sequential second-price auction.

The essay is consisted of 4 sections. Section 1 is introduction, section 2 is literature review, section 3 is the model, and the last section is conclusion. Most of the proofs are presented in appendix A.

2. RELATED LITERATURE

This essay is closely related to the literature on multiple-object auction. Auction models are distinguished not only by the rules of the auction, such as open versus sealed bid, but by the auction environment. Important features, including the number of objects being allocated, the preferences of the bidders, and the form of the private information bidders have about preferences all determine the auction environment.

The benchmark environment is the private value model, introduced by Vickrey (1961). In the private value model, each bidder has a private value for each objects, and these values do not depend on the private information of the other bidders. Each bidder knows her values, but not the values of the other bidders. Our model retains the assumption on private value.

Vickrey's paper demonstrated equilibrium bidding behavior in a first-price auction, and then showed that truth-telling bid could be induced as a dominant strategy by modifying the payment rule: let each bidder pays the opportunity cost of her winnings, rather than her bid. Finally, he showed in an example what would later be proven generally as the revenue equivalence theorem.

In contemporary literature, the famous multiple-object auction is the generalization of the Vickrey auction, the Vickrey-Clarke-Groves (VCG) mechanism.

Ausubel and Milgrom (2002) have studied the question of why the VCG mechanism is seen so little in practice. In a VCG mechanism, bidders report their valuations for all packages; objects are allocated efficiently to maximize auctioneer's revenue. Each winner pays the opportunity cost of her winnings. In this way, a winner achieves a profit equal to her incremental contribution to revenue, and it is a dominant strategy for the bidder to truthfully reports her values. Achieving efficiency in truth-dominant strategies is remarkable. Nonetheless, there are shortcomings. That is bidders are asked to express values without the aid of any information about prices.

The stating point of modeling synergy structure in bidder's preference came from Weber (1983). Weber considers a sequential auction of multiple objects and shows that the expected prices follow a martingale i.e., bidders expect price will remain constant on average throughout the auction round. In Weber's model, bidders only bid one of a fixed number of objects. That is the marginal value of a second object is zero. This is not the case in our model. Since bidders in our model can bid for more than one spectrum licenses, hence the marginal value of a second license is positive.

The essence of Weber's result is that there are two opposite and exactly offsetting effects on price as the auction proceeds; a reduction in competition with fewer bidders puts downward pressure on price, while increased competition with fewer objects put upward pressure on price. There is, however, empirical evidence (see, for example, Ashenfelter (1989)) which now becomes a puzzle that in allocation of multiple (but identical) objects prices are not constant throughout the sequential auction.

Menezes and Monteiro (1999) have explained this puzzle. *Menezes* and Monteiro impose the synergy structure in the bidder's preferences and show that the existence of synergy generally implies in declining expected price. However, the Menezes and Monteiro's model is restricted since their result is based on the strong assumption of complete information of object's value. Our model, on the other hand, assumes that the license's value is a bidder's private information.

On recent multiple-object auction model, Parkes (2001, 2006) has examined iterative combinatorial auctions for a set of large K of objects. A motivation for an iterative process is to help the bidders express their preferences by providing provisional pricing and allocation information. Parkes finds out that the equilibrium bidding strategy inheriting the allocation efficiency property. That is a bidder with the highest value on object is a winner by optimally bids the price in the marginality condition.

Unfortunately, the Parke's model does not specify functional form of bidder preference, so the model's predictions construct possible results which lack economics intuition. Base on this reason, this essay is the simple version of Parkes (2006) multiple-object auction model by assuming explicit functional form of bidder utility function to incorporate synergy structure in Menezes and Monteiro (1999)'s sense and reduce number of objects to be allocated.

3. THE MODEL

This section constructs a simple case of Parkes (2006) by incorporating synergy structure in Menezes and Monteiro (1999)'s sense. The first section describes the environment of the model. The second section solves for equilibrium bidding strategy. The last section is a discussion on allocation efficiency property.

3.1 Environment

We consider the simple case of Parkes (2006) in which there are n bidders, $N = \{1,2,....,n\}$ and 2 objects, $K = \{1,2\}$ spectrum licenses to allocate. The spectrum licenses are allocated in the second-price auctions with sequential manner. The payment rule is second-price auctions (see, for example, Vickrey (1961), Ausubel & Milgrom (2002)) in the sense that the bidder with the highest bid wins the object, paying a price equal to the amount of the second highest bid. An auction rule is sequential (see, for example, Parkes (2006)) since the bidder bids for the first spectrum license in the first round and bids for the second license in the second round.

The essay proposes the sequentially second-price auctions for spectrum allocation from the following reason; sequential auction present a complication but realistic that does not arise in simultaneous auction that is the first auction round give information about bidders' type that can be used in the second round. And the essay chooses the second-price rather than the first-price auction because we can use the concept of dominant strategy to find out equilibrium bidding strategy.

The further assumption is the spectrum licenses are indivisible and identical objects in the sense that bidders only care about how many licenses they obtain. By construction, we are dealing with multi-unit demand model, that is bidders may bid for k = 1 or k = 2 spectrum licenses.

Time does matter in our model, we consider the case such that the private value of the licenses (the bidder's "type") are redrawn in the second round. We identify a bidder's "type" with the private value of license, a number $x_{t,i} \in [0,1]$ where index $t \in \{1,2\}$ represent the first and second auction round. That is any bidder $i \in N$

receives $x_{1,i} \in [0,1]$ in the first round and receives $x_{2,i} \in [0,1]$ in the second round. By these setting, as the number of bidder getting larger, types of any bidder are unlikely to be the same in each round, that is $x_{1,i} \neq x_{2,i}$ for any $i \in N$.

At the time of the auction each bidder knows her value in each round, a private value. However, she only knows that the values of the licenses of all other bidders are independently drawn from a common distribution $F(.)$ with support [0, 1] (which is common knowledge). We assume that distribution function $F(.)$ is continuously differentiable. Let $f(.)$ be the corresponding density function, where $f(x) > 0$ for all $x_{t,i} \in (0,1)$. Therefore, our model is multiple objects with independent private value.

To impose synergy structure as in Menezes and Monteiro (1999)'s sense, if the bidder with type $x_{t,i} \in [0,1]$ receives one spectrum license her utility is $U_1(x_{t,i}) = x_{1,i}$ in the first round and $U_1(x_{t,i}) = x_{2,i}$ in the second round. Put this generally, the bidder with type $x_{t,i}$ receives $U_1(x_{t,i}) = x_{t,i}$ where $t \in \{1,2\}$ if she receives one license. On the contrary, if she receives two spectrum licenses her utility is $U_2(x_{t,i}) = \alpha \bar{x}_{t,i}$ where $\bar{x}_{t,i} = \dfrac{x_{1,i} + x_{2,i}}{2}$ is the mean value of type of bidder i. We assume a positive real number $\alpha \geq 2$ to represent positive synergies across the spectrum licenses. That is two licenses together generate more utility than twice of the utility generated from a single license[2].

Let define what we mean by positive synergy in context of our model;

Definition : If $\alpha > 2$, the utility function exhibits *strict positive synergy*. But if $\alpha = 2$, the utility function exhibits *weak positive synergy*.

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[2] Since the set of object is $K = \{1,2\}$, by definition of positive synergy, we must have

$$U(1,2) \geq U(1,0) + U(0,2)$$

Since $U(1,0) = U_1(x_{t,i}) = x_{1,i}$, $U(0,2) = U_1(x_{t,i}) = x_{2,i}$ and $U(1,2) = U_2(x_{t,i}) = \alpha \bar{x}_{t,i}$ where $\bar{x}_{t,i} = \dfrac{x_{1,i} + x_{2,i}}{2}$. So, the inequality holds, only if $\alpha \in [2, \infty)$.

Decision tree of the model is as follows;

$t=0$	$t=1$	$t=2$	$t=3$
Bidder receives private information $x_{1,i} \in [0,1]$ in the first round	Bidder realizes a positive real $\alpha \in [2,\infty)$	Bidder submits bid for the first round	Outcome in the first round is observed

$t=4$	$t=5$	$t=6$
Bidder receives private information $x_{2,i} \in [0,1]$ in the second round	Bidder submits bid for the second round	Outcome in the second round is observed

From the decision tree, at $t=0$ each bidder $i \in N$ receives private information $x_{1,i} \in [0,1]$ from the distribution function F(.) with density f(.) > 0. At $t=1$, the bidder realizes a positive real $\alpha \in [2,\infty)$ which represent positive synergies of her preferences. At $t=2$, given sequential second-price auction rule, each bidder submits her bid for the first round of auction. At $t=3$, the outcome in the first round is observed. At $t=4$, each bidder receives private information in the second round $x_{2,i} \in [0,1]$ from the common distribution function F(.). At $t=5$, each bidder submits her bid for the second round. At $t=6$, the outcome in the second round is observed and the auction is completed.

Since our model tries to capture the case that each bidder types are redrawn in the second round hence, as number of bidder getting larger, types of any bidder are unlikely to be the same in each round. The primer notation on order statistics in each round and its properties are as follows; We use notation $x_{1,(k)}^{n-1}$ to represent the k^{th} highest value of type of bidder i's competitors in the $t \in \{1,2\}$ round (the superscript $n-1$ shows that for any bidder $i \in N$, she has $n-1$ competitors), that is

$$\begin{aligned}
x_{t,(1)}^{n-1} &= \max\{x_{t,1},...,x_{t,i-1},x_{t,i+1},...,x_{t,n}\} \\
&= \max\{X_{t,-i}\} \\
x_{t,(2)}^{n-1} &= \max\{\{X_{t,-i}\} \setminus \{x_{t,(1)}^{n-1}\}\} \\
&. \\
x_{t,(k)}^{n-1} &= \max\{\{X_{t,-i}\} \setminus \{x_{t,(1)}^{n-1},...,x_{t,(k-1)}^{n-1}\}\} \\
&. \\
x_{t,(n-1)}^{n-1} &= \max\{\{X_{t,-i}\} \setminus \{x_{t,(1)}^{n-1},...,x_{t,(n-2)}^{n-1}\}\} \\
&= \min\{X_{t,-i}\}
\end{aligned}$$

Since $x_{t,i} \in [0,1]$ for all $i \in N$, we must have following relations;

$$1 \geq x_{t,(1)}^{n-1} \geq x_{t,(2)}^{n-1} \geq ... \geq x_{t,(k)}^{n-1} \geq ... \geq x_{t,(n-1)}^{n-1} \geq 0$$

Note that in modeling the independent second drawn, as number of bidder getting larger, types of any bidder are unlikely to be the same in each round, that is as $n \to \infty$ we have $x_{1,i} \neq x_{2,i}$ for any $i \in N$. However, as the types of bidders are independently drawn from a common distribution $F(.)$ with the same support [0, 1] in both rounds, we have $x_{1,(1)}^{n-1} = x_{2,(1)}^{n-1}$ and $x_{1,(n-1)}^{n-1} = x_{2,(n-1)}^{n-1}$. Put this in word, the common distribution with the same bound support overtimes implies the same bound support of the order statistics in both rounds.

The following part deals with how rational bidder behaves in this auction rule. Specifically, we study the equilibrium bidding strategy of the bidder in the second and first auction round.

3.2 Equilibrium Bidding Strategy

Since we have $k \in K = \{1,2\}$ spectrum licenses to allocate in sequential auction, we have at most 2 auction rounds. The bidder must consider optimal strategy in the second round first then think about the first round. This process of thinking is known as backward induction.

Let consider equilibrium bidding strategy of the second-price auctions in backward induction. The equilibrium bidding strategy in the second round is straightforward, since the payment rule is second-price auction and we consider independent private value type model, it is well known that truthful revelation of type is dominant strategy of the bidder. That is if bidder i does not win one spectrum license in the first round, she will bid $b_i^2(x_{2,i}) = U_2(x_{2,i}) = x_{2,i}$ in the second round.

But if she does win one spectrum license in the first round, she will bid $b_i^2(x_{t,i}) = U_2(x_{t,i}) - U_1(x_{1,i}) = \alpha \bar{x}_{t,i} - x_{1,i} \equiv M(x_{t,i})$ (this bid is equal to her value for an additional unit of spectrum license) in the second round. Note that if $\alpha > 2$, then $M(x_{t,i}) > x_{2,i}$. But if $\alpha = 2$, then $M(x_{t,i}) = x_{2,i}$. The finding can be summarized in lemma 1.1.

Lemma 1.1 (Equilibrium bidding strategy in the second round) :

In the sequentially second-price auctions, for identical $k \in K = \{1,2\}$ and utility of the bidder exhibits a positive synergy, the equilibrium bidding strategy in the second round is

$b_i^2(x_{2,i}) = x_{2,i}$ if bidder i does not win one spectrum license in the first round.

$b_i^2(x_{t,i}) = \alpha \bar{x}_{t,i} - x_{1,i} \equiv M(x_{t,i})$ if bidder i does win one spectrum license in the first round.

proof : see appendix A.1.

Lemma 1.1 predicts that the equilibrium bidding strategy in the second round will become aggressive which enhances the auctioneer's revenue only if (1) bidder does win one spectrum license in the first round and (2) the utility function exhibits strict rather than weak positive synergy. Moreover, when synergy is weak positive, equilibrium bidding strategy in the second round is the same in either bidder does or does not win one spectrum license in the first round.

Intuitively, in the case of strict positive synergy, the value of the bundle of the licenses generate strictly higher payoffs to the bidder than sum of the payoffs that generated by an exclusive license. Hence, the synergy structure of bidder preferences drives up the equilibrium bidding strategy in the second round only in this case.

Now let consider equilibrium bidding strategy in the first round. We consider symmetric Bayesian Nash equilibrium by assume that bidder $-i = 1,...,i-1,i+1,...,n$

bids according to the continuously and strictly increasing bidding strategy $b(x_{1,-i})$ in the first round.

By definition of order statistics, recall that $x_{1,(k)}^{n-1}$ is the k^{th} highest value of type of bidder i's competitors in the first round, that is

$$x_{1,(k)}^{n-1} = \max\left\{\left\{X_{1,-i}\right\} \setminus \left\{x_{1,(1)}^{n-1},\ldots,x_{1,(k-1)}^{n-1}\right\}\right\}$$

This statistics are such that,

$$1 \geq x_{1,(1)}^{n-1} \geq x_{1,(2)}^{n-1} \geq \ldots \geq x_{1,(k)}^{n-1} \geq \ldots \geq x_{1,(n-1)}^{n-1} \geq 0$$

In the following discussion, to simplify the notation, we drop the superscripts $n-1$ from the order statistics. So, $x_{t,(1)}$ is the maximal value of spectrum license of bidder i's competitors in round $t \in \{1,2\}$ and $x_{t,(2)}$ is the second highest value and so on. And since we have a small set of object, i.e. $K = \{1,2\}$, the relevant order statistics of value of the license that we need to consider is only $x_{t,(1)}$ and $x_{t,(2)}$.

Thus, if bidder i, with type $x_{1,i} \in [0,1]$, bid with $b(z)$ for some $z \in [0,1]$ her expected utility is

$$E\left[\left(x_{1,i} - b\left(x_{1,(1)}\right) + (M\left(x_{t,i}\right) - x_{2,(1)})^{+}\right)\right] \; if \; b(z) > b\left(x_{1,(1)}\right)$$

$$\text{and } E\left[\left(x_{2,i} - \max\left\{M\left(x_{t,W}\right), x_{2,(1)}^{L}\right\}\right)^{+}\right] \; if \; b(z) < b\left(x_{1,(1)}\right)$$

Where $M\left(x_{t,i}\right) = \alpha \bar{x}_{t,i} - x_{1,i}$ is marginal value of bidder i with the properties that $M\left(x_{t,i}\right) > x_{2,i}$ if $\alpha > 2$ and $M\left(x_{t,i}\right) = x_{2,i}$ if $\alpha = 2$. $M\left(x_{t,W}\right)$ is the marginal value of the winner of the first round with the properties that $M\left(x_{t,W}\right) > x_{2,W}$ if $\alpha > 2$ and $M\left(x_{t,W}\right) = x_{2,W}$ if $\alpha = 2$. $x_{2,(1)}^{L}$ is the highest value of type in the second round which drawn from the group of loser in the first round.

Note that if $\alpha > 2$ (the case of strict positive synergy), then $M\left(x_{t,W}\right) > x_{2,(1)}^{L}$, which implies that $\max\left\{M\left(x_{t,W}\right), x_{2,(1)}^{L}\right\} = M\left(x_{t,W}\right)$, but if $\alpha = 2$ (the case of weak positive synergy), then we can have the case that $M\left(x_{t,W}\right) < x_{2,(1)}^{L}$, so $\max\left\{M\left(x_{t,W}\right), x_{2,(1)}^{L}\right\} = x_{2,(1)}^{L}$.

Since $b(.)$ is strictly increasing function over [0,1], we have the following equivalent conditions;

$$b(z) > b(x_{1,(1)}) \Leftrightarrow z > x_{1,(1)}$$

$$\text{and } b(z) < b(x_{1,(1)}) \Leftrightarrow z < x_{1,(1)}$$

Hence, we can express the expected utility of bidder i, with type $x_{1,i} \in [0,1]$, bid with $b(z)$ for some $z \in [0,1]$ as function $\phi(z)$;

$$\begin{aligned}\phi(z) = E\Big[\big(x_{1,i} - b(x_{1,(1)}) + (M(x_{t,i}) - x_{2,(1)})^{+}\big).\chi_{z > x_{1,(1)}}\Big] \\ + E\Big[\big(x_{2,i} - \max\{M(x_{t,W}), x_{2,(1)}^{L}\}\big)^{+}.\chi_{z < x_{1,(1)}}\Big] \qquad (1)\end{aligned}$$

Note that, we use notation x^{+} to denote nonnegative part of x that is the maximum between x and 0 or $x^{+} = \max\{x,0\}$. Notation $\chi_{z > x_{1,(1)}}$ is an indicator variable which assign number 1 if the event z > $x_{1,(1)}$ occur and 0 otherwise. Notation $\chi_{z < x_{1,(1)}}$ is defined in similar manner.

Equation (1) is expected utility of bidder i, with type $x_{1,i} \in [0,1]$, bid with $b(z)$. The first part of (1) is for the case that if bidder i's type is higher than $x_{1,(1)}$, then she wins the first spectrum license paying price equal $b(x_{1,(1)})$ (recall that the payment rule is second-price auctions) and bids equal $M(x_{t,i})$ in the second round.

The term $\max\{M(x_{t,W}), x_{2,(1)}^{L}\}$ in the second part of (1) appears for the following reason. If bidder i does not win the first license, we know that her type is smaller than $x_{1,(1)}$. The bidder with type $x_{1,(1)}$ wins the license and therefore bids $M(x_{t,W})$ in the second round. The non-winning bidders from the first round bid their types, the highest of which is $x_{2,(1)}^{L}$, in the second round.

If we have strict positive synergy then $M(x_{t,W}) > x_{2,(1)}^{L}$. Which implies that $\max\{M(x_{t,W}), x_{2,(1)}^{L}\} = M(x_{t,W})$. If we have weak positive synergy then $M(x_{t,W}) < x_{2,(1)}^{L}$ and $\max\{M(x_{t,W}), x_{2,(1)}^{L}\} = x_{2,(1)}^{L}$. Hence, there are two possible cases to consider that are bidder i's type is higher or lower than $x_{1,(1)}$. We will consider both possible cases together by write (1) in term of density of $x_{1,(1)}$, i.e. $f_{X_1}(x)$.

First, consider the conditional expectation of $\left(x_{2,i}-\max\left\{M\left(x_{t,W}\right),x_{2,(1)}^{L}\right\}\right)^{+}$ given $x_{1,(1)}=x$, express as function $\psi(x)$;

$$\psi(x)=E\left[\left(x_{2,i}-\max\left\{M\left(x_{t,W}\right),x_{2,(1)}^{L}\right\}\right)^{+}.\mid x_{1,(1)}=x\right] \tag{2}$$

By (2), and since we know that

$$\begin{aligned}E\left[\psi(x_{1,(1)})\chi_{z<x_{1,(1)}}\right]&=E\left[E\left[\left(x_{2,i}-\max\left\{M\left(x_{t,W}\right),x_{2,(1)}^{L}\right\}\right)^{+}\mid x_{1,(1)}=x_{1,(1)}\right]\chi_{z<x_{1,(1)}}\right]\\&=E\left[\left(x_{2,i}-\max\left\{M\left(x_{t,W}\right),x_{2,(1)}^{L}\right\}\right)^{+}\chi_{z<x_{1,(1)}}\right]\end{aligned}$$

by iterated expectation, we can rewrite (1) as

$$\phi(z)=\int_{0}^{z}(x_{1,i}-b(x)+(M\left(x_{t,i}\right)-x)^{+})f_{X_1}(x)dx+\int_{z}^{1}\psi(x)f_{X_1}(x)dx \tag{3}$$

To write (3) we use property of order statistics that the common distribution with the same bound support overtimes implies the same bound support of the order statistics in both rounds. The first part of (3) we integrate $x_{1,(1)}=x$ from 0 to z, i.e. the case that bidder i's type is higher than $x_{1,(1)}$. The second part of (3) we integrate $x_{1,(1)}=x$ from z to 1, i.e. the case that bidder i's type is lower than $x_{1,(1)}$.

Recall that $\phi(z)$ is the expected utility of bidder i, with type $x_{1,i}\in[0,1]$, bid with $b(z)$ for some $z\in[0,1]$. Now we want to find optimal z that maximize $\phi(z)$ by differentiate $\phi(z)$ with respect to z , we get a continuous function

$$\phi'(z)=(x_{1,i}-b(z)+(M\left(x_{t,i}\right)-z)^{+}-\psi(z))f_{X_1}(z) \tag{4}$$

Thus by (4), if the optimal z is $x_{1,i}$ then $\phi'(x_{1,i})=0$ and we have

$$b(x_{1,i})=x_{1,i}+\left(M\left(x_{t,i}\right)-x_{1,i}\right)^{+}-\psi(x_{1,i}) \tag{5}$$

Equation (5) implies,

$$b(z)=z+\left(M\left(z\right)-z\right)^{+}-\psi(z) \tag{6}$$

Substituting (6) into (4) yields

$$\theta(z) \equiv \frac{\phi'(z)}{f_{X_1}(z)} = x_{1,i} - \left(z + \left(M(z) - z\right)^+ - \psi(z)\right) + \left(M\left(x_{t,i}\right) - z\right)^+ - \psi(z)$$
$$= x_{1,i} - z - \left(M(z) - z\right)^+ + \left(M\left(x_{t,i}\right) - z\right)^+ \qquad if\ \alpha > 2$$
$$= x_{1,i} - z \qquad if\ \alpha = 2 \qquad (7)$$

By (7), now we can show that z = $x_{1,i}$ can be supported to be optimal in both weak and strict positive synergy by following arguments;

For weak positive synergy, we have $\alpha = 2$, so if z < $x_{1,i}$ then $\phi'(z) > 0$ and if z > $x_{1,i}$ then $\phi'(z) < 0$. And with the fact that $\phi'(z)$ is continuous function and density function $f_{X_1}(z) > 0$, by virtue of an intermediate value theorem, therefore z = $x_{1,i}$ maximizes $\phi(.)$.

For strict positive synergy, we have $\alpha > 2$, so $M\left(x_{t,i}\right) > x_{2,i}$ and $M(z) > x_{2,i}$, if z < $x_{1,i}$ then

$$\theta(z) \equiv \frac{\phi'(z)}{f_{X_1}(z)} = x_{1,i} - z - \left(M(z) - z\right)^+ + \left(M\left(x_{t,i}\right) - z\right)^+ > 0$$

If z > $x_{1,i}$ then

$$\theta(z) \equiv \frac{\phi'(z)}{f_{X_1}(z)} = x_{1,i} - z - \left(M(z) - z\right)^+ + \left(M\left(x_{t,i}\right) - z\right)^+ < 0$$

Again, since $\phi'(z)$ is continuous function and density function $f_{X_1}(z) > 0$ therefore by an intermediate value theorem, there exist z = $x_{1,i} \in [0,1]$ such that $\theta\left(x_{1,i}\right) = 0$, hence we can conclude that z = $x_{1,i}$ maximizes $\phi(.)$.

Hence, by reasoning of optimization above, we can show that the equilibrium bidding strategy in the first round is

$$b(x_{1,i}) = x_{1,i} + \left(M\left(x_{t,i}\right) - x_{1,i}\right)^+ - E\left[\left(x_{2,i} - \max\left\{M\left(x_{t,W}\right), x_{2,(1)}^L\right\}\right)^+ . \mid x_{1,(1)} = x_{1,i}\right]$$

This is stated formally in the following proposition;

Proposition 1.1 (Equilibrium bidding strategy in the first round) :

In the sequentially second-price auctions, for identical $k \in K = \{1,2\}$ and utility of the bidder exhibits a positive synergy, the equilibrium bidding strategy in the first round is given by

$$b(x_{1,i}) = x_{1,i} + \left(M\left(x_{t,i}\right) - x_{1,i}\right)^{+} - E\left[\left(x_{2,i} - \max\left\{M\left(x_{t,W}\right), x_{2,(1)}^{L}\right\}\right)^{+} . \mid x_{1,(1)} = x_{1,i}\right]$$

proof : follow from the discussion above.

The equilibrium bidding strategy in the first round as suggested in proposition 1.1 may be complicate at the first glance. To clarify this bidding strategy, by proposition 1.1 and definition of strict and weak positive synergy, we obtain the following corollaries;

Corollary 1.1 (Equilibrium bidding strategy in the first round, in case of strict positive synergy) :

In the sequentially second-price auctions, for identical $k \in K = \{1,2\}$ and utility of the bidder exhibits strict positive synergy, the equilibrium bidding strategy in the first round is given by

$$\begin{aligned} b(x_{1,i}) &= M\left(x_{t,i}\right) && \textit{if } M\left(x_{t,i}\right) > x_{1,i} \\ &= x_{1,i} && \textit{if } M\left(x_{t,i}\right) < x_{1,i} \end{aligned}$$

proof :

If there is strict positive synergy, we have that $\alpha > 2$, this implies $M\left(x_{t,i}\right) > x_{2,i}$. And hence, $\max\left\{M\left(x_{t,W}\right), x_{2,(1)}^{L}\right\} = M\left(x_{t,W}\right)$ where $M\left(x_{t,W}\right) > x_{2,W}$ so it is follow that $E\left[\left(x_{2,i} - \max\left\{M\left(x_{t,W}\right), x_{2,(1)}^{L}\right\}\right)^{+} . \mid x_{1,(1)} = x_{1,i}\right] = 0$ which implies that the equilibrium bidding strategy in proposition 1.1, is given by

$$\begin{aligned} b(x_{1,i}) &= x_{1,i} + \left(M\left(x_{t,i}\right) - x_{1,i}\right)^{+} \\ &= M\left(x_{t,i}\right) && \textit{only if } M\left(x_{t,i}\right) > x_{1,i} \Rightarrow \left(M\left(x_{t,i}\right) - x_{1,i}\right)^{+} = M\left(x_{t,i}\right) - x_{1,i} \, . \\ &= x_{1,i} && \textit{only if } M\left(x_{t,i}\right) < x_{1,i} \Rightarrow \left(M\left(x_{t,i}\right) - x_{1,i}\right)^{+} = 0 \end{aligned}$$

and the proof is done.

Corollary 1.2 (Equilibrium bidding strategy in the first round, in case of weak positive synergy) :

In the sequentially second-price auctions, for identical $k \in K = \{1,2\}$ and utility of the bidder exhibits weak positive synergy, the equilibrium bidding strategy in the first round is given by

$$b(x_{1,i}) = E\left[\left(\max\{x_{2,w}, x_{2,(1)}^{L}\}\right)^{+} . \mid x_{1,(1)} = x_{1,i}\right] \; if \;\; x_{2,i} > x_{1,i}$$

and

$$b(x_{1,i}) = x_{1,i} - E\left[\left(x_{2,i} - \max\{x_{2,w}, x_{2,(1)}^{L}\}\right)^{+} . \mid x_{1,(1)} = x_{1,i}\right] \; if \;\; x_{1,i} > x_{2,i}$$

proof :

If there is weak positive synergy, by definition, we have that $\alpha = 2$, this implies $M(x_{t,i}) = x_{2,i}$ and $M(x_{t,W}) = x_{2,W}$. And hence, it is follow that $b(x_{1,i}) = x_{1,i} + (x_{2,i} - x_{1,i})^{+} - E\left[\left(x_{2,i} - \max\{x_{2,w}, x_{2,(1)}^{L}\}\right)^{+} . \mid x_{1,(1)} = x_{1,i}\right]$ which implies that the equilibrium bidding strategy in proposition 1.1, is given by

$$b(x_{1,i}) = E\left[\left(\max\{x_{2,w}, x_{2,(1)}^{L}\}\right)^{+} . \mid x_{1,(1)} = x_{1,i}\right] \; if \;\; x_{2,i} > x_{1,i} \Rightarrow (x_{2,i} - x_{1,i})^{+} = x_{2,i} - x_{1,i}$$

and

$$b(x_{1,i}) = x_{1,i} - E\left[\left(x_{2,i} - \max\{x_{2,w}, x_{2,(1)}^{L}\}\right)^{+} . \mid x_{1,(1)} = x_{1,i}\right] \; if \;\; x_{1,i} > x_{2,i} \Rightarrow (x_{2,i} - x_{1,i})^{+} = 0$$

and the proof is done.

3.3 Efficient Allocation Property

It is our interest to know whether the efficient allocation property, which is a nice property in existing sequential multiple-object auction model (see for example, Weber, 1983, Ausubel & Milgrom, 2002, and Parke, 2001; 2006), always holds in context of our model. Unfortunately, once we incorporate synergy structure into the multiple-objects auction model, the answer is no.

Base on corollary 1.1 and 1.2, both synergy structures of bidder's preferences as well as independent second drawn are jointly determine the equilibrium bidding strategy in our model. This result has alternate predictions with exiting multiple-objects auction model without synergy (see Weber, 1983, Parke, 2001, 2006).

One can observe that for the case of strict positive synergy, the marginal value of any bidder $i \in N$ has the property that this value is strictly higher that the private value of the license in the second round i.e. $M\left(x_{t,i}\right) > x_{2,i}$.

Moreover, the equilibrium bidding strategy in the first round is either the marginal value or the private value of the license in the first round. In this case the bidder with the highest value of spectrum license wins the license by optimally bids the price in either marginality condition or truth-telling bid. Hence, the allocation efficiency is guaranteed in this case. That is we can allocate the spectrum licenses into the hands of the bidder who values them the most.

The opposite result appears in the case of weak positive synergy. If the bidder's preferences exhibit weak positive synergy, then the marginal value of any bidder $i \in N$ has the property that this value is equal to the private value of the license in the second round i.e. $M\left(x_{t,i}\right) = x_{2,i}$.

In this case the equilibrium bidding strategy in the first round depended on two mutually exclusive events that can occur. If the bidder expects that the private value of the license in the second round is higher than the first round, the equilibrium bidding strategy is to choose the maximal value between either the winner's value of the license or the highest of the loser's value. When this event happens, the auctioneer cannot allocate the spectrum license into the hands of the bidder who values them the most. Hence, the efficient allocation property is failed in this case.

The situation is poorer for the auctioneer if the other event occurs. If the bidder expects that the private value of the license in the second round is lower than the first round, this expectation will further drives down the equilibrium bidding strategy. The intuition is that for any bidder that tends to create pessimistic expectation on the future value of the spectrum license also tends to submit uncompetitive bid today.

4. CONCLUSION

In this essay, we have analyzed a sequential second-price auction of spectrum licenses. Our model is the simple case of Parkes (2006) by incorporating synergy structure in Menezes and Monteiro (1999)'s sense. The bidder's preferences in this model exhibit positive synergies. That is two licenses together generate more utility to the bidder than twice of the utility generated from a single license. Moreover, each bidder's private values in this model are redrawn in the second round. We have shown that the synergy structure drives up the bidding strategy only in case of strictly positive synergy. We have found that any bidder that tends to create pessimistic expectation on the future value also tends to submit uncompetitive bid today.

Our main conclusion is that it is not always efficient to allocate the spectrum licenses in sequential second-price auction. This conclusion reveals the ineffective of the prediction used in the literature that the auctioneer can always guarantee allocation efficiency in the sequential auction.

In our model, we have assumed that the set of objects is small and exogenous. It may be more realistic to relax this assumption and consider more combinatorial complexity. This direction of model extension is left for the future research.